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D-recurrence of Operators on Banach Spaces

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ABSTRACT: An operator T is called recurrent if for any non empty set U there exists $n \in \mathbb{N}$ such that $T^n U \cap U \neq \phi$. In this paper, we generalize this concept by using recurrence of operators in the closed unit disk $\mathbb{D}$ and we call it $\mathbb{D}$-recurrent operators. We extend some properties of recurrence to $\mathbb{D}$-recurrence ones. In particular, we show that an operator is $\mathbb{D}$-recurrent if and only if the set of all its recurrent vectors is dense. Also, we show that every operator T and its iterates T^n $(n \in \mathbb{N})$ shares the same $\mathbb{D}$-recurrent vectors. Unlike the case of recurrent operators, we show that the inverse of some $\mathbb{D}$-recurrent operators are not $\mathbb{D}$-recurrent, while others are $\mathbb{D}$-recurrent too. According to the strong connection between $\mathbb{D}$-recurrent operators and both diskcyclic and recurrent operators, we give an example to show that not every $\mathbb{D}$-recurrent operator is recurrent and diskcyclic. The later results rely on a nice characterization for an operator to be $\mathbb{D}$-recurrent which we call $\mathbb{D}$- recurrent criterion. Finally, we provide the relation between power boundedness and $\mathbb{D}$-recurrence. In particular, if T is power bounded then the set of all D-recurrent vectors is closed. Also, T^{-1} is power bounded and $\mathbb{D}$-recurrent whenever T is.

Key Words: recurrent operator, $\mathbb{D}$-recurrent operator, hypercyclic operator, diskcyclic operator, power bounded.

Contents

1. Introduction

One of the main concepts in Linear Dynamics is that of hypercyclicity. An operator T in $\mathcal{B}(\mathcal{X})$ is said to be hypercyclic if there exists a vector x in $\mathcal{X}$ such that the orbit $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in $\mathcal{X}$. Such a vector x is called a hypercyclic vector for T. The first example of a hypercyclic operator on a Banach space was provided in 1969 by Rolewicz [11] who showed that if T is the unilateral backward shift on the sequence space $\ell^p(\mathbb{N})$ and if c is a scalar with $|c| > 1$, then the operator cT is hypercyclic. In Banach spaces, being hypercyclic is equivalent to a property called topological transitive. An operator T is called topological transitive if for any pair G, H of non empty open subsets of $\mathcal{X}$ there is a positive integer n such that $T^n G \cap H \neq \phi$.

The notion of supercyclicity was invented by Hilden and Wallen [10]. An operator T is called supercyclic if there is a vector x such that $\mathbb{C}Orb(T, x)\, \{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}$ is dense in $\mathcal{X}$. Such a vector x is called a supercyclic vector for T. Good sources to learn about hypercyclic and supercyclic operators are [3], [7] and [9].

Moreover, diskcyclicity concept was introduced by Zeana [12] which is the mid way between hypercyclicity and supercyclicity. An operator T is called diskcyclic if there is a vector $x \in \mathcal{X}$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \geq 0\}$ is dense in $\mathcal{X}$, such a vector x is called diskcyclic for T. For more information about diskcyclic operators, the reader may refer to [1], [2] and [12].

Systems with a dense orbit play a fundamental role in essentially all branches of the area of Dynamical Systems and are related to the recurrence. In the context of linear dynamics, Costakis et al. [6] were the first to study recurrent operators. An operator is called recurrent if for any non-empty open subset G of $\mathcal{X}$ there exists $n \in \mathbb{N}$ such that $T^n G \cap G \neq \phi$. It is clear from its definition that every topological transitive (hypercyclic) operator is recurrent. However, the converse is not true as shown in [6]. Some

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properties and examples on recurrent operators can be found in [4], [5] and [6].

The purpose of this paper is to define a new notion in linear dynamics which we call $\mathbb{D}$-recurrence and it is a weaker form of recurrence. First, we find the relation between $\mathbb{D}$-recurrence and its corresponding concept of diskcyclicity. Second, we extend some recurrence properties to $\mathbb{D}$-recurrence. The two most interesting properties are: an operator T is $\mathbb{D}$-recurrent if and only if the set of all its recurrent vectors is dense, and every operator T and its iterates T^n ($n \in \mathbb{N}$) shares the same $\mathbb{D}$-recurrent vectors.

It is proved in [6] that an operator is recurrent if and only if its inverse is recurrent. For this reason, it is natural to ask if the inverse of $\mathbb{D}$-recurrent operators is again $\mathbb{D}$-recurrent. We show that the answer is in negative. In particular, we provide a nice characterization for an operator to be $\mathbb{D}$-recurrent which we call $\mathbb{D}$- recurrent criterion. We use this criterion to show that the inverse of some $\mathbb{D}$-recurrent operators may fail to be $\mathbb{D}$-recurrent, while others may be $\mathbb{D}$-recurrent too. Also, we use $\mathbb{D}$- recurrent criterion to show that not every $\mathbb{D}$-recurrent operator is recurrent.

