Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.63209

The existence results for a fractional Riemann-Liouville boundary value problem involving the p-Laplace operator

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ABSTRACT: In the present paper, we will study the multiplicity of solutions for some classes of boundary value problems involving the Riemann Liouville operators and the p-Laplacian operator. The proofs are based on the variational method combined with the Nehari manifold method and the fibering map analysis.

Key Words: Variational methods, Riemann Liouville derivative, Nehari manifold, boundary value problems.

Contents

1. Introduction

Fractional differential equations have played an important role in many fields such as physics, chemistry, aerodynamics, and electrodynamics of complex mediums. We refer the readers to [39,32] and references therein.
Recently, there have been significant works in boundary value problems involving fractional derivatives. The interested readers can see the monographs [1,2,3,4,7,11,24,26,30,31,32,40]. Also, Some recent contributions on the existence of solutions for fractional boundary value problems have been investigated, we refer the interested readers to [5,7,9,10,16,18,19,24,32,40]. Moreover, problems including both left and right fractional derivatives are discussed. In this topic, existence results are obtained by using different techniques, such as fixed-point theory [21], critical points theory [26,27] and comparison method [5]. It should be noted that Variational methods have also turned out to be very effective tools in determining the existence and multiplicity of solutions for fractional boundary value problems, like the Nehari manifold combined with fibering maps [13,17,20,35,36] and Mountain pass theorem [15,37].
Inspired by the above-mentioned papers, the goal of this work is to look at the existence and the multiplicity of nontrivial solutions to the following p-fractional boundary value problem

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_T^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}\ {}_0D_t^\alpha u(t)\right)=\lambda g(t)|u(t)|^{r-1}u(t)+f(t,u(t))\ t\in(0,T);\\ u(0)=u(T)=0,\end{cases}$$

where $T>0$, $p,r>1$, $\frac{1}{p}<\alpha\leq 1$, λ is a positive parameter, g is a bounded function and $f\in C([0,T]\times\mathbb{R},\mathbb{R})$ is positively homogeneous of degree $q-1$ that is $f(x,tu)=t^{q-1}f(x,u)$ hold for all $(x,u)\in[0,T]\times\mathbb{R}$. In addition, we assume the following hypotheses:
(H_1) The function F homogeneous of degree q that is, for all $s>0$, we have

$$F(t,su)=s^qF(t,u),\quad \text{for all}\quad (t,u)\in[0,T]\times\mathbb{R}.$$

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Submitted April 09, 2022. Published May 23, 2025
2010 *Mathematics Subject Classification:* 34A08, 34B10, 47H10.

Note that, from $(\mathbf{H}_2)$, f leads to the so-called Euler identity

$$uf(x,u) = qF(x,u),$$

and

$$|F(x,u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1.1}$$

Theorem 1.1 Under hypothesis (H_1), if $1 < q < p < r$, then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, problem (P_λ) admits at least two nontrivial solutions.

The following is a breakdown of the paper's structure. Preliminaries on fractional calculus are covered in Section 2. In Section 3, we set up the problem's variable framework (P_λ) and provide the relevant lemmas. Section 4 concludes with the proof of the main result (Theorem 1.1).

2. Preliminaries

In this section, we provide some basic theory on the idea of fractional calculus, focusing on the Riemann-Liouville operators and results, which will be used throughout the article. Let us begin by defining the term the fractional integral in Riemann-meaning Liouville's.

Definition 2.1 Let $\alpha > 0$ and u be a function defined a.e. on $(a,b) \subset \mathbb{R}$. The left (respectively, the right) Riemann Liouville fractional integral with an inferior limit a (resp. superior limit b) of order α

$${}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)ds, \quad t \in (a,b],$$

respectively

$${}_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} u(s)ds, \quad t \in [a,b).$$

Provided that the right side is defined point-wise on $[a,b]$, where Γ is the Euler's Gamma function.
We notes that, if $u \in L^1(a,b)$, then ${}_aI_t^\alpha u$ and ${}_tI_b^\alpha u$ are defined a.e. on (a,b).

Now, we define in the following, the fractional derivatives in the sense of Riemann-Louville.

