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Statistically Convergent Δ^3 Difference Triple Sequence Spaces on a Seminormed Space

Bimal Chandra Das and Binod Chandra Tripathy*

ABSTRACT: In this paper we define and study the difference triple sequence spaces $\ell^3_{\infty st}(\Delta^3,\mathsf{q})$, $c^3_{st}(\Delta^3,\mathsf{q})$, $c^{3B}_{st}(\Delta^3,\mathsf{q})$, $c^{3R}_{st}(\Delta^3,\mathsf{q})$ and $c^{3BR}_{st}(\Delta^3,\mathsf{q})$ defined over a seminormed space (X,q), seminormed by q. Some algebraic and topological properties of these classes of sequences are established and certain inclusion results have been obtained. Several examples are also provided to support the results and notions introduced.

Key Words: Statistical convergence, difference operator, triple sequence, symmetric space, solid space.

Contents

1. Introduction

Throughout the article $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers, respectively. A triple sequence in a seminormed space (X,q), seminormed by q is a function $\mathbf{x}:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\longrightarrow X$. In this article a triple sequence $\mathbf{x}$ will be denoted by $\mathbf{x}=(x_{lmn})_{l,m,n\in\mathbb{N}}$ (shortly $\mathbf{x}=(x_{lmn})$). Different types of triple sequences were introduced and investigated in the literature available. Sahiner et al. [24], studied this type of sequences at the initial stage and then many authors investigated convergence of such sequences (Savaş and Esi [25], Debnath et al. [16], [17] are a few to be named).

Savaş and Esi [25], have introduced statistical convergence of triple sequences on probabilistic normed space. Debnath et al. [16], [17], generalized these concepts by using the difference operator and regular matrix transformation, respectively.

In this paper we study triple sequence spaces over a semi-normed space (X,q). Throughout $\mathsf{w}^3(\mathsf{q})$ denotes the class of all triple sequences on (X,q).

Further, $\ell^3_{\infty-st}(\mathsf{q})$, $c^3_{0-st}(\mathsf{q})$, $c^{3B}_{0-st}(\mathsf{q})$, $c^{3R}_{0-st}(\mathsf{q})$, $c^{3BR}_{0-st}(\mathsf{q})$, $c^3_{st}(\mathsf{q})$, $c^{3B}_{st}(\mathsf{q})$, $c^{3R}_{st}(\mathsf{q})$ and $c^{3BR}_{st}(\mathsf{q})$ denote the following subclasses of $\mathsf{w}^3(\mathsf{q})$: statistically bounded, statistically null (in Pringsheim's sense), bounded statistically null (in Pringsheim's sense), regularly statistically null, bounded regularly statistically null, statistically convergent (in Pringsheim's sense), bounded statistically convergent in Pringsheim's sense, regularly statistically convergent, and bounded regularly statistically convergent triple sequence spaces, respectively.

The statistical convergence was introduced independently by Fast [19], and Steinhaus [26], as a generalization of ordinary convergence of real sequences (although the idea of statistical convergence appeared initially as almost convergence, in 1935 in the first edition of Zygmund's monograph [30], The notion of statical convergence of sequences is based on the notion of asymptotic or natural density of a set $A\subset\mathbb{N}$ (see, for example, [22].). Statistical convergence has been developed by many authors in several directions (for instance, see [18], [21], [27], [27]).

A subset A of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have the (triple) natural density $\delta_3(A)$ if the limit

$$\delta_3(A):=\lim_{l,m,n\to\infty}\frac{1}{lmn}\Sigma_{p\le l}\Sigma_{q\le m}\Sigma_{r\le n}\chi_A(p,q,r)$$

* Corresponding author

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exists [24]. Here χ_A is the characteristic function of A.

A triple sequence (x_{lmn}) is said to be statistically convergent to L (in Pringshiem's sense) if for every $\varepsilon > 0$,

$$\delta_3(\{(l,m,n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{lmn} - L| \geq \varepsilon\}) = 0.$$

We denote this fact by $st - \lim_{l,m,n\to\infty} x_{lmn} = L$.

A statistically convergent sequence (x_{lmn}) is said to be regularly statistically convergent if the following limits exist:

1. $\mathrm{st} - \lim_{n\to\infty} f(|x_{lmn} - L_{lm}|) = 0, \quad (l,m \in \mathbb{N})$;
2. $\mathrm{st} - \lim_{m\to\infty} f(|x_{lmn} - L_{ln}|) = 0, \quad (l,n \in \mathbb{N})$;
3. $\mathrm{st} - \lim_{l\to\infty} f(|x_{lmn} - L_{mn}|) = 0, \quad (m,n \in N)$.

The idea of difference sequence spaces was introduced by Kizmaz [20] as follows:

$$Z(\Delta) = \{\mathbf{x} = (x_n) \in w : (\Delta x_n) \in Z\}$$

for $Z = \mathsf{c}, \ \mathsf{c}_0, \ \ell_\infty$, the spaces of convergent, null, and bounded single sequences, respectively, where $\Delta x_n = x_n - x_{n+1}$ for all $n \in \mathbb{N}$. He proved that these spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{n\in\mathbb{N}} |\Delta x_n|.$$

Tripathy and Sarma [29] introduced difference double sequence spaces on a seminormed space and difference double sequence spaces as follows:

$$Z(\Delta) = \{(x_{mn}) \in \mathsf{w}^2 : (\Delta x_{mn}) \in Z\}$$

for $Z = \mathsf{c}^2, \ \mathsf{c}_0^2, \ \ell_\infty^2$, the spaces of convergent, null, and bounded double sequences, respectively, where

$$\Delta x_{mn} = x_{mn} - x_{m(n+1)} - x_{(m+1)n} + x_{(m+1)(n+1)}, \ m,n \in \mathbb{N}.$$

Another approach to difference double sequence spaces was proposed in [16] by using a matrix transformation.