Finally, we provide the relation between power boundedness and $\mathbb{D}$-recurrence. In particular, if T is power bounded then the set of all D-recurrent vectors is closed. We proved that if T is a contraction operator and $\mathbb{D}$-recurrent then it is a surjective isometry. We use the latest result to prove that T^{-1} is power bounded and $\mathbb{D}$-recurrent whenever T is.

2. Main Results

In all that follows, we use $\mathcal{X}$ to denote a separable infinite dimensional Banach space over the complex field $\mathbb{C}$ unless otherwise stated. In addition, we use $\mathcal{B}(\mathcal{X})$ to denote the operator algebra consisting of all continuous linear operators $T : \mathcal{X} \to \mathcal{X}$ and $\mathbb{D}$ be the closed unit disk.

Definition 2.1 Let $T \in \mathcal{B}(\mathcal{X})$. Then T is called $\mathbb{D}$-recurrent if for every open set G there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{D}\backslash\{0\}$ such that $\lambda T^n(G) \cap G \neq \phi$ or equivalently there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}, |\alpha| \geq 1$ such that $\lambda T^{-n}(G) \cap G \neq \phi$.

Definition 2.2 Let $T \in \mathcal{B}(\mathcal{X})$, a vector $x \in \mathcal{X}$ is called a $\mathbb{D}$-recurrent vector for T if there exist an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k T^{n_k} x \to x$ as $k \to \infty$. The set of all recurrent vectors for an operator is denoted by $\mathbb{D}R(T)$

The following proposition is an immediate consequence of [2, Proposition 2.10.]

Proposition 2.1 *Every diskcyclic operator is $\mathbb{D}$-recurrent.*

Proof: Let T be a diskcyclic operator on a Banach space $\mathcal{X}$, then T is disk transitive and so for every non-empty open sets G and H there exist $n \in \mathbb{N}$ and $0 \neq \lambda \in \mathbb{D}$ such that $\lambda T^n(G) \cap H \neq \phi$ and so T is $\mathbb{D}$-recurrent. □

The following example shows that the converse of the above proposition is not true in general.

Example 2.1 *Let $T \in \mathcal{B}(\mathbb{C}^2)$ defined by $Tx = cx$ where $x = (x_1, x_2) \in \mathbb{C}^2$ and $c \in \mathbb{C}; |c| \geq 1$. Then T is $\mathbb{D}$-recurrent but not diskcyclic.*

Proof: Suppose that $\lambda_k = \frac{1}{c^k}$ then $\lambda_k \in \mathbb{D}\backslash\{0\}$ for each $k \in \mathbb{N}$. Let G be a non-empty open subset of $\mathbb{C}^2$ and $y \in G$ then there exists $r \in \mathbb{N}$ such that $\lambda_r T^r y = \lambda_r c^r y = \frac{1}{c^r} c^r y = y \in G$ which follows that $\lambda_r T^r G \cap G \neq \phi$. Hence T is $\mathbb{D}$-recurrent.
However, by [2, Proposition 3.9] T cannot be diskcyclic. □

We extend some properties for recurrent operators to $\mathbb{D}$-recurrent ones in the following theorem.

Theorem 2.1 *Let $T \in \mathcal{B}(\mathcal{X})$, then $\mathbb{D}R(T)$ is dense in $\mathcal{X}$ if and only if T is $\mathbb{D}$-recurrent.*

Proof: Let $\mathbb{D}R(T)$ be a dense set in $\mathcal{X}$ and G be an open set in $\mathcal{X}$. Let y be a point of intersection of G with $\mathbb{D}R(T)$ and $\epsilon > 0$ then there exists an open ball $B_\epsilon(y)$ such that $y \in B_\epsilon(y) \subseteq G$. Since y is a $\mathbb{D}$-recurrent vector for T, then there exist an increasing sequence of positive integers $(n_k)_{n\in\mathbb{N}}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k T^{n_k} y \to y$ as $k \to \infty$ i.e. $\|\lambda_m T^m y - y\| \leq \epsilon$ for some $m \in \mathbb{N}$ which implies that $\lambda_m T^m y \in G$ and so$\lambda_m T^m(G) \cap G \neq \phi$. It follows that T is $\mathbb{D}$-recurrent.
Conversely: Let T be a $\mathbb{D}$-recurrent operator, we will prove that $\mathbb{D}R(T)$ is dense by showing that it intersects every open ball in $\mathcal{X}$. Suppose that $B_1 = B_\epsilon(x)$ be an open ball with center $x \in \mathcal{X}$ and radius $\epsilon < 1$. Since T is $\mathbb{D}$-recurrent, then there exist $x_1 \in \mathcal{X}$, $r_1 \in \mathbb{N}$ and $\alpha_1 \in \mathbb{C}, |\alpha_1| \geq 1$ such that $x_1 \in \alpha_1 T^{-r_1}(B_1) \cap B_1 \neq \phi$ and so $(1/\alpha_1)T^{r_1}x_1 \in B_1 \cap (1/\alpha_1)T^{r_1}(B_1)$, but T is continuous then there exists an open ball $B_2 = B_{\epsilon_1}(x_1)$ with center x_1 and radius $\epsilon_1 < 1/2$ such that $(1/\alpha_1)T^{r_1}(B_2) \subseteq B_1 \cap (1/\alpha_1)T^{r_1}(B_1)$ which follows that $B_2 \subseteq \alpha_1 T^{-r_1}(B_1) \cap B_1$. Therefore,

