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Invariant Submanifolds of A (κ, μ)-Paracontact Metric Manifold

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ABSTRACT: In this paper, we research some geometric conditions for an invariant submanifold of a (κ, μ)-paracontact metric manifold to be totally geodesic. Besides this, we characterize an invariant submanifold of a (κ, μ)-paracontact metric manifold satisfying the conditions $Q(S, \sigma) = 0$, $Q(S, \nabla\sigma) = 0$, $Q(S, R \cdot \sigma) = 0$, $Q(g, C \cdot \sigma) = 0$ and $Q(S, C \cdot \sigma) = 0$ under the some conditions.

Key Words: Paracontact metric manifold, invariant submanifold, totally geodesic submanifold.

Contents

1. Introduction

The study of paracontact geometry was initiated by Kaneyuki and Williams [4]. After then, Zamkovoy started working paracontact metric manifolds and their subclasses [9]. Since several geometers interested paracontact metric manifolds and researched various important properties of these manifolds and some interesting results have been found.

The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. One of the class of paracontact manifolds for which the characteristic vector field ξ-belongs to the (κ, μ)-nullity condition for some real constants κ and μ. Such manifolds are known as (κ, μ)-paracontact metric manifolds [7].

In [8], authors gave some characterizations for an invariant submanifold of an LP-Sasakian manifold to be totally geodesic. Also, in [6], a generalized (κ, μ) paracontact metric manifold satisfying the curvature conditions $Q(S, R) = Q(S, g) = 0$ are characterized.

Recently, we have studied an invariant submanifold of a (κ, μ) paracontact metric manifold and obtained some new results [1]. In this paper, we research the conditions $Q(S, \sigma) = Q(S, \nabla\sigma) = Q(S, R \cdot \sigma) = 0$ and $Q(g, C \cdot \sigma) = Q(S, C \cdot \sigma) = 0$ for an invariant submanifold of a (κ, μ)-paracontact metric manifold and we think that new results are obtained contribute to geometry.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ is said to be a paracontact metric manifold if it admits a (1,1)-type tensor field ϕ, a unit spacelike vector field ξ, 1-form η and a semi-Riemannian metric tensor g which satisfy

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(X) = g(X, \xi) \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad \eta \circ \phi = 0 \tag{2.2}$$

and

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.3}$$

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for all $X, Y \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of the differentiable vector fields on $\widetilde{M}$.

In a paracontact metric manifold $(\widetilde{M}, \phi, \eta, \xi, g)$, we define a $(1,1)$-type tensor field by $h = \frac{1}{2}\ell_\xi \phi$, where ℓ denotes the Lie-derivative. One can easily to see that h is a symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h \;\; and \;\; Trh = 0. \tag{2.4}$$

$$2hX = (\ell_\xi \phi)X = \ell_\xi \phi X - \phi \ell_\xi X = [\xi, \phi X] - \phi[\xi, X] \tag{2.5}$$

By $\widetilde{\nabla}$, we denote the Levi-Civita connection of g, then we have

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X, \tag{2.6}$$

for all $X \in \Gamma(T\widetilde{M})$.

A paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be a (κ, μ)-space form if its the Riemannian curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.7}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where κ, μ are real constant. In a (κ, μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$, we have

$$\begin{aligned}
(\widetilde{\nabla}_X \phi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX), && (2.8)\\
S(X,Y) &= [2(1-n) + n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\
&+ [2(n-1) + n(2\kappa - \mu)]\eta(X)\eta(Y) && (2.9)\\
h^2 &= (1+\kappa)\phi^2, && (2.10)\\
Q\phi - \phi Q &= 2[2(n-1) + \mu]h\phi, && (2.11)
\end{aligned}$$

where S and Q denote the Ricci tensor and Ricci operatory defined $S(X,Y) = g(QX,Y)$.

In [5], author studied the Riemannian manifold satisfying $Q(S,R) = Q(S,g) = 0$. On the other hand, De and Samui studied pseudo-parallel, generalized Ricci pseudo-parallel and bi-recurrent, $C(X,Y)\cdot\sigma = fQ(g,\sigma)$ and $C(X,Y)\cdot\sigma = fQ(S,\sigma)$ [2].

Recently, Hu and Wang obtained the geometric conditions of invariant submanifolds of a trans-Sasakian manifold to be totally geodesic [3].