Debnath and Das [14] introduced p^{th} order difference triple sequence spaces in the following way:

$$Z(\Delta_p) = \{(x_{lmn}) \in \mathsf{w}^3 : (\Delta_p x_{lmn}) \in Z\}$$

for $Z = \mathsf{c}^3, \ \mathsf{c}_0^3, \ \ell_\infty^3, \ \mathsf{c}^{3B}, \ \mathsf{c}^{3R}$, the spaces of convergent, null, bounded, bounded convergent and regularly convergent triple sequences respectively, where $\Delta_p x_{lmn} = x_{lmn} - x_{lm(n+p)} - x_{l(m+p)n} + x_{l(m+p)(n+p)} - x_{(l+p)mn} + x_{(l+p)m(n+p)} + x_{(l+p)(m+p)n} - x_{(l+p)(m+p)(n+p)}$ for all $l,m,n \in \mathbb{N}$.

Recently for the above mentioned spaces, 2^{nd} order difference triple sequence spaces was introduced by Debnath and Das [15] as follows:

$$Z(\Delta^2) = \{(x_{lmn}) \in \mathsf{w}^3 : (\Delta^2 x_{lmn}) \in Z\},$$

where $\Delta^2 x_{lmn}$ was defined in the following way:

$$\begin{aligned}
\Delta^2 x_{lmn} = x_{lmn} &- 2x_{(l+1)mn} + x_{(l+2)mn} - 2x_{l(m+1)n} + 4x_{(l+1)(m+1)n} \\
&- 2x_{(l+2)(m+1)n} + x_{l(m+2)n} - 2x_{(l+1)(m+2)n} + x_{(l+2)(m+2)n} \\
&- 2x_{lm(n+1)} + 4x_{(l+1)m(n+1)} - 2x_{(l+2)m(n+1)} + 4x_{l(m+1)(n+1)} \\
&- 8x_{(l+1)(m+1)(n+1)} + 4x_{(l+2)(m+1)(n+1)} - 2x_{l(m+2)(n+1)} \\
&+ 4x_{(l+1)(m+2)(n+1)} - 2x_{(l+2)(m+2)(n+1)} + x_{lm(n+2)} \\
&- 2x_{(l+1)m(n+2)} + x_{(l+2)m(n+2)} - 2x_{l(m+1)(n+2)} \\
&+ 4x_{(l+1)(m+1)(n+2)} - 2x_{(l+2)(m+1)(n+2)} + x_{l(m+2)(n+2)} \\
&- 2x_{(l+1)(m+2)(n+2)} + x_{(l+2)(m+2)(n+2)}
\end{aligned}$$

2. Definitions and Preliminaries

In 1900, Pringsheim [23] introduced convergence of double sequences. Following this idea one obtains the following definition.

Definition 2.1 A real or complex triple sequence (x_{lmn}) is said to be convergent to L (in the Pringsheim sense) if for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{lmn} - L| < \varepsilon$, whenever $l \geq n_0$, $m \geq n_0$, $n \geq n_0$. In this case we write $\lim_{l,m,n\to\infty} x_{lmn} = L$.

Note. A triple sequence convergent in Pringsheim's sense may not be bounded [24]..

The statistical version of this definition for triple sequences in a semi-normed space (X, q) is:

Definition 2.2 A triple sequence (x_{lmn}) in a semi-normed space (X, q) is said to be statistically convergent to L (in the Pringsheim sense) if for every $\varepsilon > 0$ we have

$$\delta_3\left(\{(l,m,n) \in \mathbb{N}^3 : q(x_{lmn} - L) \geq \varepsilon\}\right) = 0.$$

In this case we write $\text{st}-\lim_{l,m,n\to\infty} x_{lmn} = L$.

Definition 2.3 A triple sequence (x_{lmn}) in a semi-normed space (X, q) is said to be statistically bounded if there exists $M > 0$ such that $\delta_3(\{(l,m,n) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : q(x_{lmn}) > M\}) = 0$.

Let E be a triple sequence space and let $K = \{(l_i, m_i, n_i) : i \in \mathbb{N}; l_1 < l_2 < \ldots; m_1 < m_2 < \ldots; n_1 < n_2 < \ldots\} \subset E$. A K-step space of E is a sequence space $\lambda_K(E) = \{(x_{l_i m_i n_i}) \in \mathsf{w}^3 : (x_{lmn}) \in E\}$.