Definition 2.2 Let $0 < \alpha < 1$. Then, the Left (resp. right) fractional derivative in the sense of Riemann-Liouville with an inferior limit a (resp. superior limit b) of order α of u is given by

$${}_aD_t^\alpha u(t) = \frac{d}{dt}\left({}_aI_t^{1-\alpha}u\right)(t), \ \forall\, t \in (a,b],$$

respectively

$${}_tD_b^\alpha u(t) = \frac{d}{dt}\left({}_tI_b^{1-\alpha}u\right)(t), \ \forall\, t \in [a,b],$$

provided that the right-hand side is point-wise defined.

Remark 2.1 If u is an absolutely continuous function in $[a,b]$. Then ${}_aD_t^\alpha u$ and ${}_tD_b^\alpha u$ are defined a.e. on (a,b) and fulfill.

$${}_aD_t^\alpha u(t) = \ {}_aI_t^{1-\alpha}u'(t) + \frac{u(a)}{(t-a)^\alpha \Gamma(1-\alpha)} \tag{2.1}$$

and

$${}_tD_b^\alpha u(t) = \ -{}_tI_b^{1-\alpha}u'(t) + \frac{u(b)}{(b-t)^\alpha \Gamma(1-\alpha)}. \tag{2.2}$$

In addition, if $u(a) = u(b) = 0$, then ${}_aD_t^\alpha u(t) ={}_a I_t^{1-\alpha}u'(t)$ and ${}_tD_b^\alpha u(t) = -{}_tI_b^{1-\alpha}u'(t)$. So, in this case, the Riemann-Liouville fractional derivative and the Caputo derivative are equivalent.

$${}_a^cD_t^\alpha u(t) = \ {}_aI_t^{1-\alpha}u'(t)$$

and

$$ {}_t^c D_b^{\alpha} u(t) = - {}_t I_b^{1-\alpha} u'(t). $$

Consequently, one gets

$$ {}_a D_t^{\alpha} u(t) = {}_a^c D_t^{\alpha} u(t) + \frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)}, $$

and

$$ {}_t D_b^{\alpha} u(t) = {}_t^c D_b^{\alpha} u(t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)}. $$

Following that, we discuss certain properties of Riemann-left Liouville's fractional operators. We suggest the reader to [28] for more information on Fractional integral.

Proposition 2.1 The following equality holds for all $\alpha,\ \beta > 0$ and any $u \in L^1(a,b)$,

$$ {}_a I_t^{\alpha} \circ {}_a I_t^{\beta} u = {}_a I_t^{\alpha+\beta}. $$

From proposition 2.1 and the equations (2.1) and (2.2), it is simple to deduce the following results concerning the composition between fractional integral and fractional derivative. That is, for any $0 < \alpha < 1$, if $u \in L^1(a,b)$ we have

$$ {}_a D_t^{\alpha} \circ {}_a I_t^{\alpha} u = u, $$

and if u is absolutely continuous such that $u(a) = 0$. Then, one has

$$ {}_a I_t^{\alpha} \circ {}_a D_t^{\alpha} u = u. $$

Now, we presented an important result on the bounds of the left fractional integral from $L^p(a,b)$ to $L^p(a,b)$.

Proposition 2.2 This proves the bounds of the left fractional integral from $L^p(a,b)$ to $L^p(a,b)$. For any $\alpha > 0$ and any $p \geq 1$, ${}_a I_t^{\alpha}$ it is linear and continuous. In addition, for any $u \in L^p(a,b)$, we have

$$ \|{}_a I_t^{\alpha} u\|_p \leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} \|u\|_p. $$

Similarly, we prove a traditional conclusion on the bounds of the left fractional integral from $L^p(a,b)$ to $C_a(a,b)$ which completes Proposition 2.2 in the case $0 < \frac{1}{p} < \alpha < 1$.

Proposition 2.3 Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then, for any $u \in L^p(a,b)$, ${}_a I_t^{\alpha} u$ is Hölder continuous on $(a,b]$ with exponent $\alpha - \frac{1}{p} > 0$, moreover, ${}_a I_t^{\alpha} u(t) = 0$. Consequently, ${}_a I_t^{\alpha} u$ can be continuously extended by 0 into $t = a$. Finally, ${}_a I_t^{\alpha} u \in C_a(a,b)$, and

$$ \|{}_a I_t^{\alpha} u\|_{\infty} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((a-1)q+1\right)^{\frac{1}{q}}} \|u\|_p. \tag{2.3} $$

Also, we will need the following formula for integration by parts:

Proposition 2.4 Let $0 < \alpha < 1$ and p, q are such that

$$ p \geq 1,\ q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \alpha \quad \text{or} \quad p \neq 1,\ q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \alpha. $$

Then, for all $u \in L^p(a,b)$ and all $v \in L^q(a,b)$, one has

$$ \int_a^b v(t)\ {}_a I_t^{\alpha} u(t) dt = \int_a^b u(t)\ {}_a I_t^{\alpha} v(t) dt, \tag{2.4} $$

and

$$ \int_a^b u(t)\ {}_a^c D_t^{\alpha} v(t) dt = v(t) {}_t I_b^{1-\alpha} u(t)\Big|_{t=a}^{t=b} + \int_a^b v(t)\ {}_a D_t^{\alpha} u(t) dt. \tag{2.5} $$

Moreover, if $v(a) = v(b) = 0$, then, one gets

$$ \int_a^b u(t)\ {}_a D_t^{\alpha} v(t) dt = \int_a^b v(t)\ {}_a^c D_t^{\alpha} u(t) dt. \tag{2.6} $$

3. A Variational setting and proof of the main results

Critical point theory will be used to demonstrate the existence of solutions to the problem (P_λ). We introduce some basic notations and results for this purpose, which we will use to prove our main findings. The set of all functions $\phi \in C^\infty([0,T],\mathbb{R})$ with $\phi(0) = \phi(T) = 0$ is denoted by $C_0^\infty([0,T],\mathbb{R})$. For, $\alpha > 0$ we define the fractional derivative space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0,T],\mathbb{R})$ under the norm

$$\|\phi\|_{\alpha,p} = \left(\|\phi\|_p^p + \|{}_0^cD_t^\alpha \phi\|_p^p\right)^{\frac{1}{p}}. \tag{3.1}$$

Remark 3.1 (i) The fractional derivative space $E_0^{\alpha,p}$ is clearly defined as the space of functions $\phi \in L^p([0,T])$ with an α-order Caputo fractional derivative ${}_0^cD_t^\alpha \phi \in L^p([0,T])$ and $\phi(0) = \phi(T) = 0$.

(ii) We have $\phi \in E_0^{\alpha,p}$, for each u e *Enoting* the fact $\phi(0) = 0$, we have

$$ {}_0^cD_t^\alpha \phi(t) = {}_0D_t^\alpha \phi(t),\ t \in [0,T].$$

This means that the left and right Riemann-Liouville fractional derivatives of order α are equivalent to the left and right Caputo fractional derivatives of order α.

(iii) The fractional space $E_0^{\alpha,p}$ is reflexive and a separable Banach space.

Lemma 3.1 Let $0 < \alpha \le 1$, and $1 < p < \infty$. Then, for all $\phi \in E_0^{\alpha,p}$, one has

$$\|\phi\|_p \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha \phi\|_p. \tag{3.2}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{\widetilde{p}} = 1$, we have

$$\|\phi\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}}\|{}_0D_t^\alpha \phi\|_p. \tag{3.3}$$

According to Eq. (3.2), we can consider $E_0^{\alpha,p}$ with respect to the equivalent norm

$$\|\phi\| = \|{}_0D_t^\alpha \phi\|_p.$$

Lemma 3.2 Let $0 < \alpha \le 1$, and $1 < p < \infty$ be your starting points. Assume that $\alpha > \frac{1}{p}$ and that the sequence $\{\phi_n\} \rightharpoonup \phi$ appears only faintly in $E_0^{\alpha,p}$. Then, $\{\phi_n\} \to \phi$ in $C([0,T])$, which is

$$\|\phi_n - \phi\|_\infty \to 0 \quad \text{as } n \to \infty.$$

We set the definition of functional with relation to the problem (P_λ).

Definition 3.1 A weak solution of problem (P_λ) is one in which a function $\phi \in E_0^{\alpha,p}$ such that $f(t,\phi(t)) \in L^1([0,T]\mathbb{R})$, satisfies the following equation

$$\begin{aligned}\int_0^T |{}_0D_t^\alpha(\phi(t)|^{p-2}{}_aD_t^\alpha\phi(t){}_aD_t^\alpha\psi(t)dt &= \lambda\int_0^T g(t)|\phi(t)|^{r-1}\phi(t)\psi(t)dt \\ &\quad + \int_0^T f(t,\phi(t))\psi(t)dt,\ \text{for all } \psi \in E_0^{\alpha,p}.\end{aligned}$$

