Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–5.
ISSN-0037-8712 in press
doi:10.5269/bspm.63283

On Symmetric Generalized bi-semiderivations of Prime Rings

Faiza Shujat

abstract: In the present note, we inaugrate the idea of symmetric generalized bi-semiderivation on rings and prove some classical commutativity results for generalized bi-semiderivation. Moreover, our main objective is to extend the main theorem in [7] for biderivation to the case of symmetric generalized bi-semiderivation on prime ring.

Key Words: Prime ring, homomorphism, bi-derivation, generalized bi-semiderivation.

Contents

1. Introduction

The idea of symmetric bi-derivations inaugrated by Maksa [3]. A mapping $D : R \times R \to R$ is said to be symmetric if $D(a, b) = D(b, a)$ for all $a, b \in R$. A mapping $D : R \times R \to R$ is called biadditive if it is additive in both slots. Now we introduce the concept of symmetric biderivations as follows: A biadditive mapping $D : R \times R \longrightarrow R$ is called a biderivation if for every $a \in R$, the map $b \mapsto D(a, b)$ as well as for every $b \in R$, the map $a \mapsto D(a, b)$ is a derivation of R. For ideational reading in the related matter one can turn to [3]. For a symmetric biadditive mapping D, a map $h : R \to R$ defined as $h(a) = D(a, a)$, for every a in R is called the trace of D.

Bergen [5] defined the concept of semiderivations of a ring R. An additive mapping $f : R \longrightarrow R$ is called a semiderivation if there exists a function $g : R \longrightarrow R$ such that $f(ab) = f(a)g(b) + af(b) = f(a)b + g(a)f(b)$ and $f(g(a)) = g(f(a))$ for each $a, b \in R$. In case g is an identity map of R; then all semiderivations associated with g are merely ordinary derivations. On the other hand, if g is a homomorphism of R such that $g \neq 1$; then $f = g - 1$ is a semiderivation which is not a derivation. In case R is prime and $f \neq 0$; it has been shown by Chang [6] that g must necessarily be a ring endomorphism. Let f be a semiderivation of R, associated with an endomorphism g. The additive map F on R is a generalized semiderivation of R if $F(ab) = F(a)b + g(a)f(b) = F(a)g(b) + af(b)$ and $F(g) = g(F)$, for every a, b in R. Of course any semiderivation is a generalized semiderivation. Moreover, if g is the identity map of R, then all generalized semiderivations associated with g are merely generalized derivations of R. The most natural example of generalized semiderivation, we consider a semiderivation f on a ring R associated with a function g and define the two map as $F(a) = f(a) - a$ and $H(a) = f(a) + a$, a in R. With such construction the map F and H are generalized semiderivations on R with associated function g.

Following [8], a symmetric bi-additive function $\vartheta : R\times R \longrightarrow R$ is called a symmetric bi-semiderivation associated with a function $f : R \longrightarrow R$ (or simply a symmetric bi-semiderivation of a ring R) if

$$\vartheta(ab, c) = \vartheta(a, c)f(b) + a\vartheta(b, c) = \vartheta(a, c)b + f(a)\vartheta(b, c)$$

$$\vartheta(a, bc) = \vartheta(a, b)f(c) + b\vartheta(a, c) = \vartheta(a, b)c + f(b)\vartheta(a, c)$$

and $\vartheta(f) = f(\vartheta)$ for all $a, b, c \in R$.

2010 *Mathematics Subject Classification:* 16U80, 16N60, 16W25.

Submitted April 16, 2022. Published December 16, 2022

Example 1.1. *Consider R is a commutative ring and* $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\}$ *will be a ring under matrix addition and multiplication. Define* $\vartheta : S \times S \longrightarrow S$ *such that* $\vartheta\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & bd \\ 0 & 0 \end{pmatrix}$ *and* $f : S \longrightarrow S$ *by* $f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. *Then ϑ represents bi-semiderivation on S with associated function f.*

Motivated by the above explanation, we introduce the concept of generalized bi-semiderivation on ring as follows: Consider the maps $\delta, \vartheta : R \times R \longrightarrow R$ and $f : R \longrightarrow R$. Now define if for every $a \in R$, $b \mapsto \delta(a, b)$ and for every $b \in R$, $a \mapsto \delta(a, b)$ are generalized semi-derivation of R with associated function ϑ, f (defined as above), and satisfying $\delta(f) = f(\delta)$, then δ will be called generalized bi-semiderivation on R. More precisely, we say that δ, ϑ, f satisfying the following:

1. $\delta(ab, c) = \delta(a, c)f(b) + a\vartheta(b, c) = \delta(a, c)b + f(a)\vartheta(b, c)$
2. $\delta(a, bc) = \delta(a, b)f(c) + b\vartheta(a, c) = \delta(a, b)c + f(b)\vartheta(a, c)$
3. $f(\delta) = \delta(f)$ for every $a, b, c \in R$.