A canonical pre-image of a sequence $(x_{l_i m_i n_i}) \in \lambda_K(E)$ is a sequence $(y_{lmn}) \in E$ defined as follows:

$$y_{lmn} = \begin{cases} x_{lmn}, & \text{if } (l,m,n) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4 A triple sequence space E is said to be:

1. *solid* (or *normal*) if $(\alpha_{lmn} \cdot x_{lmn}) \in E$ whenever $(x_{lmn}) \in E$ and (α_{lmn}) is a triple sequence of scalars with $|\alpha_{lmn}| \leq 1$ for all $l, m, n \in \mathbb{N}$;

2. *monotone* if it contains the canonical pre-images of all its step spaces;

3. *symmetric* if $(x_{lmn}) \in E$ implies $(x_{\sigma(l,m,n)}) \in E$, for any permutation σ of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$;

4. a *sequence algebra* if $(x_{lmn}) \star (y_{lmn}) = (x_{lmn} \cdot y_{lmn}) \in E$, whenever $(x_{lmn}) \in E$ and $y_{lmn} \in E$.

Remark 2.1 *If a triple sequence space is solid, then it is monotone.*

Now we introduce the following difference operator on triple sequence spaces, over a seminormed space (X, q) by

$$Z(\Delta^3, q) = \{(x_{lmn}) \in w^3(q) : (\Delta^3 x_{lmn}) \in Z(q)\}$$

for $Z=\ell^3_{\infty-\mathsf{st}}$, $\mathsf{c}^3_{0-\mathsf{st}}$, $\mathsf{c}^{3B}_{0-\mathsf{st}}$, $\mathsf{c}^{3R}_{0-\mathsf{st}}$, $\mathsf{c}^{3BR}_{0-\mathsf{st}}$, $\mathsf{c}^3_{\mathsf{st}}$, $\mathsf{c}^{3B}_{\mathsf{st}}$, $\mathsf{c}^{3R}_{\mathsf{st}}$ and $\mathsf{c}^{3BR}_{\mathsf{st}}$, where

$$\begin{aligned}
\Delta^3 x_{lmn} &= x_{lmn} - 3x_{lm(n+1)} + 3x_{lm(n+2)} - x_{lm(n+3)} - 3x_{l(m+1)n} + 9x_{l(m+1)(n+1)} \\
&- 9x_{l(m+1)(n+2)} + 3x_{l(m+1)(n+3)} + 3x_{l(m+2)n} - 9x_{l(m+2)(n+1)} \\
&+ 9x_{l(m+2)(n+2)} - 3x_{l(m+2)(n+3)} - x_{l(m+3)n} + 3x_{l(m+3)(n+1)} \\
&- 3x_{l(m+3)(n+2)} + x_{l(m+3)(n+3)} - 3x_{(l+1)mn} + 9x_{(l+1)m(n+1)} \\
&- 9x_{(l+1)m(n+2)} + 3x_{(l+1)m(n+3)} + 9x_{(l+1)(m+1)n} - 27x_{(l+1)(m+1)(n+1)} \\
&+ 27x_{(l+1)(m+1)(n+2)} - 9x_{(l+1)(m+1)(n+3)} - 9x_{(l+1)(m+2)n} \\
&+ 27x_{(l+1)(m+2)(n+1)} - 27x_{(l+1)(m+2)(n+2)} + 9x_{(l+1)(m+2)(n+3)} \\
&+ 3x_{(l+1)(m+3)n} - 9x_{(l+1)(m+3)(n+1)} + 9x_{(l+1)(m+3)(n+2)} \\
&- 3x_{(l+1)(m+3)(n+3)} + 3x_{(l+2)mn} - 9x_{(l+2)m(n+1)} + 9x_{(l+2)m(n+2)} \\
&- 3x_{(l+2)m(n+3)} - 9x_{(l+2)(m+1)n} + 27x_{(l+2)(m+1)(n+1)} - 27x_{(l+2)(m+1)(n+2)} \\
&+ 9x_{(l+2)(m+1)(n+3)} + 9x_{(l+2)(m+2)n} - 27x_{(l+2)(m+2)(n+1)} \\
&+ 27x_{(l+2)(m+2)(n+2)} - 9x_{(l+2)(m+2)(n+3)} - 3x_{(l+2)(m+3)n} \\
&+ 9x_{(l+2)(m+3)(n+1)} - 9x_{(l+2)(m+3)(n+2)} + 3x_{(l+2)(m+3)(n+3)} \\
&- x_{(l+3)mn} + 3x_{(l+3)m(n+1)} - 3x_{(l+3)m(n+2)} + x_{(l+3)m(n+3)} \\
&+ 3x_{(l+3)(m+1)n} - 9x_{(l+3)(m+1)(n+1)} + 9x_{(l+3)(m+1)(n+2)} \\
&- 3x_{(l+3)(m+1)(n+3)} - 3x_{(l+3)(m+2)n} + 9x_{(l+3)(m+2)(n+1)} \\
&- 9x_{(l+3)(m+2)(n+2)} + 3x_{(l+3)(m+2)(n+3)} + x_{(l+3)(m+3)n} \\
&- 3x_{(l+3)(m+3)(n+1)} + 3x_{(l+3)(m+3)(n+2)} - x_{(l+3)(m+3)(n+3)}, \ldots\ldots\ldots\ldots\ldots\ldots(1)
\end{aligned}$$

for all $l,m,n\in\mathbb{N}$ (see [1]).