$$B_2 \subseteq B_1 \text{ and } (1/\alpha_1)T^{r_1}(B_2) \subseteq B_1$$

Since T is $\mathbb{D}$-recurrent, then there exist $x_2 \in \mathcal{X}$, $r_2 \in \mathbb{N}; r_2 > r_1$ and $\alpha_2 \in \mathbb{C}, |\alpha_2| \geq 1$ such that $x_2 \in \alpha_2 T^{-r_2}(B_2) \cap B_2 \neq \phi$ and so $(1/\alpha_2)T^{r_2}x_2 \in B_2 \cap (1/\alpha_2)T^{r_2}(B_2)$. Again since T is continuous then there exists an open ball $B_3 = B_{\epsilon_2}(x_2)$ with center x_2 and radius $\epsilon_2 < 1/2^2$ such that $(1/\alpha_2)T^{r_2}(B_3) \subseteq B_2 \cap (1/\alpha_2)T^{r_2}(B_2)$ which follows that $B_3 \subseteq \alpha_2 T^{-r_2}(B_2) \cap B_2$. Therefore,

$$B_3 \subseteq B_2 \text{ and } (1/\alpha_2)T^{r_2}(B_3) \subseteq B_2$$

By continuing this process, we get an open ball $B_{n+1} = B_{\epsilon_n}(x_n)$ with center x_n and radius $\epsilon_n < 1/2^n$ such that $(1/\alpha_n)T^{r_n}(B_{n+1}) \subseteq B_n \cap (1/\alpha_n)T^{r_n}(B_n)$ for some $x_n \in \mathcal{X}$, $r_n \in \mathbb{N}; r_n > r_{n-1}$ and $\alpha_n \in \mathbb{C}, |\alpha_n| \geq 1$. It follows that $B_{n+1} \subseteq \alpha_n T^{-r_n}(B_n) \cap B_n$. Now, for all $n \geq 1$

$$B_{n+1} \subseteq B_n \text{ and } (1/\alpha_n)T^{r_n}(B_{n+1}) \subseteq B_n \tag{2.1}$$

Moreover, $\bigcap_n B_n \neq \phi$ and by cantor intersection theorem there exists $y \in \mathcal{X}$ such that

$$\bigcap_n B_n \subseteq \bigcap_n cl(B_n) = \{y\} \tag{2.2}$$

From Equation (2.2), we get $y \in B_n$ for all n and hence $\|x_n - y\| \leq \epsilon_n$. Also, be Equation (2.1), $(1/\alpha_n)T^{r_n}y \in B_n$; i.e. $\|x_n - (1/\alpha_n)T^{r_n}y\| \leq \epsilon_n$. It follows that $(1/\alpha_n)T^{r_n}y \to y$ and hence y is a $\mathbb{D}$-recurrent vector for T such that $y \in \mathbb{D}R(T) \cap B_n$ which shows that $\mathbb{D}R(T)$ is dense.

□

Proposition 2.2 *If $x \in \mathbb{D}R(T)$, then $T^k x \in \mathbb{D}R(T)$ for all $k \in \mathbb{N}$.*

Proof: Let $x \in \mathbb{D}R(T)$, then there exist an increasing sequence of positive integers n_k and a sequence $\lambda_k \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k T^{n_k} x \to x$. By continuity of T, we get $\lambda_k T^{n_k}(T^k x) \to T^k x$ and thus $T^k x \in \mathbb{D}R(T)$. □

Proposition 2.3 *The set of all recurrent vectors for an operator T is a G_δ set.*

Proof: From the proof of the second part of Theorem (2.1), since $y \in \mathbb{D}R(T)$ if and only if $y \in \alpha_n T^{-r_n} B_n$; i.e $y \in \bigcap_n \left(\bigcup_{\substack{r_n \in \mathbb{N} \\ 0 \neq \alpha_n \in \mathbb{D}}} T^{-r_n}(\alpha_n B_n) \right)$. It follows that

$$\mathbb{D}R(T) = \bigcap_n \left(\bigcup_{\substack{r_n \in \mathbb{N} \\ 0 \neq \alpha_n \in \mathbb{D}}} T^{-r_n}(\alpha_n B_n) \right)$$

which implies that $\mathbb{D}R(T)$ is a G_δ set. □

It is shown in [6] that every operator T and its iterates $T^n (n \geq 1)$ shares the same recurrent-vectors. An analogous result is presented below.