Example 1.2. *Consider the set* $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in 2\mathbb{Z}_8 \right\}$. *Then R represents a ring under matrix addition and matrix multiplication. Define $\delta, \vartheta : R \times R \longrightarrow R$ such that*

$$\delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & k \\ g & h \end{pmatrix} \right) = \begin{pmatrix} 0 & bk \\ cg & 0 \end{pmatrix},$$

$$\vartheta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & k \\ g & h \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix}$$

and $f : R \longrightarrow R$ *by* $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. *Therefore, δ is a generalized bi-semiderivation with associated function ϑ and f on R. In above defined concept of generalized bi-semiderivation, we have the following observations:*

1. *δ will be considered as bi-semiderivation, if we assume $\delta = \vartheta$.*
2. *If we consider f an identity map, then δ acting as generalized biderivation on R with associated map ϑ.*
3. *If we take f an identity map and $\delta = \vartheta$, then δ acting as biderivation.*
4. *Construct a new function $F(a, x) = \delta(a, x)$ and $f =$ identity, for some fixed a and for all x in R, then F behaves as generalized derivation with associated derivation $\vartheta(a, x)$.*

It is obvious that the definition of generalized bi-semiderivations unifies the notions of bi-semiderivation and generalized bi-derivation and covers the concepts of derivations, generalized derivations, left (right) centralizers, and semiderivations. In this paper, we prove some theorems on symmetric generalized bi-semiderivation of a ring that is the extension of Vukman's results which states that: Let R be a prime ring of characteristic not two and three. If D_1, D_2 are the symmetric biderivations of R with trace h_1, h_2, respectively, such that $h_1(a)h_2(a) = 0$ for every a in R, then either $D_1 = 0$ or $D_2 = 0$. In this note, we obtain some commutativity results associated with symmetric generalized bi-semiderivations in the setting of prime ring, that is the extension of some theorems proved in [2]. For more related literature reader can look inside [1,3,5,7,8] and the references therein.

2. Main results

Throughout we consider f a surjective function on a prime ring R. Next, we use the characteristic restriction without mentioning it, wherever, it is needed. To prove our main theorems, we need the following lemmas.

Lemma 2.1. *Let δ be a generalized bi-semiderivation with associated bi-semiderivation ϑ and associated function f on R. If $\delta \neq 0$, then f acting as a homomorphism on R.*

Proof: We have by the definition of δ,

$$\delta(u(a+b),c) = \delta(u,c)f(a+b) + u\vartheta((a+b),c) = \delta(u,c)f(a+b) + u\vartheta(a,c) + u\vartheta(b,c).$$

Reword the left hand side of above equation as

$$\delta(ua+ub,c) = \delta(ua,c) + \delta(ub,c) = \delta(u,c)f(a) + u\vartheta(a,c) + \delta(u,c)f(b) + u\vartheta(b,c)$$

$$= \delta(u,c)\{f(a)+f(b)\} + u\vartheta(a,c) + u\vartheta(b,c).$$

Now comparing the above two expansion of $\delta(u(a+b),c)$ to find

$$\delta(u,c)f(a+b) = \delta(u,c)\{f(a)+f(b)\}$$

and hence we can conclude $f(a+b) = f(a)+f(b)$. Since $\delta \neq 0$ is a generalized bi-semiderivation, we have

$$\delta(w(ab),c) = \delta(w,c)f(ab) + wD(ab,c)f(ab) + wD(a,c)f(b) + waD(b,c).$$

Also we express above relation as

$$\delta(wab,c) = \delta(wa,c)f(b) + waD(b,c)$$

$$\delta(w,c)f(a)f(b) + wD(a,c)f(b) + waD(b,c).$$

Compare the right hand side of above two expansion of δ, we arrive at $\delta(w,c)f(ab) = \delta(w,c)f(a)f(b)$, for all $a,b,c \in R$. This implies that

$$\delta(w,c)\{f(ab) - f(a)f(b\} = 0.$$

Given that $\delta \neq 0$, and hence we find $f(ab) - f(a)f(b) = 0$. That is $f(ab) = f(a)f(b)$. This completes the proof. □