Remark 2.2 *If a triple sequence is convergent in Pringsheim's sense then it is statistically convergent but the converse is not necessarily true (see [24]).*

Example 2.1 *We define a triple sequence*

$$x_{lmn} = l+m+n-2, \quad \textit{for all } l,m,n\in\mathbb{N},$$

It can be express as follows:

$$x_{lmn} = \begin{pmatrix}
1 & 2 & 3 & 4 & \ldots & \ldots \\
2 & 3 & 4 & 5 & \ldots & \ldots \\
3 & 4 & 5 & 6 & \ldots & \ldots \\
4 & 5 & 6 & 7 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
2 & 3 & 4 & 5 & \ldots & \ldots \\
3 & 4 & 5 & 6 & \ldots & \ldots \\
4 & 5 & 6 & 7 & \ldots & \ldots \\
5 & 6 & 7 & 8 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
3 & 4 & 5 & 6 & \ldots & \ldots \\
4 & 5 & 6 & 7 & \ldots & \ldots \\
5 & 6 & 7 & 8 & \ldots & \ldots \\
6 & 7 & 8 & 9 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
4 & 5 & 6 & 7 & \ldots & \ldots \\
5 & 6 & 7 & 8 & \ldots & \ldots \\
6 & 7 & 8 & 9 & \ldots & \ldots \\
7 & 8 & 9 & 10 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}$$

Here the sequence is neither statistically bounded nor statistically convergent, so this sequence is not convergent also. Now we calculate the triple difference sequence for $x_{lmn} = l + m + n - 2$*, using result (1) in the following way:*

$$\begin{aligned}\Delta^3 x_{lmn} &= (l+m+n-2) - 3(l+m+n-1) + 3(l+m+n) - (l+m+n+1)\\ &- 3(l+m+n-1) + 9(l+m+n) - 9(l+m+n+1) + 3(l+m+n+2)\\ &+ 3(l+m+n) - 9(l+m+n+1) + 9(l+m+n+2) - 3(l+m+n-3)\\ &- (l+m+n+1) + 3(l+m+n+2) - 3(l+m+n+3) + (l+m+n+4)\\ &- 3(l+m+n-1) + 9(l+m+n) - 9(l+m+n+1) + 3(l+m+n+2)\\ &+ 9(l+m+n) - 27(l+m+n+1) + 27(l+m+n+2) - 9(l+m+n+3)\\ &- 9(l+m+n+1) + 27(l+m+n+2) - 27(l+m+n+3) + 9(l+m+n+4)\\ &+ 3(l+m+n+2) - 9(l+m+n+3) + 9(l+m+n+4) - 3(l+m+n+5)\\ &+ 3(l+m+n) - 9(l+m+n+1) + 9(l+m+n+2) - 3(l+m+n+3)\\ &- 9(l+m+n+1) + 27(l+m+n+2) - 27(l+m+n+3) + 9(l+m+n+4)\\ &+ 9(l+m+n+2) - 27(l+m+n+3) + 27(l+m+n+4) - 9(l+m+n+5)\\ &- 3(l+m+n+3) + 9(l+m+n+4) - 9(l+m+n+5) + 3(l+m+n+6)\\ &- (l+m+n+1) + 3(l+m+n+2) - 3(l+m+n+3) + (l+m+n+4)\\ &+ 3(l+m+n+2) - 9(l+m+n+3) + 9(l+m+n+4) - 3(l+m+n+5)\\ &- 3(l+m+n+3) + 9(l+m+n+4) - 9(l+m+n+5) + 3(l+m+n+6)\\ &+ (l+m+n+4) - 3(l+m+n+5) + 3(l+m+n+6) - (l+m+n+7) = 0\end{aligned}$$

This result can be expressed in triple sequence notation as follows:

$$\Delta^3 x_{lmn} = \begin{pmatrix} 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ 0 & 0 & 0 & 0 & ... & ...\\ ... & ... & ... & ... & ... & ...\\ ... & ... & ... & ... & ... & ... \end{pmatrix}$$

Which is statistically convergent to zero i.e. $\mathrm{st} - \lim_{l,m,n\to\infty} \Delta^3 x_{lmn} = 0$ *as well as bounded.*

Example 2.2 *Let*

$$x_{lmn} = \begin{cases} n, & \text{when } m = 1, l \in \mathbb{N};\\ 2, & \text{otherwise.}\end{cases}$$

Then $\mathrm{st}-\lim_{l,m,n\to\infty} x_{lmn} = 2,$ *but* $\mathrm{st}-\lim_{l,m,n\to\infty} \Delta^3 x_{lmn} = 0.$

Example 2.3 *Let*

$$x_{lmn} = \begin{cases} 2, & \text{when } l,m,n \text{ are even;} \\ -2, & \text{otherwise.} \end{cases}$$

It can be express in triple sequence notation as follows:

$$x_{lmn} = \begin{pmatrix} -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & 2 & -2 & 2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & 2 & -2 & 2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & 2 & -2 & 2 & \dots & \dots \\ -2 & -2 & -2 & -2 & \dots & \dots \\ -2 & 2 & -2 & 2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Here the sequence (x_{lmn}) *is not statistically convergent and after taking difference operator on this sequence using equation (1) we obtain the following triple sequence representation:*

$$\Delta^3 x_{lmn} = \begin{pmatrix} -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ -256 & 256 & -256 & 256 & \dots & \dots \\ 256 & -256 & 256 & -256 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This sequence $(\Delta^3 x_{lmn})$ *is not statiscally convergent but bounded.*

3. Main Results

Theorem 3.1 *The triple sequence spaces* $\ell^3_{\infty-st}(\Delta^3, q)$, $c^3_{0-st}(\Delta^3, q)$, $c^{3B}_{0-st}(\Delta^3, q)$, $c^{3R}_{0-st}(\Delta^3, q)$, $c^{3BR}_{0-st}(\Delta^3, q)$, $c^3_{st}(\Delta^3, q)$, $c^{3B}_{st}(\Delta^3, q)$, $c^{3R}_{st}(\Delta^3, q)$ *and* $c^{3BR}_{st}(\Delta^3, q)$ *are all linear spaces over* $\mathbb{C}$.