Proposition 2.4 *Let $T \in \mathbb{D}R(\mathcal{X})$, then $\mathbb{D}R(T) = \mathbb{D}R(T^i)$ for all $i \in \mathbb{N}$.*

Proof: First we will show that $\mathbb{D}R(T^i) \subseteq \mathbb{D}R(T)$. Let $x \in \mathbb{D}R(T^i)$ then there exist an increasing sequence $(n_k) \in \mathbb{N}$ and a sequence $(\lambda_k) \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k (T^i)^{n_k} x \to x$ which means that $\lambda_k T^{in_k} x \to x$ and so $x \in \mathbb{D}R(T)$.
Now to show $\mathbb{D}R(T) \subseteq \mathbb{D}R(T^i)$, suppose that $x \in \mathbb{D}R(T)$, then there exist an increasing sequence $(n_k) \in \mathbb{N}$ and a sequence $(\lambda_k) \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k T^{n_k} x \to x$. By division algorithm theorem for $\mathbb{Z}$, there exist an increasing sequence $(q_k) \in \mathbb{Z}$ and $(r_k) \in \mathbb{Z}$ such that $0 \leq r_k < i$ and $n_k = iq_k + r_k$ for all $k \in \mathbb{N}$ and so $\lambda_k T^{iq_k + r_k} x \to x$. By boundedness of (r_k), there is $r \in (r_k)$ such that $\lambda_k T^{iq_k + r} x \to x$. Now, let G be an open set containing x then there exist a positive integer $j_1 = q_{k_1}$, $\alpha_1 = \lambda_{k_1}$ for some $k_1 \in \mathbb{N}$ such that $\alpha_1 T^{ij_1 + r} x \in G$. Also we have

$$\lambda_k T^{i(q_k + j_1) + 2r} x = \lambda_k T^{ij_1 + r} \left(\alpha_1 T^{iq_k + r} x \right) \to \left(\alpha_1 T^{ij_1 + r} x \right) \in G$$

then there exist a positive integer $j_2 = q_{k_2} + j_1 > j_1$, $\alpha_2 = \lambda_{k_2}$ for some $k_2 \in \mathbb{N}$ such that $\alpha_2 T^{ij_2 + 2r} x \in G$. By continuing this process, we get a positive integer $j_i = q_{k_i} + j_{i-1}$, $\alpha_i = \lambda_{k_i}$; $k_i \in \mathbb{N}$ such that $\alpha_i T^{ij_i + ir} x = \alpha_i (T^i)^{j_i + r} x \in G$ for an increasing sequence of positive integers $(j_i + r)$ and a sequence $(\alpha_i) \in \mathbb{D}\backslash\{0\}$, that is $x \in \mathbb{D}R(T^i)$. □

The following proposition shows the connection between the direct sum of finite operators and recurrence.

Proposition 2.5 *Let $T_i \in \mathcal{B}(\mathcal{X}_i)$ for each $1 \leq i \leq n$, and let $\bigoplus_{i=1}^n T_i$ be $\mathbb{D}$-recurrent in $\bigoplus_{i=1}^n \mathcal{X}_i$ then T_i is $\mathbb{D}$-recurrent in $\mathcal{X}_i$ for all $1 \leq i \leq n$.*

Proof: Let G_i be an open set in $\mathcal{X}_i$ for each $1 \leq i \leq n$. , then $\bigoplus_{i=1}^n G_i$ is open in $\bigoplus_{i=1}^n \mathcal{X}_i$. Since $\bigoplus_{i=1}^n T_i$ be $\mathbb{D}$-recurrent in $\bigoplus_{i=1}^n \mathcal{X}_i$ then there exist an increasing sequence of positive integers (n_k) and $\lambda_k \in \mathbb{D}\backslash\{0\}$ such that $\lambda_k \bigoplus_{i=1}^n T_i^{n_k} \bigoplus_{i=1}^n G_i \cap \bigoplus_{i=1}^n G_i \neq \phi$ which follows that $\lambda_k T_i^{n_k} G_i \cap G_i \neq \phi$ for each $1 \leq i \leq n$ and thus $T_i \in \mathbb{D}R(\mathcal{X}_i)$. □

The next theorem will be useful in the sequel, it gives some equivalent assertions for $\mathbb{D}$-recurrence.