We define the following energy functional in relation to problem (P_λ).

$$\mathcal{J}_\lambda(\phi) = \frac{1}{p}\|\phi\|^p - \frac{\lambda}{r}\int_0^T g(t)|\phi|^r dt - \frac{1}{q}\int_0^T F(t,\phi(t))dt,$$

is the result of a simple calculation $\mathcal{J}_\lambda \in C^1(E_0^{\alpha,p},\mathbb{R})$. Furthermore, for each $\phi,\psi \in E_0^{\alpha,p}$, we get

$$\begin{aligned}\langle \mathcal{J}_\lambda^{'}(\phi),\psi\rangle &= \int_0^T |{}_0D_t^\alpha(\phi(t)|^{p-2}{}_aD_t^\alpha\phi(t){}_aD_t^\alpha\psi(t)dt\\ &-\lambda\int_0^T g(t)|\phi(t)|^{r-1}\phi(t)\psi(t)dt-\int_0^T f(t,\phi(t))\psi(t)dt.\end{aligned}$$

Therefore, the critical points of $\mathcal{J}_\lambda$ are the weak solution for the problem (P_λ).
Since $\mathcal{J}_\lambda$ is not bounded below on the space $E_0^{\alpha,p}$, then we will work in the following subset is called Nehari manifold.

$$\mathcal{E}_\lambda := \left\{\phi\in E_0^{\alpha,p}\setminus\{0\} : \langle \mathcal{J}_\lambda^{'}(\phi),\phi\rangle = 0\right\}.$$

Lemma 3.3 If assumptions $\lambda\in(0,\lambda_0)$) are true, then the functional $\mathcal{J}_\lambda$ is coercive and bounded bellow on $\mathcal{E}_\lambda$ and $\lambda>0$.

Proof: Let $\phi\in\mathcal{E}_\lambda$, for all $\lambda>0$, and using Eqs. (1.1) and (3.3) we obtain

$$\int_0^T f(t,\phi(t))dt \le K\int_0^T|\phi(t)|^q \le \frac{KT^{1+q(\alpha-\frac{1}{p})}}{\Gamma(\alpha)^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}}\|\phi\|^q, \tag{3.4}$$

and

$$\int_0^T g(t)|\phi|^r dt \le \|g\|_\infty\int_0^T|\phi|^r dt \le \|g\|_\infty\frac{T^{1+r(\alpha-\frac{1}{p})}}{\Gamma(\alpha)^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{p}}}\|\phi\|^r. \tag{3.5}$$

Consequently, from Eqs. (3.4) and (3.5) we obtain

$$\begin{aligned}\mathcal{J}_\lambda(\phi) &= \frac{1}{p}\|\phi\|^p-\frac{\lambda}{r}\int_0^T g(t)|\phi|^r dt-\frac{1}{q}\int_0^T F(t,\phi(t))dt.\\ &= \frac{r-p}{rp}\|\phi\|^p-\frac{r-q}{qr}\int_0^T F(t,\phi(t))dt\\ &\ge \frac{r-p}{rp}\|\phi\|^p-\frac{K(r-q)T^{1+q(\alpha-\frac{1}{p})}}{qr\Gamma(\alpha)^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}}\|\phi\|^q.\end{aligned}$$

So, $\mathcal{J}_\lambda$ is coercive and bounded below on $\mathcal{E}_\lambda$. □

For $\phi\in E_0^{\alpha,p}$, we define the fibering map $\Phi_\lambda:(0,\infty)\to\mathbb{R}$, by

$$\Phi_\phi(t)=\mathcal{J}_\lambda(t\phi).$$

A simple calculation shows that

$$\Phi_\phi(t)=\frac{t^p}{p}\|\phi\|^p-\lambda\frac{t^r}{r}\int_0^T g(t)|\phi|^r dt-\frac{t^q}{q}\int_0^T F(t,\phi(t))dt, \tag{3.6}$$

then, we have

$$\Phi'_\phi(t) = t^{p-1}\|\phi\|^p-\lambda t^{r-1}\int_0^T g(t)|\phi|^r dt-t^{q-1}\int_0^T F(t,\phi(t))dt,$$

and

$$\begin{aligned}\Phi''_\phi(t) &= (p-1)t^{p-2}\|\phi\|^p-\lambda(r-1)t^{r-2}\int_0^T g(t)|\phi|^r dt\\ &- (q-1)t^{q-2}\int_0^T F(t,\phi(t))dt.\end{aligned}$$

Remark 3.2 Then, it is easy to see that $t\phi \in \mathcal{E}_\lambda$ if and only if $\Phi'_\phi(t) = 0$ and in particular, $\phi \in \mathcal{E}_\lambda$ if and only if $\Phi'_\phi(1) = 0$.