Proof: The proof is easy, so omitted. □

Theorem 3.2 *For the triple sequence spaces above we have:*

(1) $c^3_{0-st}(\Delta^3, q) \subsetneqq c^3_{st}(\Delta^3, q)$;

(2) $c^{3B}_{0-st}(\Delta^3, q) \subsetneqq c^{3B}_{st}(\Delta^3, q)$;

(3) $c^{3R}_{0-st}(\Delta^3, q) \subsetneqq c^{3R}_{st}(\Delta^3, q)$;

(4) $c^{3BR}_{0-st}(\Delta^3, q) \subsetneqq \ell^3{}_{\infty-st}(\Delta^3, q)$;

(5) $c^{3BR}_{st}(\Delta^3, q) \subsetneqq \ell^3{}_{\infty-st}(\Delta^3, q)$;

(6) $c^{3R}_{st}(\Delta^3, q) \subsetneqq c^3_{st}(\Delta^3, q)$;

(7) $c^{3R}_{0-st}(\Delta^3, q) \subsetneqq c^{3B}_{0-st}(\Delta^3, q) \subsetneqq \ell^3{}_{\infty-st}(\Delta^3, q)$;

(8) $c^{3R}_{st}(\Delta^3, q) \subsetneqq c^{3B}_{st}(\Delta^3, q) \subsetneqq \ell^3{}_{\infty-st}(\Delta^3, q)$.

Proof: (1) Let $X = C$ with the usual norm $q(x) = |x|$. Consider the sequence (x_{lmn}) defined by

$$x_{lmn} = \begin{cases} 1, & \text{if } l, m, n \text{ are prime numbers,} \\ mn, & \text{otherwise.} \end{cases}$$

Then $(x_{lmn}) \in c^3_{st}(\Delta^3, q)$ but the sequence $(x_{lmn}) \notin c^3_{0-st}(\Delta^3, q)$.
Hence $c^3_{0-st}(\Delta^3, q) \subsetneqq c^3_{st}(\Delta^3, q)$. □

Proof: (2) Let $X = C$ with the usual norm $q(x) = |x|$. Consider the sequence (x_{lmn}) defined by

$$x_{lmn} = \begin{cases} 2, & \text{if } l, m, n \text{ are square,} \\ 5, & \text{otherwise.} \end{cases}$$

Then $(x_{lmn}) \in c^{3B}_{st}(\Delta^3, q)$ but the sequence $(x_{lmn}) \notin c^{3B}_{0-st}(\Delta^3, q)$.
Similarly the others. □

Theorem 3.3 *The classes of sequences* $\ell^3{}_{\infty-st} c^{3B}_{0-st}(\Delta^3, q)$, $c^{3R}_{0-st}(\Delta^3, q)$, c^{3BR}_{0-st} (Δ^3, q), $c^3_{st}(\Delta^3, q)$, $c^{3B}_{st}(\Delta^3, q)$, $c^{3R}_{st}(\Delta^3, q)$ *and* $c^{3BR}_{st}(\Delta^3, q)$ *are seminormed spaces with the seminorm* φ *defined by*

$$\varphi(x_{lmn}) = \sup_l q(x_{l11}) + \sup_m q(x_{1m1}) + \sup_n q(x_{11n}) + \sup_{l,m,n} q(\Delta^3 x_{lmn}).$$

Proof: We prove the theorem for the space $c^{3BR}_{st}(\Delta^3, q)$; the proofs for other cases are similar.

Since q is a seminorm, we have $\varphi(x_{lmn}) \geq 0$ for each $(x_{lmn}) \in c^{3BR}_{st}(\Delta^3, q)$. Consider the zero triple sequence denoted by $\underline{\theta}^3$. Clearly $\varpi(\underline{\theta}^3) = 0$. Also, by the definition of ϖ it follows $\varphi(\lambda(x_{lmn})) = |\lambda|\varphi(x_{lmn})$ for each $(x_{lmn} \in c^{3BR}_{st}(\Delta^3, q)$ and any scalar λ.

Let (x_{lmn}) and (y_{lmn}) be two sequences in $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3, \mathsf{q})$. Then

$$\begin{aligned}\varphi((x_{lmn})+(y_{lmn})) &= \sup_l q(x_{l11}+y_{l11}) + \sup_m q(x_{1m1}+y_{1m1}) + \sup_n q(x_{11n} \\ &+ y_{11n}) + \sup_{l,m,n} q(\Delta^3 x_{lmn} + \Delta^3 y_{lmn}) \\ &\le (sup_l q(x_{l11}) + sup_m q(x_{1m1}) + sup_n q(x_{11n}) \\ &+ sup_{l,m,n} q(\Delta^3 x_{lmn})) + (\sup_l q(y_{l11}) + \sup_m q(y_{1m1}) \\ &+ \sup_n q(y_{11n}) + \sup_{l,m,n} q(\Delta^3 y_{lmn})) = \varphi(x_{lmn}) + \varphi(y_{lmn}).\end{aligned}$$

Therefore, φ is a seminorm. □

Theorem 3.4 *Let (X,q) be a complete seminormed space. Then the spaces $\ell^3{}_{\infty-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^3_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3B}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3R}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3BR}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3}_{\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3B}_{\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$ and $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3,\mathsf{q})$ are all complete under the seminorm φ.*

Proof: We establish the result for the sequence space $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$; the other cases can be established similarly.