In this connection, we attempt to study invariant submanifolds of a (κ, μ)-paracontact metric manifold satisfying geometric conditions $Q(S,\sigma) = 0$, $Q(S, \nabla\cdot\sigma) = 0$, $Q(S, R\cdot\sigma) = 0$, $Q(g, C\cdot\sigma) = 0$ and $Q(S, C\cdot\sigma) = 0$. Finally, we see that these conditions are equivalent to $\sigma = 0$ under the some conditions.

On a semi-Riemannian manifold (M,g), for a (o,k)-type tensor field T and $(0,2)$-type tensor field A, $(0,k+2)$-type tensor field $Q(A,T)$ is defined as

$$\begin{aligned}
Q(A,T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, ..., X_k) \\
&- T(X_1, (X \wedge_A Y)X, X_3, ..., X_k) \\
&\cdot \\
&\cdot \\
&\cdot \\
&- T(X_1, X_2, ..., (X \wedge_A Y)X_k), && (2.12)
\end{aligned}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)X_1 = A(Y, X_1)X - A(X, X_1)Y. \tag{2.13}$$

For a Riemannian manifold (M^n, g), the concircular curvature tensor C is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{2.14}$$

for all $X, Y, Z \in \Gamma(TM)$, where τ denote the scalar curvature of M^n.

3. Invariant Submanifolds of A (κ,μ)-Paracontact Metric Manifold

Now, let M be an immersed submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, by ∇ and $\nabla^{\perp}$, we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{3.1}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{3.2}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where σ and A are called the second fundamental form and shape operator of M, respectively [1].

For an immersed submanifold M of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$, M is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for all $X \in \Gamma(TM)$ at every point on M, that is, $\phi(T_xM) \subseteq T_xM$ at each point $x \in M$. We will assume that M is an invariant submanifold in the rest of this paper unless say otherwise.

Lemma 3.1 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then the following relations hold.*

$$\nabla_X \xi = -\phi X + \phi h X \tag{3.3}$$
$$\sigma(\phi X, Y) = \sigma(X, \phi Y) = \phi\sigma(X,Y) \tag{3.4}$$
$$\sigma(X,\xi) = 0, \tag{3.5}$$

for all $X, Y \in \Gamma(TM)$.

Proof: Since the proof is a result of direct calculations, we will omit to it. □

Theorem 3.1 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(S,\sigma) = 0$ if and only if M is totally geodesic provided $\kappa \neq 0$.*

Proof: Let us assume $Q(S,\sigma) = 0$ which implies that

$$Q(S,\sigma)(U,V;X,Y) = -\sigma((X \wedge_S Y)U, V) - \sigma(U, (X \wedge_S Y)V) = 0,$$

for all $X, Y, U, V \in \Gamma(TM)$. Also, this implies

$$\sigma(S(Y,U)X - S(X,U)Y, V) + \sigma(U, S(Y,V)X - S(X,V)Y) = 0. \tag{3.6}$$

Here putting $Y = U = \xi$ in (3.6), using (2.9) and (3.5), we obtain

$$\sigma(S(\xi,\xi)X - S(X,\xi)\xi, V) + \sigma(\xi, S(\xi,V)X - S(V,X)\xi) = 2n\kappa\sigma(X,V) = 0.$$

The converse of the proof is obvious. Thus the proof is completed. □

Theorem 3.2 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(S,\widetilde{\nabla}\sigma) = 0$ if and only if M is totally geodesic provided $\kappa \neq 0$.*

Proof: let us suppose that $Q(S,\widetilde{\nabla}\sigma)=0$, that is,

$$\begin{aligned}-Q(S,\widetilde{\nabla}\sigma)(U,V,Z;X,Y) &= (\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\\ &= 0,\end{aligned}$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. This implies that

$$\begin{aligned}S(Y,U)(\widetilde{\nabla}_X\sigma)(V,Z) &- S(X,U)(\widetilde{\nabla}_Y\sigma)(V,Z)+(\widetilde{\nabla}_U\sigma)(S(Y,V)X,Z)\\ &- S(X,V)Y,Z)+(\widetilde{\nabla}_U\sigma)(V,S(Y,Z)X-S(X,Z)Y)\\ &= 0.\end{aligned} \tag{3.7}$$

The relation (3.7) yields for $Y=Z=\xi$,

$$\begin{aligned}S(\xi,U)(\widetilde{\nabla}_X\sigma)(V,\xi) &- S(X,U)(\widetilde{\nabla}_\xi\sigma)(V,\xi)+(\widetilde{\nabla}_U\sigma)(S(\xi,V)X,\xi)\\ &- (\widetilde{\nabla}_U\sigma)S(X,V)\xi,\xi)+2\kappa(\widetilde{\nabla}_U\sigma)(V,X)\\ &- (\widetilde{\nabla}_U\sigma)(S(X,\xi)\xi,V)=0.\end{aligned} \tag{3.8}$$