Theorem 2.2 *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent:*

1. *$T \in \mathbb{D}R(\mathcal{X})$*
2. *For each $x \in \mathcal{X}$, there are sequences $(x_k) \in \mathcal{X}$, $(n_k) \in \mathbb{N}$ and $\lambda_k \in \mathbb{D}\backslash\{0\}$ such that $x_k \to x$ and $\lambda_k T^{n_k} x_k \to x$,*
3. *For each $x \in \mathcal{X}$ and each neighborhood $\mathcal{N}$ for zero in $\mathcal{X}$, there are $z \in \mathcal{X}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{D}\backslash\{0\}$ such that $x - z \in \mathcal{N}$ and $\lambda T^n z - x \in \mathcal{N}$.*

Proof: 1⇒ 2: Let $x \in \mathcal{X}$ and $A_1 = B_1(x)$ be an open ball with center x and radius 1. By (1), there are $n_1 \in \mathbb{N}$, $\lambda_1 \in \mathbb{D}\backslash\{0\}$ such that $\lambda_1 T^{n_1} A_1 \cap A_1 \neq \phi$. Thus there is $x_1 \in A_1$ such that $\lambda_1 T^{n_1} x_1 \in A_1$. Now take $A_2 = B_{\frac{1}{2}}(x)$ be an open ball with center x and radius $\frac{1}{2}$. Then, there are $n_2 \in \mathbb{N}$, $\lambda_2 \in \mathbb{D}\backslash\{0\}$ such that $\lambda_2 T^{n_2} A_2 \cap A_2 \neq \phi$; therefore, there is $x_2 \in A_2$ such that $\lambda_2 T^{n_2} x_2 \in A_1$. By continuing this process we get sequences $(n_k) \in \mathbb{N}$, $\lambda_k \in \mathbb{D}\backslash\{0\}$ and $(x_k) \in \mathcal{X}$ such that $x_k \in A_k$ and $\lambda_k T^{n_k} x_k \in A_k$ for all $k \geq 1$. It follows that $\|x_k - x\| \leq \frac{1}{k}$ and $\|\lambda_k T^{n_k} x_k - x\| \leq \frac{1}{k}$ for all $k \geq 1$. Now, the proof follows by letting $k \to \infty$.
2⇒ 3: Let $x \in \mathcal{X}$ and let $\mathcal{N}$ be a neighborhood for zero in $\mathcal{X}$. By(2), there are sequences $(x_k) \in \mathcal{X}$, $(n_k) \in \mathbb{N}$ and $(\lambda_k) \in \mathbb{D}\backslash\{0\}$ such that $x_k \to x$ and $\lambda_k T^{n_k} x_k \to x$. Thus there is $k \in \mathbb{N}$ such that $x_k - x \in \mathcal{N}$ and $\lambda_k T^{n_k} x_k - x \in \mathcal{N}$. The result follows by letting $z = x_k$.
3 ⇒ 1: Let G be an open set and $x \in G$. Let $\epsilon > 0$ and $\mathcal{N} = B_\epsilon(0)$ be a neighborhood for zero in $\mathcal{X}$, then by (3) there are $z \in \mathcal{X}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{D}\backslash\{0\}$ such that $x - z \in \mathcal{N}$ and $\lambda T^n z - x \in \mathcal{N}$. It follows that $z \in G$ and $\lambda T^n z \in G$ which shows that T is $\mathbb{D}$-recurrent. □

It is proved in [6] that an operator is recurrent if and only if its inverse is. In contrast to the recurrence, the following example shows that the inverse of a $\mathbb{D}$-recurrent operator may fail to be $\mathbb{D}$-recurrent.

Example 2.2 *Let $T \in \mathcal{B}(\mathbb{C})$ such that $T(x) = kx$ for some $k \in \mathbb{C}; |k| > 1$. Then T is $\mathbb{D}$-recurrent but its inverse is not.*

Proof: For each $y \in \mathbb{C}; y \neq 0$, we can find a sequence of the form $(y + \frac{1}{n}) \to y$ and a sequence $\frac{1}{k^n} \in \mathbb{D}$ such that $\frac{1}{k^n}T^n(y + \frac{1}{n}) = \frac{1}{k^n}k^n(y + \frac{1}{n}) \to y$ and so by theorem (2.2) part (2), T is $\mathbb{D}$-recurrent.
However, For each $y \in \mathbb{C}; y \neq 0$ and each sequence $(x_n) \to y$, there exist no sequence $\lambda_n \in \mathbb{D}\backslash\{0\}$ such that $\lambda_n (T^{-1})^n x_n = \lambda_n \frac{1}{k^n} x_n$ which converges to y and so by theorem (2.2) part (2), T^{-1} is not $\mathbb{D}$-recurrent. □

Observe that the previous example is in finite dimension. It would be more interesting if we can construct a $\mathbb{D}$-recurrent operator in infinite dimensions whose inverse is not $\mathbb{D}$-recurrent. For this reason, the next theorem gives a nice criterion for an operator to be $\mathbb{D}$-recurrent which will be used to find such an example. First, we need the following lemma.