To demonstrate the multitude of solutions, we divided $\mathcal{E}_\lambda$, into the three subsets below.

$$\mathcal{E}_\lambda^+ = \left\{ \phi \in \mathcal{E}_\lambda : p\|\phi\|^p - \lambda r \int_0^T g(t)|\phi|^r dt - q \int_0^T F(t,\phi(t))dt > 0 \right\},$$

$$\mathcal{E}_\lambda^- = \left\{ \phi \in \mathcal{E}_\lambda : p\|\phi\|^p - \lambda r \int_0^T g(t)|\phi|^r dt - q \int_0^T F(t,\phi(t))dt < 0 \right\},$$

and

$$\mathcal{E}_\lambda^0 = \left\{ \phi \in \mathcal{E}_\lambda : p\|\phi\|^p - \lambda r \int_0^T g(t)|\phi|^r dt - q \int_0^T F(t,\phi(t))dt = 0 \right\}.$$

Next, we present some important properties of $\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-$ and $\mathcal{E}_\lambda^0$. Put

$$\lambda_0 = \frac{(p-q)\Gamma(\alpha)^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{p}}}{(r-q)\|g\|_\infty T^{1+r(\alpha-\frac{1}{p})}} \left(\frac{(r-p)\Gamma(\alpha)^q(\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}}{K(r-q)T^{1+q(\alpha-\frac{1}{p})}} \right)^{\frac{r-q}{p-q}}.$$

Lemma 3.4 Under assumptions H_1, H_2 there exists λ_0 such that for each $\lambda \in (0, \lambda_0)$, we have $\mathcal{E}_\lambda^0 = \emptyset$.

Proof:
To prove that $\mathcal{E}_\lambda^0 = \emptyset$ for a supposing that there exists to $\phi_0 \in \mathcal{E}_\lambda^0$ we proceed by contradiction. Then we get from Eq. (3.6)

$$(p-q)\|\phi_0\|^p - \lambda(r-q) \int_0^T g(t)|\phi_0|^r dt = 0, \tag{3.7}$$

and

$$(r-p)\|\phi_0\|(r-q) \int_0^T F(t,\phi_0(t))dt = 0. \tag{3.8}$$

From Eqs. (3.3) and (3.7) one has

$$\|\phi_0\| \geq \left(\frac{(p-q)(\Gamma(\alpha))^r(\alpha - 1\widetilde{p}+1)^{\frac{r}{p}}}{\lambda(r-q)\|g\|_\infty T^{1+r(\alpha-\frac{1}{p})}} \right)^{\frac{1}{r-p}}, \tag{3.9}$$

from Eqs. (1.1) and (3.3)

$$\|\phi_0\| \leq \left(\frac{K(r-q)T^{1+q(\alpha-\frac{1}{p})}}{(r-p)(\Gamma(\alpha))^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}} \right)^{\frac{1}{p-q}}. \tag{3.10}$$

Combining Eqs. (3.9) and (3.10), we obtain $\lambda \geq \lambda_0$. This contradicts our choice of $\lambda \in (0, \lambda_0)$. □

Lemma 3.5 For all $\phi \in E_0^{\alpha,p}$, there exist $s_1 > 0$ and $s_2 > 0$, such that $s_1\phi \in \mathcal{E}_\lambda^+$ and $s_2\phi \in \mathcal{E}_\lambda^-$.