Let (x^i_{lmn}) be a Cauchy sequence in $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$.

Let $\varepsilon > 0$, then there exist n_0 such that $\varphi(x^i_{lmn} - x^j_{lmn}) < \varepsilon$ for all $i,j \ge n_0$. Hence we have $\sup_{l,m,n} q(x^i_{lmn} - x^j_{lmn}) < \varepsilon$ for all $i,j \ge n_0 \Rightarrow x^i_{lmn}$ is a Cauchy sequence in (X,q), (X,q) being complete, so is convergent in (X,q). let $\lim_{i\to\infty} x^i_{l11} = x_{l11}$ for $l \in N$, say. Similarly we find x_{1m1} such that $\lim_{i\to\infty} x^i_{1m1} = x_{1m1}$ and $\lim_{i\to\infty} x^i_{11n} = x_{11n}$. Also we can find x_{lmn} for $l,m,n > 1$ such that $\lim_{i\to\infty} x^i_{lmn} = x_{lmn}$. Thus we have (x_{lmn}) such that $\lim_{i\to\infty} x^i_{lmn} = x_{lmn} \in X$. Hence $x^i_{lmn} - x_{lmn} \in \mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$, for all $i \ge n_0$. Further, $x_{lmn} = x^i_{lmn} - (x^i_{lmn} - x_{lmn}) \in c^{3R}_{st}(\Delta^3, q)$, by the linearity of the space. Hence $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$ is complete. □

Result 3.5. The triple sequence spaces $\ell^3{}_{\infty-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^3_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3B}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3R}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $c^{3BR}_{0-\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3}_{\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3B}_{\mathsf{st}}(\Delta^3,\mathsf{q})$, $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3,\mathsf{q})$ and $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3,\mathsf{q})$ are not solid in general. It follows from the following example.

Example 3.1 *Let $X = C$ with the usual norm $q(x) = |x|$. Consider the sequence (x_{lmn}) defined by*

$$x_{lmn} = \begin{cases} 3 & \text{if } l,m,n \text{ are cubes,} \\ 0, & \text{otherwise.} \end{cases}$$

Then the direct calculation gives $\Delta^3 x_{lmn} = 0$ for all $l,m,n \in \mathbb{R}$, Consider the triple sequence of scalars defined by

$$\alpha_{lmn} = (-1)^{l+m+n} \quad \text{for all } l,m,n \in \mathbb{N}.$$

Then the sequence $(\alpha_{lmn} \cdot x_{lmn})$ takes the following form

$$x_{lmn} = \begin{cases} 3(-1)^{l+m+n}, & \text{if } l,m,n \text{ are cubes,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$(x_{lmn}) \in c^3_{0-st}(\Delta^3,q),\ c^{3B}_{0-st}(\Delta^3,q),\ c^{3R}_{0-st}(\Delta^3,q),\ c^{3BR}_{0-st}(\Delta^3,q),$$

but $(\alpha_{lmn} \cdot x_{lmn}) \notin c^3_{0-st}(\Delta^3,q)$, $c^{3B}_{0-st}(\Delta^3,q)$, $c^{3R}_{0-st}(\Delta^3,q)$, c^{3BR}_{0-st} (Δ^3,q), hence the mentioned spaces are not solid in general.

Example 3.2 *Let $X = \mathbb{C}$ and consider the sequence (x_{lmn}) defined by*

$$x_{lmn} = \begin{cases} l^2, & \text{if } m = 1 \text{ for all, } l, n \in \mathbb{N}; \\ mn, & \text{otherwise.} \end{cases}$$

Consider the triple sequence of scalars defined by

$$\alpha_{lmn} = (-1)^{l+n}, \quad \text{for all } l, m, n \in \mathbb{N}.$$

Then the sequence $(\alpha_{lmn} \cdot x_{lmn})$ takes the following form:

$$\alpha_{lmn} \cdot x_{lmn} = \begin{cases} l^2(-1)^{l+n}, & \text{if } m = 1 \text{ for all } l, n \in \mathbb{N}; \\ mn(-1)^{l+n}, & \text{otherwise.} \end{cases}$$

Hence it follows that

$$(x_{lmn}) \in l^3_{\infty st}(\Delta^3, q),\ c^3_{st}(\Delta^3, q),\ c^{3B}_{st}(\Delta^3, q),\ c^{3R}_{st}(\Delta^3, q),\ c^{3BR}_{st}(\Delta^3, q)$$

, but

$$(\alpha_{lmn} \cdot x_{lmn}) \notin l^3_{\infty st}(\Delta^3, q),\ c^3_{st}(\Delta^3, q),\ c^{3B}_{st}(\Delta^3, q),\ c^{3R}_{st}(\Delta^3, q),\ c^{3BR}_{st}(\Delta^3, q).$$

Therefore, the spaces $l^3_{\infty st}(\Delta^3, q)$, $c^3_{st}(\Delta^3, q)$, $c^{3B}_{st}(\Delta^3, q)$, $c^{3R}_{st}(\Delta^3, q)$, $c^{3BR}_{st}(\Delta^3, q)$ are not solid in general.