Lemma 2.1 *Let $T \in \mathcal{B}(\mathcal{X})$. Let (n_k) be an increasing of positive integers, $(\lambda_k) \in \mathbb{D}\backslash\{0\}$ and A be a dense subset of $\mathcal{X}$ such that for every $x \in A$,*

1. *there exists a sequence $(x_k) \in \mathcal{X}$ such that $\left\|{\lambda_k}^{-1} x_k\right\| \to 0$ and $T^{n_k}x_k \to x$ as $k \to \infty$,*
2. *$\|\lambda_k T^{n_k} x\| \to 0$ as $k \to \infty$.*

Then T is $\mathbb{D}$-recurrent.

Proof: Let G be an open subset of $\mathcal{X}$, then there exists $x \in A \cap G$. By hypothesis, there exist a small positive number ϵ and a large positive integer r, such that

$$\|\lambda_r T^{n_r} x\| \leq \frac{\epsilon}{2} \quad \left\|{\lambda_r}^{-1} x_r\right\| \leq \epsilon \text{ and } \|T^{n_r}x_r - x\| < \frac{\epsilon}{2}$$

Let $y = x + \lambda_r^{-1} x_r$, then $\|y - x\| = \left\|\lambda_r^{-1} x_r\right\| \leq \epsilon$ which follows that $y \in G$. Now since $\lambda_r T^{n_r} y = \lambda_r T^{n_r} x + T^{n_r} x_r$, then $\|\lambda_r T^{n_r} y - x\| = \|\lambda_r T^{n_r} x + T^{n_r} x_r - x\| \leq \|\lambda_r T^{n_r} x\| + \|T^{n_r} x_r - x\| < \epsilon$. Therefore $\lambda_r T^{n_r} y \in G$; i.e, $\lambda_r T^{n_r} G \cap G \neq \phi$ and hence T is $\mathbb{D}$-recurrent. □

Theorem 2.3 ($\mathbb{D}$-recurrent Criterion) *Let $T \in \mathcal{B}(\mathcal{X})$. Suppose that (n_k) is an increasing sequence of positive integers and $A \in \mathcal{X}$ is a dense subset of $\mathcal{X}$ such that for each $x \in D$*

1. *there is a sequence (x_k) in $\mathcal{X}$ such that $\|x_k\| \to 0$ and $T^{n_k}x_k \to x$ as $k \to \infty$,*
2. *$\|T^{n_k}x\| \|x_k\| \to 0$ as $k \to \infty$,*

Then T is $\mathbb{D}$-recurrent.

Proof: Let $\{\epsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0, and let $\{y_n\}_{n\in\mathbb{N}} \subset D$ be a countable dense subset in $\mathcal{X}$. By hypothesis, there exists a sequence $(x_k^{(j)})$ in $\mathcal{X}$ such that

$$\left\|x_k^{(j)}\right\| < \epsilon_k \tag{2.3}$$

$$T^{n_k} x_k^{(j)} \to y_j \tag{2.4}$$

and

$$\|T^{n_k} y_i\| \left\|x_k^{(j)}\right\| < \epsilon_k^2 \tag{2.5}$$

where $j = 1, \cdots, k, i = 1, \cdots, k$. Suppose that for each $k \geq 1$,

$$\lambda_{n_k} = \frac{1}{\epsilon_k} \max_{1\leq j\leq k} \left\{\left\|x_k^{(j)}\right\|\right\}$$

It follows that $\lambda_{n_k} \in \mathbb{D}\backslash\{0\}$, and by equation (2.3) we have

$$\frac{1}{\lambda_{n_k}} \left\| x_k^{(j)} \right\| \leq \frac{1}{\lambda_{n_k}} \max_{1\leq j\leq k} \left\{ \left\| x_k^{(j)} \right\| \right\} = \epsilon_k \text{ for all } j \leq k \tag{2.6}$$

By equation (2.5), we get

$$\lambda_{n_k} \left\| T^{n_k} y_i \right\| = \frac{1}{\epsilon_k} \max_{1\leq j\leq k} \left\{ \left\| x_k^{(j)} \right\| \right\} \left\| T^{n_k} y_i \right\| < \epsilon_k \text{ for all } i \leq k. \tag{2.7}$$

As $k \to \infty$, now since A is dense then by equations (2.4), (2.6) and (2.7) all hypothesizes of proposition 2.1 hold and so T is $\mathbb{D}$-recurrent. □

The following example shows that even in infinite dimensional Banach spaces there exists a $\mathbb{D}$-recurrent operator whose inverse is not. Also, it shows that a $\mathbb{D}$-recurrent operator may not be recurrent.

Example 2.3 *Let $F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral forward weighted shift with the weight sequence*

$$w_n = \begin{cases} c_1 & \text{if } n \geq 0, \\ c_1 & \text{if } n < 0. \end{cases}$$

where $c_1, c_2 \in \mathbb{R}^+; 1 < c_1 < c_2$. Then F is $\mathbb{D}$-recurrent but not recurrent and F^{-1} is not $\mathbb{D}$-recurrent.