In view of (2.9), it follows that

$$\begin{aligned}2n\kappa\eta(U)(\widetilde{\nabla}_X\sigma)(V,\xi) &- S(X,U)(\widetilde{\nabla}_\xi\sigma)(V,\xi)+2n\kappa(\widetilde{\nabla}_U\sigma)(\eta(V)X,\xi)\\ &- (\widetilde{\nabla}_U\sigma)(S(V,X)\xi,\xi)+2n\kappa(\widetilde{\nabla}_U\sigma)(S(\xi,\xi)V,X)\\ &- 2n\kappa(\widetilde{\nabla}_U\sigma)(\eta(X)\xi,V)=0.\end{aligned}$$

Here if the necessary covariant derivatives are calculated, we have

$$\begin{aligned}&2n\kappa\eta(U)\{\nabla^\perp_X\sigma(V,\xi)-\sigma(\nabla_X V,\xi)-\sigma(\nabla_X\xi,V)\}\\ -\;& S(X,U)\{\nabla^\perp_\xi\sigma(V,\xi)-\sigma(\nabla_\xi V,\xi)-\sigma(V,\nabla_\xi\xi)\}\\ +\;& 2n\kappa\{\nabla^\perp_U\sigma(\eta(V)X,\xi)-\sigma(\nabla_U\eta(V)X,\xi)-\sigma(\eta(V)X,\nabla_U\xi)\}\\ -\;& \{\nabla^\perp_U\sigma(S(V,X)\xi,\xi)-\sigma(\nabla_U S(V,X)\xi,\xi)-\sigma(S(V,X)\xi,\nabla_U\xi)\}\\ -\;& 2\kappa\{\nabla^\perp_U\sigma(\eta(X)\xi,V)-\sigma(\nabla_U\eta(X)\xi,V)-\sigma(\eta(X)\xi,\nabla_U V)\}\\ +\;& 2\kappa(\widetilde{\nabla}_U\sigma)(V,X)=0.\end{aligned} \tag{3.9}$$

Taking into account of (3.3) and (3.5), we reach at

$$\begin{aligned}-\;& 2n\kappa\eta(U)\sigma(-\phi X+\phi hX,V)-2n\kappa\eta(V)\sigma(-\phi U+\phi hU,X)\\ +\;& 2n\kappa\sigma(U\eta(X)\xi+\eta(X)\nabla_U\xi,V)+2n\kappa(\widetilde{\nabla}_U\sigma)(V,X)=0,\end{aligned}$$

that is,

$$\begin{aligned}&2n\kappa\eta(U)\phi\{\sigma(V,X)-\sigma(hX,V)\}+2n\kappa\eta(V)\phi\{\sigma(U,X)-\sigma(hU,X)\}\\ -\;& 2n\kappa\eta(X)\phi\{\sigma(U,V)-\sigma(hU,V)\}+2n\kappa(\widetilde{\nabla}_U\sigma)(V,X)=0.\end{aligned} \tag{3.10}$$

Putting $V=\xi$ in (3.10), we obtain

$$2n\kappa\phi\{\sigma(X,U)-\sigma(X,hU)\}+2n\kappa\{\nabla^\perp_U\sigma(X,\xi)-\sigma(\nabla_U X,\xi)-\sigma(X,\nabla_U\xi)\}=0,$$

that is,

$$\kappa\phi\{\sigma(X,U)-\sigma(X,hU)\}=0. \tag{3.11}$$

If hU is written instead of U at (3.11) and using (2.10), we have

$$\begin{aligned}\kappa\{\sigma(X,hU) &- \sigma(X,h^2U)\}=0\\ \kappa\{\sigma(X,hU) &- (1+\kappa)\sigma(X,U)\}=0.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we conclude that $\kappa^2\sigma(X,U)=0$. The converse obvious. This completes the proof. □

Theorem 3.3 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(S,\widetilde{R}\cdot\sigma)=0$ if and only if M is totally geodesic submanifold provided $\mu^2(1+\kappa)-\kappa^2\neq 0$.*