Result 3.7. The triple sequence spaces $l^3_{\infty-st}(\Delta^3, q)$, $c^3_{0-st}(\Delta^3, q)$, $c^{3B}_{0-st}(\Delta^3, q)$, $c^{3R}_{0-st}(\Delta^3, q)$, $c^{3BR}_{0-st}(\Delta^3, q)$, $c^3_{st}(\Delta^3, q)$, $c^{3B}_{st}(\Delta^3, q)$, $c^{3R}_{st}(\Delta^3, q)$ and $c^{3BR}_{st}(\Delta^3, q)$ are not symmetric in general. This follows from the following example.

Example 3.3 *Let $X = \mathbb{C}$ and $q(x) = |x|$. We consider the sequence (x_{lmn}) defined by*

$$x_{lmn} = n, \text{ for all } l, m, n \in \mathbb{N}.$$

Consider a rearranged sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} n + 2, & \text{if } n = l \text{ and } m \text{ is odd;} \\ n - 2, & \text{if } n = l + 2 \text{ and } m \text{ is odd;} \\ n, & \text{otherwise.} \end{cases}$$

Clearly,

$$(x_{lmn}) \in c^3_{0-st}(\Delta^3, q),\ c^{3B}_{0-st}(\Delta^3, q),\ c^{3R}_{0-st}(\Delta^3, q),\ c^{3BR}_{0-st}(\Delta^3, q)$$

but

$$(y_{lmn}) \notin c^3_{0-st}(\Delta^3, q),\ c^{3B}_{0-st}(\Delta^3, q),\ c^{3R}_{0-st}(\Delta^3, q),\ c^{3BR}_{0-st}(\Delta^3, q).$$

Hence the spaces $c^3_{0-st}(\Delta^3, q)$, $c^{3B}_{0-st}(\Delta^3, q)$, $c^{3R}_{0-st}(\Delta^3, q)$, $c^{3BR}_{0-st}(\Delta^3, q)$ are not symmetric in general.

Example 3.4 *Let $X = \mathbb{C}$ and consider the sequence (x_{lmn}) defined by*

$$x_{lmn} = lmn, \quad \text{for all } l, m, n \in \mathbb{N},$$

and a rearranged sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} m + 2, & \text{if } m = l \text{ and } n \text{ is even;} \\ m - 2, & \text{if } m = l + 2 \text{ and } n \text{ is even;} \\ m, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(x_{lmn}) \in l^3_{\infty-st}(\Delta^3, q),\ c^3_{st}(\Delta^3, q),\ c^{3B}_{st}(\Delta^3, q),\ c^{3R}_{st}(\Delta^3, q),\ c^{3BR}_{st}(\Delta^3, q)$$

but

$$(y_{lmn}) \notin l^3_{\infty st}(\Delta^3, q),\ c^3_{st}(\Delta^3, q),\ c^{3B}_{st}(\Delta^3, q),\ c^{3R}_{st}(\Delta^3, q),\ c^{3BR}_{st}(\Delta^3, q).$$

Hence the spaces $l^3_{\infty st}(\Delta^3, q)$, $c^3_{st}(\Delta^3, q)$, $c^{3B}_{st}(\Delta^3, q)$, $c^{3R}_{st}(\Delta^3, q)$, $c^{3BR}_{st}(\Delta^3, q)$ need not be symmetric.

Proposition 3.1 *The inclusion relation* $Z(q) \subsetneq Z(\Delta^3, q)$ *holds for the following triple sequence spaces:*

(1) $Z = l^3_{\infty-st}$;

(2) c^3_{0-st}, c^{3B}_{0-st};

(3) c^{3R}_{0-st}, c^{3BR}_{0-st},

(4) c^3_{st}, c^{3B}_{st};

(5) c^{3R}_{st} *and* c^{3BR}_{st}.

Theorem 3.5 *The triple sequence spaces* $\ell^3_{\infty-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3\mathsf{B}}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3\mathsf{R}}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3\mathsf{BR}}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3\mathsf{B}}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3\mathsf{R}}_{\mathsf{st}}(\Delta^3, \mathsf{q})$ *and* $\mathsf{c}^{3\mathsf{BR}}_{\mathsf{st}}(\Delta^3, \mathsf{q})$ *are not monotone in general.*

Proof: The proof follow from the following Example.