Proof: We will verify $\mathbb{D}$-recurrent criterion. Let

$$A = \left\{ x \in \ell^2(\mathbb{Z}) : x \text{ has only finitely many non–zero coordinates} \right\},$$

and (n) be the sequence of all non-negative integers. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$, and let $x \in A$, we have to find a sequence $x_n \in \mathcal{X}$ such that $\|x_n\| \to 0$. Since (w_n) is bounded, then the bilateral backward weighted shift $S = F^{-1}$ where $Se_n = (1/w_{n-1})\, e_{n-1}$ and weight sequence

$$\frac{1}{w_{n-1}} = \begin{cases} \frac{1}{c_1} & \text{if } n > 0, \\ \frac{1}{c_2} & \text{if } n \leq 0. \end{cases}$$

Now, take $x_n = S^n x$, without loss of generality we will suppose that $x = e_0$ then by [8, Lemma 3.1.], if $S^n e_0 \to 0$ as $n \to \infty$ then $S^n e_k \to 0$ for all $k \in \mathbb{Z}$ and so by triangle inequality, $S^n x \to 0$.
Since $\lim_{n\to\infty} \|S^n e_0\| = \lim_{n\to\infty} \prod_{i=1}^{n} (1/c_2) = \lim_{n\to\infty} (1/c_2^n) = 0$ then

$$\|x_n\| = \|S^n x\| \to 0 \tag{2.8}$$

Again by [8, Lemma 3.3.], if $\|F^n e_0\| \|S^n e_0\| \to 0$ as $n \to \infty$ then $\|F^n e_k\| \|S^n e_k\| \to 0$ for all $k \in \mathbb{Z}$ and so by triangle inequality, $\|F^n x\| \|S^n x\| = \|F^n x\| \|x_n\| \to 0$. Since

$$\lim_{n\to\infty} \|F^n e_0\| \|S^n e_0\| = \lim_{n\to\infty} \left(\prod_{k=1}^{n} c_1 \right) \left(\prod_{k=1}^{n} \frac{1}{c_2} \right) = \lim_{n\to\infty} (c_1^n)(\frac{1}{c_2^n}) = 0,$$

Then

$$\lim_{n\to\infty} \|F^n x\| \|x_n\| = 0, \tag{2.9}$$

Then by equations (2.8) and (2.9), and the fact that $F^n x_n = F^n S^n x = x$, F satisfies $\mathbb{D}$-recurrent criterion and so it is $\mathbb{D}$-recurrent.
On the other hand, let $x \in A$ then for any sequences $(x_k) \in \mathcal{X}$, $(n_k) \in \mathbb{N}$ and $\lambda_k \in \mathbb{D}\backslash\{0\}$ such that $x_k \to x$, we have $\lambda_k S^{n_k} x_k \to 0$; therefore by Theorem 2.2, S is not $\mathbb{D}$-recurrent. It follows that S is not recurrent too and so by [6, Proposition 2.6.] F is not recurrent.

□

We now give an example to show that the inverse of some $\mathbb{D}$-recurrent operators may be recurrent too.

Example 2.4 *Let $F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral forward weighted shift with the weight sequence*

$$w_n = \begin{cases} \frac{1}{3} & \text{if } n \geq 0, \\ 4 & \text{if } n < 0. \end{cases}$$

Then both F and F^{-1} are $\mathbb{D}$-recurrent.

Proof: We will verify $\mathbb{D}$-recurrent criterion. By using the same dense set and same process of Example 2.3, we have $x_n = S^n x$ where $S = F^{-1}$ is the bilateral backward weighted shift and weight sequence

$$\frac{1}{w_{n-1}} = \begin{cases} 3 & \text{if } n > 0, \\ \frac{1}{4} & \text{if } n \leq 0. \end{cases}$$

Then $\lim_{n\to\infty} \|S^n e_0\| = \lim_{n\to\infty} \prod_{i=1}^{n} (1/4) = \lim_{n\to\infty} \frac{1}{4^n} = 0$ then

$$\|x_n\| = \|S^n x\| \to 0 \tag{2.10}$$

Again since

$$\lim_{n\to\infty} \|F^n e_0\| \, \|S^n e_0\| = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \frac{1}{3} \right) \left(\prod_{k=1}^{n} \frac{1}{4} \right) = \lim_{n\to\infty} (\frac{1}{3^n})(\frac{1}{4^n}) = 0,$$