Proof: From (2.12) and (2.13), we obtain

$$\begin{aligned}Q(S,\widetilde{R}(X,Y)\cdot\sigma)(U,V,W,Z) &= (\widetilde{R}(X,Y)\cdot\sigma)((W\wedge_S Z)U,V)\\ &+ (\widetilde{R}(X,Y)\cdot\sigma)(U,(W\wedge_S Z)V)=0,\end{aligned}$$

for all $X,Y,Z,U,V,W\in\Gamma(TM)$. This implies that

$$\begin{aligned}&(\widetilde{R}(X,Y)\cdot\sigma)(S(Z,U)W,V)-(\widetilde{R}(X,Y)\cdot\sigma)(S(W,U)Z,V)\\ +\ &(\widetilde{R}(X,Y)\cdot\sigma)(U,S(V,Z)W)\\ -\ &(\widetilde{R}(X,Y)\cdot\sigma)(U,S(W,V)Z)=0.\end{aligned} \tag{3.13}$$

Putting $Y=U=V=Z=\xi$ in (3.13), we have

$$\begin{aligned}&(\widetilde{R}(X,\xi)\cdot\sigma)(S(\xi,\xi)W,\xi)-(\widetilde{R}(X,\xi)\cdot\sigma)(S(W,\xi)\xi,\xi)\\ +\ &(\widetilde{R}(X,\xi)\cdot\sigma)(\xi,S(\xi,\xi)W)-(\widetilde{R}(X,\xi)\cdot\sigma)(\xi,S(W,\xi)\xi)=0.\end{aligned}$$

Also, by using (2.10), we obtain

$$\begin{aligned}&2n\kappa(\widetilde{R}(X,\xi)\cdot\sigma)(W,\xi)-2n\kappa(\widetilde{R}(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi)\\ +\ &2n\kappa(\widetilde{R}(X,\xi)\cdot\sigma)(\xi,W)-2n\kappa(\widetilde{R}(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi)=0.\end{aligned}$$

After the necessary arrangements are made, we reach at

$$n\kappa\{(\widetilde{R}(X,\xi)\cdot\sigma)(W,\xi)-(\widetilde{R}(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi)\}=0,$$

that is,

$$\begin{aligned}&R^{\perp}(X,\xi)\sigma(W,\xi)-\sigma(R(X,\xi)W,\xi)-\sigma(W,R(X,\xi)\xi)\\ -\ &R^{\perp}(X,\xi)\sigma(\eta(W)\xi,\xi)+\sigma(\eta(W)R(X,\xi)\xi,\xi)+\sigma(\eta(W)\xi,R(X,\xi)\xi)=0.\end{aligned}$$

Thus we have

$$\kappa\sigma(X,W)+\mu\sigma(hX,W)=0. \tag{3.14}$$

If hX is written instead of X at (3.14) and using (2.10), we reach at

$$\kappa\sigma(hX,W)+\mu\sigma(h^2X,W)=\mu(1+\kappa)\sigma(X,W)+\kappa\sigma(hX,W)=0. \tag{3.15}$$

From (3.14) and (3.15), we conclude that

$$(\mu^2(1+\kappa)-\kappa^2)\sigma(W,X)=0. \tag{3.16}$$

This proves our assertions. □

Theorem 3.4 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(g,C\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold or the scalar curvature τ of $\widetilde{M}^{2n+1}$ satisfies the equality $\tau=2n(2n+1)(\kappa\pm\mu\sqrt{1+\kappa})$.*

Proof: We suppose that $Q(g,C\cdot\sigma)=0$. This means that

$$Q(g,C(X,Y)\cdot\sigma)(U,V,Z,W)=0,$$

for all $X,Y,Z,U,V,W\in\Gamma(T\widetilde{M})$. This implies that

$$(C(X,Y)\cdot\sigma)((Z\wedge_g W)U,V)+(C(X,Y)\cdot\sigma)((U,Z\wedge_g W)V),$$

that is,

$$\begin{aligned} & (C(X,Y)\cdot\sigma)(g(U,W)Z,V)-(C(X,Y)\cdot\sigma)(g(Z,U)W,V)\\ + & (C(X,Y)\cdot\sigma)(U,g(V,W)Z)-(C(X,Y)\cdot\sigma)(U,g(Z,V)W)\\ = & 0. \end{aligned} \tag{3.17}$$

Again, putting $Y=V=Z=U=\xi$ in (3.17), we obtain

$$(C(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi)-(C(X,\xi)\cdot\sigma)(W,\xi)=0. \tag{3.18}$$