Proof:

$$\begin{aligned} \Phi'_\phi(t) &= t^{p-1}\|\phi\|^p - \lambda t^{r-1} \int_0^T g(t)|\phi|^r dt - t^{q-1} \int_0^T F(t,\phi(t))dt \\ &= t^{q-1}\left[t^{p-q}\|\phi\|^p - \lambda t^{r-q} \int_0^T g(t)|\phi|^r dt - \int_0^T F(t,\phi(t))dt \right]. \end{aligned}$$

$$\Psi_{\phi}(t) = t^{p-q}\|\phi\|^{p} - \lambda t^{r-q}\int_{0}^{T} g(t)|\phi|^{r}dt,$$

$$\Psi_{\phi}^{'}(t) = (p-q)t^{p-q-1}\|\phi\|^{p} - \lambda(r-q)t^{r-q-1}\int_{0}^{T} g(t)|\phi|^{r}dt.$$

$$\Psi_{\phi}^{'}(t) = -\lambda\int_{0}^{T} g(t)|\phi|^{r}dt(r-q)t^{p-q-1}\left(t^{r-p} - \frac{\|\phi\|^{p}(p-q)}{\lambda\int_{0}^{T} g(t)|\phi|^{r}dt(r-q)}\right).$$

$$\Phi_{\phi}^{''}(t) = (q-1)t^{q-2}\left(\Psi_{\phi}(t) - \int_{0}^{T} F(t,\phi(t))dt\right) + t^{q-1}\Psi_{\phi}^{'}(t).$$

Now we get that, Ψ_{ψ} is a maximum at

$$S = \left(\frac{\|\phi\|^{p}(p-q)}{\lambda\int_{0}^{T} g(t)|\phi|^{r}dt(r-q)}\right)^{\frac{1}{r-p}}.$$

Suppose $B = \int_{0}^{T} F(t,\phi(t))dt$.
$\Psi(S)$ is local maximum. Moreover $\Psi_{\phi}(s_1) = B = \Psi_{\phi}(s_2)$ and From Eq. (1.1) we get $\Psi_{\phi}(S) > B > 0$.
$\Phi_{\phi}^{'}(s_1)$ is increasing and has local minimum, so $s_1 \in \mathcal{E}_{\lambda}^{+}$
$\Phi_{\phi}^{'}(s_2) = 0$ is decreasing and has local maximum, therefore $s_2 \in \mathcal{E}_{\lambda}^{-}$ □

4. Proof of Theorem 1.1

In this section, we will prove the main result of this paper (Theorem 1.1) which is divided into several steps.

Step 1 We claim that $\mathcal{J}_{\lambda}$ has a minimum on $\mathcal{E}_{\lambda}^{+}$ from Lemma 3.3 we have that $\mathcal{J}_{\lambda}$ is coercive and bounded below on $\mathcal{E}_{\lambda}^{+} \subset \mathcal{E}_{\lambda}$ there exists sequence $\{\phi_k\} \subset \mathcal{E}_{\lambda}^{+}$

$$\lim_{k\to\infty} \mathcal{J}_{\lambda}(\phi_k) = \inf_{\phi\in\mathcal{E}_{\lambda}^{+}} \mathcal{J}_{\lambda}(\phi)$$

ϕ_k is a bounded sequence in $E_0^{\alpha,p}$, reflexive and Banach space.

$$\phi_k \rightharpoonup \phi_{\lambda}\text{weakly in } E_0^{\alpha,p}.$$

Then from Lemma 3.5, there exists $s_1\phi \in \mathcal{E}_{\lambda}^{+}$ since $\phi_k \subset \mathcal{E}_{\lambda}$ we get

$$\mathcal{J}_{\lambda}(\phi_k) = \left(\frac{1}{p} - \frac{1}{q}\right)\|\phi_k\|^{p} - \left(\frac{1}{q} - \frac{1}{r}\right)\int_{0}^{T} F(t,\phi_k(t))dt,$$

now we get that

$$\left(\frac{1}{q} - \frac{1}{r}\right)\int_{0}^{T} F(t,u(t))dt = \left(\frac{1}{p} - \frac{1}{q}\right)\|\phi_k\|^{p} - \mathcal{J}_{\lambda}(\phi_k),$$

when $k \to \infty$, we get $\left(\frac{1}{p} - \frac{1}{q}\right)\|\phi_{\infty}\|^{p} - \mathcal{J}_{\lambda}(\phi_{\infty}) > 0$. We conclude that

$$\int_{0}^{T} F(t,\phi_{\lambda}(t))dt > 0,$$

if $\phi_k \rightharpoonup \phi_{\lambda}$ and is weakly in $E_0^{\alpha,p}$.