Example 3.5 *Let* $X = \mathbb{C}$ *and* $q(x^i) = |x|$. *Consider the sequence* (x_{lmn}) *defined by*

$$x_{lmn} = l, \quad \text{for all } l, m, n \in \mathbb{N}$$

It can be express as follows:

$$x_{lmn} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots \\ 1 & 1 & 1 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & 2 & \dots & \dots \\ 2 & 2 & 2 & 2 & \dots & \dots \\ 2 & 2 & 2 & 2 & \dots & \dots \\ 2 & 2 & 2 & 2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 3 & 3 & 3 & 3 & \dots & \dots \\ 3 & 3 & 3 & 3 & \dots & \dots \\ 3 & 3 & 3 & 3 & \dots & \dots \\ 3 & 3 & 3 & 3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 4 & 4 & 4 & 4 & \dots & \dots \\ 4 & 4 & 4 & 4 & \dots & \dots \\ 4 & 4 & 4 & 4 & \dots & \dots \\ 4 & 4 & 4 & 4 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Taking difference operator on this sequence using result (1) we obtain the following result:

$$\Delta^3 x_{lmn} = \begin{pmatrix} 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \end{pmatrix}$$

Here $(x_{lmn}) \in \mathsf{c}^3_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3BR}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, , $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3, \mathsf{q})$.

Now we consider the sequence (y_{lmn}) *in the pre-image space defined by*

$$y_{lmn} = \begin{cases} x_{lmn}, & \textit{if } l+m+n \textit{ is odd;} \\ 0, & \textit{otherwise.} \end{cases}$$

It can be express in triple sequence notation as follows:

$$y_{lmn} = \begin{pmatrix} 1 & 0 & 1 & 0 & ... & ... \\ 0 & 1 & 0 & 1 & ... & ... \\ 1 & 0 & 1 & 0 & ... & ... \\ 0 & 1 & 0 & 1 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 2 & 0 & 2 & ... & ... \\ 2 & 0 & 2 & 0 & ... & ... \\ 0 & 2 & 0 & 2 & ... & ... \\ 2 & 0 & 2 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 3 & 0 & 3 & 0 & ... & ... \\ 0 & 3 & 0 & 3 & ... & ... \\ 3 & 0 & 3 & 0 & ... & ... \\ 0 & 3 & 0 & 3 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 4 & 0 & 4 & ... & ... \\ 4 & 0 & 4 & 0 & ... & ... \\ 0 & 4 & 0 & 4 & ... & ... \\ 4 & 0 & 4 & 0 & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \end{pmatrix}$$

Now we applying the difference operator using equation (1) in (y_{lmn}) we get the following result:

$$\Delta^3 y_{lmn} = \begin{pmatrix} 640 & -640 & 640 & -640 & \dots & \dots \\ -640 & 640 & -640 & 640 & \dots & \dots \\ 640 & -640 & 640 & -640 & \dots & \dots \\ -640 & 640 & -640 & 640 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -896 & 896 & -896 & 896 & \dots & \dots \\ 896 & -896 & 896 & -896 & \dots & \dots \\ -896 & 896 & -896 & 896 & \dots & \dots \\ 896 & -896 & 896 & -896 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1152 & -1152 & 1152 & -1152 & \dots & \dots \\ -1152 & 1152 & -1152 & 1152 & \dots & \dots \\ 1152 & -1152 & 1152 & -1152 & \dots & \dots \\ -1152 & 1152 & -1152 & 1152 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1408 & 1408 & -1408 & 1408 & \dots & \dots \\ 1408 & -1408 & 1408 & -1408 & \dots & \dots \\ -1408 & 1408 & -1408 & 1408 & \dots & \dots \\ 1408 & -1408 & 1408 & -1408 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Here $(y_{lmn}) \notin \mathsf{c}^3_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3BR}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, , $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3, \mathsf{q})$. *Thus none of these classes of spaces is monotone in general.*

Example 3.6 *Let $X = \mathbb{C}$. Consider the sequence (x_{lmn}) defined by*

$$x_{lmn} = lmn, \quad \textit{for all } l, m, n \in \mathbb{N}$$

and the sequence (y_{lmn}) in the pre-image space defined by

$$y_{lmn} = \begin{cases} x_{lmn}, & \textit{if } l+m+n \textit{ is even;} \\ 0, & \textit{otherwise.} \end{cases}$$

Clearly,

$$(x_{lmn}) \in \ell^3_{\infty\mathsf{st}}(\Delta^3, \mathsf{q}),$$

but

$$(y_{lmn}) \notin \ell^3_{\infty\mathsf{st}}(\Delta^3, \mathsf{q}),$$

which shows that the space $\ell^3_{\infty\mathsf{st}}(\Delta^3, \mathsf{q})$ is not monotone in general.

□

We state the following result without proof, since it can be easily established.

Theorem 3.6 *The triple sequence spaces* $\ell^3_{\infty-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3BR}_{0-\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^3_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3B}_{\mathsf{st}}(\Delta^3, \mathsf{q})$, $\mathsf{c}^{3R}_{\mathsf{st}}(\Delta^3, \mathsf{q})$ *and* $\mathsf{c}^{3BR}_{\mathsf{st}}(\Delta^3, \mathsf{q})$ *are sequence algebra.*

4. Conclusion

In this article, we have established martingale difference sequence in possibility theory using hybrid filtration. Also convergence relational analysis based strategy under the possibility environment. Besides, we have validated our proposed hybrid possibility Martingale strategy by real life numerical example. Further, it is hoped that the proposed hybrid possibility martingale be the potential topic for the future research.

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Bimal Chandra Das,
Department of Mathematics,
Govt. Degree College, Kamalpur-799285, Tripura
India.
E-mail address: bcdas3744@gmail.com

and

Binod Chandra Tripathy,
Department of Mathematics,
Tripura University, Agartala-799022, Tripura
India.
E-mail address: tripathybc@yahoo.com, tripathybc@gmail.com