Lemma 2.2. *Let δ be a generalized bi-semiderivation with associated bi-semiderivation ϑ and associated function f on R. If for $0 \neq \gamma \in R$, such that $\gamma\delta(u,v) = 0$, then $\vartheta = 0$.*

Proof: Since $\gamma\delta(u,v) = 0$, for all $u,v \in R$, we have by replacing u by uw, $0 = \gamma\delta(uw,v) = \gamma\delta(u,v)f(w) + \gamma u\vartheta(w,v) = \gamma u\vartheta(w,v)$. It is given that $\gamma \neq 0$, primeness implies that $\vartheta(w,v) = 0$ for all $w,v \in R$. □

Theorem 2.3. *Let R be a prime ring of characteristic not two and three. If δ is a symmetric generalized bi-semiderivation on R with associated bi-semiderivation ϑ and associated map f such that $\delta(r,r) \subseteq Z(R)$, then either R is commutative or $\vartheta = 0$.*

Proof: Since $\delta(r,r) \subseteq Z(R)$, we have

$$[\delta(r,r),s] = 0 \text{ for every } r,s \in R. \tag{2.1}$$

Linearizing the above equation in r, we find

$$[\delta(r,r),s] + [\delta(p,p),s] + 2[\delta(r,p),s] = 0 \text{ for every } r,p,s \in R. \tag{2.2}$$

Using characteristic condition of R and (2.1), we obtain

$$[\delta(r,p),s]=0 \text{ for every } r,p,s\in R. \tag{2.3}$$

Put rt for r in (2.3) to get

$$\delta(r,p)[f(t),s]+r[\vartheta(t,p),s]+[r,s]\vartheta(t,p)=0 \text{ for every } r,p,t,s\in R. \tag{2.4}$$

Substituting rs for r in (2.4) and use (2.4), we arrive at

$$\delta(s,p)f(r)[f(t),s]+s\vartheta(r,p)[f(t),s]-s\delta(r,p)[f(t),s]=0 \text{ for every } r,p,t,s\in R. \tag{2.5}$$

This implies that

$$\{\delta(s,p)f(r)+s\vartheta(r,p)-s\delta(r,p)\}[f(t),s]=0 \text{ for every } r,p,t,s\in R. \tag{2.6}$$

Particulary, we can get

$$s\vartheta(s,p)[t,s]=0 \text{ for every } p,t,s\in R. \tag{2.7}$$

From the above equation we can make the subsets $K=\{s\in R,\ \vartheta(s,p)=0, \text{ for all } p\in R\}$ and $J=\{[t,s]=0, \text{ for every } s,t\in R\}$. Clearly $(K,+)$ and $(J,+)$ are additive subgroups of R. Since a group cannot be the union of two proper subgroups. Hence primeness yields that either $J=0$ or $K=0$. If $J=0$, then R is commutative. In the second case we get $\vartheta=0$. □

Theorem 2.4. *Let R be a prime ring of characteristic not two and three, δ_1,δ_2 be generalized bi-semiderivation having associated bi-semiderivation ϑ_1,ϑ_2 and associated functions f_1,f_2 on R respectively. If $\delta_1(p,p)\delta_2(p,p)=0$, for all $p\in R$, then one of the following conditions hold:*

1. *R is commutative*
2. *$\vartheta_1=0$*
3. *$\vartheta_2=0$.*

Proof: Given that

$$\delta_1(p,p)\delta_2(p,p)=0 \text{ for every } p\in R. \tag{2.8}$$

Linearize the equation (2.8) and reuse (2.8) to find

$$\begin{aligned}\delta_1(p,p)\delta_2(t,t) &+ 2\delta_1(p,p)\delta_2(p,t)+\delta_1(t,t)\delta_2(p,p)+2\delta_1(t,t)\delta_2(p,t)\\ &+ 2\delta_1(p,t)\delta_2(p,t)+2\delta_1(p,t)\delta_2(t,t)\\ &+ 4\delta_1(p,t)\delta_2(p,t)=0 \text{ for all } p,t\in R.\end{aligned} \tag{2.9}$$

Putting $-t$ in place of t in (2.9) and subtracting from (2.9), we obtain

$$\delta_1(p,p)\delta_2(p,t)+\delta_1(t,t)\delta_2(p,t)+2\delta_1(p,t)\delta_2(t,t)=0 \text{ for every } p,t\in R. \tag{2.10}$$