Here,

$$\begin{aligned} (C(X,\xi)\cdot\sigma)(\eta(W)\xi,\xi) &= R^{\perp}(X,\xi)\sigma(\eta(W)\xi,\xi)-\sigma(\eta(W)C(X,\xi)\xi,\xi)\\ &- \sigma(\eta(W)\xi,C(X,\xi)\xi)=0, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} (C(X,\xi)\cdot\sigma)(W,\xi) &= R^{\perp}(X,\xi)\sigma(W,\xi)-\sigma(C(X,\xi)W,\xi)-\sigma(W,C(X,\xi)\xi)\\ &= -\sigma(W,C(X,\xi)\xi). \end{aligned} \tag{3.20}$$

From (2.14), (3.5), (3.18), (3.19) and (3.20) Thus we have

$$(\kappa-\frac{\tau}{2n(2n+1)})\sigma(X,W)+\mu\sigma(hX,W)=0. \tag{3.21}$$

Substituting hX into X in (3.21) and making use of (2.10), we obtain

$$\begin{aligned} & (\kappa-\frac{\tau}{2n(2n+1)})\sigma(hX,W)+\mu\sigma(h^2X,W)\\ = & \mu(1+\kappa)\sigma(X,W)+(\kappa-\frac{\tau}{2n(2n+1)})\sigma(hX,W)=0. \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we conclude that

$$\left[\left(\kappa-\frac{\tau}{2n(2n+1)}\right)^2-\mu^2(1+\kappa)\right]\sigma(X,W)=0, \tag{3.23}$$

which proves our assertion. □

Example 3.1 *We consider the 3-dimensional manifold $M=\{(x,y,t)\in\mathbb{R}^3,\ t\neq 0\}$, where (x,y,t) are standart coordinates of $\mathbb{R}^3$. The vector fields*

$$e_1=\frac{\partial}{\partial x},\quad e_2=e^{2t}\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial t}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_2) &= g(e_1, e_3) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1 \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X, e_1)$ for any $X \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = 0, \qquad \phi(e_3) = -e_2, \qquad \phi(e_2) = -e_3.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_3, e_1] = 0, \quad [e_1, e_2] = 0, \qquad [e_2, e_3] = -2e^t e_1.$$

Then we have

$$\eta(e_1) = g(e_1, e_1) = 1, \quad \phi^2 X = X - \eta(X)e_1, \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$. Hence, (ϕ, ξ, η, g) defines a paracontact metric structure on M for $e_1 = \xi$.

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using the above formula, we obtain.

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_2} e_1 &= -e^t e_3, & \nabla_{e_3} e_1 &= -e^t e_2, \\ \nabla_{e_1} e_2 &= -e^t e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_3} e_2 &= e^t e_1, \\ \nabla_{e_1} e_3 &= -e^t e_2, & \nabla_{e_2} e_3 &= -e^t e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

Comparing the above relations with $\nabla_X e_1 = -\phi X + \phi h X$, we get

$$he_2 = -(e^t + 1)e_2, \qquad he_3 = -(e^t + 1)e_3, \qquad he_1 = 0.$$

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we calculate the following:

$$\begin{aligned} R(e_1, e_2)e_1 &= (\frac{1}{(e^t + 1)^2} - 1)\{\eta(e_2)e_1 - \eta(e_1)e_2\} + ((e^t - 1) \\ &\quad + \frac{1}{(e^t + 1)^3})\{\eta(e_2)he_1 - \eta(e_1)he_2\} \\ &= e^{2t} e_2 \end{aligned}$$

$$\begin{aligned} R(e_1, e_3)e_1 &= (\frac{1}{(e^t + 1)^2} - 1)\{\eta(e_3)e_1 - \eta(e_1)e_3\} + ((e^t - 1) \\ &\quad + \frac{1}{(e^t + 1)^3})\{\eta(e_3)he_1 - \eta(e_1)he_3\} \\ &= e^{2t} e_3 \end{aligned}$$

$$\begin{aligned} R(e_2, e_3)e_1 &= (\frac{1}{(e^t + 1)^2} - 1)\{\eta(e_3)e_2 - \eta(e_2)e_3\} + ((e^t - 1) \\ &\quad + \frac{1}{(e^t + 1)^3})\{\eta(e_3)he_2 - \eta(e_2)he_3\} \\ &= 0. \end{aligned}$$

By the above expressions of the curvature tensor and using (2.10), we conclude that M is a generalized $(k, \mu)-$paracontact metric manifold with $k = (\frac{1}{(e^t+1)^2} - 1)$ and $\mu = ((e^t - 1) + \frac{1}{(e^t+1)^3})$.

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