We get

$$\|\phi_\lambda\|^p < \lim\inf_{k\to\infty} \|\phi_k\|^p.$$

Since $\Phi^{'}_{\phi_\lambda}(s_1) = 0$, it follows that $\Phi^{'}_{\phi_k}(s_1) > 0$.
Then $s_1\phi \in \mathcal{E}^+_\lambda$ and

$$\mathcal{J}_\lambda(s_1\phi_\lambda) < \mathcal{J}_\lambda(\phi_\lambda) \leq \lim_{k\to\infty} \lambda(\phi_k) = \inf_{\phi\in\mathcal{E}^+_\lambda} \mathcal{J}_\lambda(\phi),$$

which gives a contradiction. Thus

$$\phi_k \rightharpoonup \phi_\lambda \text{ is strongly in } E_0^{\alpha,p}.$$

This implies that $\phi_\lambda \in \mathcal{E}^+_\lambda \cup \mathcal{E}^0_\lambda$. Since $\mathcal{E}^0_\lambda = \emptyset$. Then
$\phi_\lambda \in \mathcal{E}^+_\lambda \phi_\lambda$ is a nontrivial solution of problem (P_λ).

Step 2 We claimed that $\mathcal{J}_\lambda$ has a minimum on $\mathcal{E}^-_\lambda$ from Lemma 3.5 we have the existence of $s_1 > 0$ such that $\mathcal{J}_\lambda(\phi) \geq s_1$ there exists sequence $\{\psi_k\} \subset \mathcal{E}^-_\lambda$

$$\lim_{k\to\infty} \mathcal{J}_\lambda(\psi_k) = \inf_{\phi\in\mathcal{E}^-_\lambda} \mathcal{J}_\lambda(\phi). \tag{4.1}$$

Moreover $\{\psi_k\}$ is a bounded sequence in $E_0^{\alpha,p}$ reflexive and Banach space,
$\psi_k \rightharpoonup \psi_\lambda$ is weakly in $E_0^{\alpha,p}$.

$$\mathcal{J}_\lambda(\psi_k) = \left(\frac{1}{p} - \frac{1}{r}\right)\|\psi_k\|^p - \lambda\left(\frac{1}{r} - \frac{1}{q}\right)\int_0^T g(t)\psi(t)dt, \tag{4.2}$$

and so

$$\lambda\left(\frac{1}{r} - \frac{1}{q}\right)\int_0^T g(t)\psi(t)dt = \left(\frac{1}{p} - \frac{1}{r}\right)\|\psi_k\|^p - \mathcal{J}_\lambda(\psi_k).$$

Let $k \to \infty$ in Eq.(4.2) and combining with Lemma 3.2, we find from Eq. (4.1).

$$\int_0^T g(t)\psi(t)dt > 0.$$

Hence, $\psi_\lambda \in \mathcal{E}^+_\lambda$ and so ϕ_{ψ_λ} has a global maximum at some point S. Consequently $S_{\psi_\lambda} \in \mathcal{E}^-_\lambda$, we claim that $\psi_k \to \psi_\lambda$. Assume it is not true, then

$$\|\psi_\lambda\|^p \liminf_{k\to\infty} \|\psi_k\|^p.$$

$$\begin{aligned}
\mathcal{J}_\lambda(S\psi_\lambda) &= \frac{S^p}{p}\|\psi_\lambda\|^p - \lambda\frac{S^r}{r}\int_0^T g(t)\|\psi_\lambda\|^r dt - \frac{S^q}{q}\int_0^T F(t,\psi_\lambda(t))dt \\
&< \inf_{k\to\infty}\left(\frac{S^p}{p}\|\psi_k\|^p - \lambda\frac{S^r}{r}\int_0^T g(t)\|\psi_k\|^r dt - \frac{S^q}{q}\int_0^T F(t,\psi_k(t))dt\right) \\
&\leq \lim_{k\to\infty} \mathcal{J}_\lambda(S\psi_\lambda) \leq \lim_{k\to\infty} \mathcal{J}_\lambda(S\psi_\lambda) = \inf_{\phi\in\mathcal{E}^-_\lambda} \mathcal{J}_\lambda(\phi).
\end{aligned}$$

which gives a contradiction $\psi_k \to \psi_\lambda$ and so $\psi_\lambda \in \mathcal{E}^-_\lambda \cup \mathcal{E}^0_\lambda$ since $\mathcal{E}^0_\lambda = \emptyset$
ψ_λ is a nontrivial solution of problem (P_λ). Finally, since $\mathcal{E}^+_\lambda \cap \mathcal{E}^-_\lambda = \emptyset$, ϕ_λ and ψ_λ are distinct. That is the result of Theorem 1.1

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