Again linearize (2.10) in p and compare with (2.10) to get

$$\delta_1(z,z)\delta_2(p,t)+2\delta_1(p,z)\delta_2(p,t)+\delta_1(p,p)\delta_2(z,t)+2\delta_1(p,z)\delta_2(z,t)=0 \text{ for every } p,t,z\in R. \tag{2.11}$$

Put $-z$ in place of z in (2.11) and adding resulting equation with (2.11) to find

$$\delta_1(z,z)\delta_2(p,t)+2\delta_1(p,z)\delta_2(z,t)=0 \text{ for every } p,t,z\in R. \tag{2.12}$$

Substitute tr for t in (2.12) and apply (2.12) to obtain

$$\delta_1(z,z)t\vartheta_2(p,r)+2\delta_1(p,z)t\vartheta_2(z,r)=0 \text{ for every } p,t,z,r\in R. \tag{2.13}$$

In particular, we can write

$$3\delta_1(z,z)t\vartheta_2(z,r) = 0 \text{ for every } t,z,r \in R. \quad (2.14)$$

Since R is not of characteristic 3, we have

$$\delta_1(z,z)t\vartheta_2(z,r) = 0 \text{ for every } t,z,r \in R. \quad (2.15)$$

By primeness arguments, above equation splits into two cases like left hand side as first case and right hand side as second case. A simple computation with application of Theorem 2.3 in first case together with second case come to an end the proof. □

Theorem 2.5. *Let R be a prime ring of characteristic not two and three, δ be a generalized bi-semiderivation having associated bi-semiderivation ϑ and associated functions f on R. If $[\delta(b,b),\vartheta(c,c)] = 0$, for all $b,c \in R$, then either R is commutative or $\vartheta = 0$.*

Proof: It is given that

$$[\delta(b,b),\vartheta(c,c)] = 0 \text{ for every } b,c \in R. \quad (2.16)$$

Linearize (2.16) to get

$$[\delta(b,b),\vartheta(c,c)] + [\delta(b,b),\vartheta(u,u)] + 2[\delta(b,b),\vartheta(c,u)] = 0 \text{ for every } b,c,u \in R. \quad (2.17)$$

In view of (2.16) and using characteristic condition, (2.17) takes the form

$$[\delta(b,b),\vartheta(c,u)] = 0 \text{ for every } b,c,u \in R. \quad (2.18)$$

Putting ur in place of u in (2.18), we obtain

$$\vartheta(c,u)[\delta(b,b),f(r)] + [\delta(b,b),u]\vartheta(c,r) = 0 \text{ for every } b,c,u,r \in R. \quad (2.19)$$

Particularly replace $f(r)$ by $\vartheta(p,p)$ in (2.19) to find

$$[\delta(b,b),u]\vartheta(c,r) = 0 \text{ for every } b,c,u,r \in R. \quad (2.20)$$

We can rewrite the above equation as $[\delta(b,b),u]R\vartheta(c,r) = 0$ for every $b,c,u,r \in R$. Primeness of R yields that either $[\delta(b,b),u] = 0$ or $\vartheta(c,r) = 0$ for all $b,c,u,r \in R$. If we take the part $[\delta(b,b),u] = 0$ for all $b,u \in R$, then we can conclude by theorem 2.3. □

References

1. F. Shujat, *Symmetric generalized biderivations of prime rings*, Bol. Soc. Paran. Mat. 39(4)(2021), 65-72.
2. F. Shujat, *Commuting symmetric bi-semiderivations on rings,* Adv. in Maths: Sci. J. 10 (9) (2021), 3233–3240
3. G. Maksa, *A remark on symmetric biadditive functions having non-negative diagonalization*, Glasnik. Mat. 15 (35) (1980), 279-282.
4. I. N. Herstein, *A note on derivations II,* Canad. Math. Bull. 22 (1979), 509-511.
5. J. Bergen, *Derivations in Prime Rings,* Canad. Math. Bull. 26 (1983), 267-270.
6. J. C. Chang, *On semiderivations of prime rings,* Chinese J. Math. 12 (1984), 255-262.
7. J. Vukman, *Two results concerning symmetric biderivations on prime rings*, Aequat. Math. 40 (1990), 181–189.
8. H. Yazrali and D. Yilmaz, *On symmetric bi-semiderivation on prime rings,* preprint (2020).
9. N. Rehman and A. Z. Ansari, *On lie ideals with symmetric bi-additive maps in rings,* Palestine J. Math. 2 (2013), 14-21.

Faiza Shujat,
Department of Mathematics,
Faculty of Science, Taibah University, Madinah
Saudi Arabia.
E-mail address: faiza.shujat@gmail.com, fullahkhan@taibahu.edu.sa