Then

$$\lim_{n\to\infty} \|F^n x\| \, \|x_n\| = 0, \tag{2.11}$$

Then by equations (2.10) and (2.11), and the fact that $F^n x_n = F^n S^n x = x$, F satisfies $\mathbb{D}$-recurrent criterion and so it is $\mathbb{D}$-recurrent.
To show S is $\mathbb{D}$-recurrent, let $x \in A$ and $x_n = F^n x$. Since $\lim_{n\to\infty} \|F^n e_0\| = \lim_{n\to\infty} \prod_{i=1}^{n} (1/3) = \lim_{n\to\infty} \frac{1}{3^n} = 0$ then

$$\|x_n\| = \|F^n x\| \to 0 \tag{2.12}$$

Again since

$$\lim_{n\to\infty} \|S^n e_0\| \, \|F^n e_0\| = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \frac{1}{4} \right) \left(\prod_{k=1}^{n} \frac{1}{3} \right) = \lim_{n\to\infty} (\frac{1}{4^n})(\frac{1}{3^n}) = 0,$$

Then

$$\lim_{n\to\infty} \|F^n x\| \, \|x_n\| = 0, \tag{2.13}$$

Then by equations (2.12) and (2.13), and the fact that $S^n x_n = S^n F^n x = x$, S satisfies $\mathbb{D}$-recurrent criterion and so it is $\mathbb{D}$-recurrent. □

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called power bounded if there exists a real number $C \geq 1$ such that $\|T^n\| \leq C$ for all $n \in \mathbb{N}$. To show the relation between power boundedness and recurrence, we have the following results.

Theorem 2.4 *Let $T \in \mathcal{B}(\mathcal{X})$ be a power bounded operator then the set of all $\mathbb{D}$-recurrent vectors for T is closed.*

Proof: Let $\|T^n\| \leq C$ for all $n \in \mathbb{N}$, and let $(x_n) \in \mathbb{D}R(T)$, such that $x_n \to x \in \mathcal{X}$. Then for each $n \in \mathbb{N}$, there exist a sequence $\lambda_r^{(n)} \in \mathbb{D}\backslash\{0\}$ and an increasing sequence $(k_r^{(n)})$ of positive integers such that $\lambda_r^{(n)} T^{k_r^{(n)}} x_n \to x_n$ as $r \to \infty$; i.e, $\lim_{r\to\infty} \left\| \lambda_r^{(n)} T^{k_r^{(n)}} x_n - x_n \right\| = 0$. Now, we have

$$\begin{aligned}\left\|\lambda_r^{(n)}T^{k_r^{(n)}}x-x\right\| &= \left\|\lambda_r^{(n)}T^{k_r^{(n)}}x-\lambda_r^{(n)}T^{k_r^{(n)}}x_n+\lambda_r^{(n)}T^{k_r^{(n)}}x_n-x_n+x_n-x\right\| \\ &\le \left\|\lambda_r^{(n)}T^{k_r^{(n)}}x-\lambda_r^{(n)}T^{k_r^{(n)}}x_n\right\|+\left\|\lambda_r^{(n)}T^{k_r^{(n)}}x_n-x_n\right\|+\|x_n-x\| \\ &\le \left\|\lambda_r^{(n)}\right\|\left\|T^{k_r^{(n)}}\right\|\|x-x_n\|+\left\|\lambda_r^{(n)}T^{k_r^{(n)}}x_n-x_n\right\|+\|x_n-x\| \\ &\le C\|x-x_n\|+\left\|\lambda_r^{(n)}T^{k_r^{(n)}}x_n-x_n\right\|+\|x_n-x\|\end{aligned}$$

as $n,r\to 0$ since $\left\|\lambda_r^{(n)}\right\|\le 1$. It follows that $x\in\mathbb{D}R(T)$ and so $\mathbb{D}R(T)$ is closed. □

Corollary 2.1 *If T is $\mathbb{D}$-recurrent and power bounded then $\mathbb{D}R(T)=\mathcal{X}$.*

Proof: By theorem 2.1 and proposition 2.4, $\overline{\mathbb{D}R(T)}=\mathbb{D}R(T)=\mathcal{X}$. □

Proposition 2.6 *If $\|T\|\le 1$ and T is $\mathbb{D}$-recurrent then T is a surjective isometry.*

Proof: Let $x\in\mathcal{X}$. By corollary, 2.1 $x\in\mathbb{D}R(T)$ so there exist a sequence $\lambda_k\in\mathbb{D}\backslash\{0\}$ and an increasing sequence of positive integers (n_k) such that $\lambda_k T^{n_k}x\to x$. Now since $\|T\|\le 1$ and $\lambda_k\in\mathbb{D}\backslash\{0\}$ then for every $k\in\mathbb{N}$

$$\begin{aligned}\left\|\lambda_k T^k x\right\| &\|\lambda_k\|\,\|T\|\left\|T^{k-1}x\right\| \\ &\le \left\|T^{k-1}x\right\|\le\cdots\le\|Tx\|\end{aligned}$$

Now since $\|\lambda_k T^{n_k}x\|\to\|x\|$ then $\left\|\lambda_k T^k x\right\|\to\|x\|$. Since $\left\|\lambda_k T^k x\right\|$ is bounded above by $\|Tx\|$ then $\|Tx\|=\|x\|$ which implies that T is isometry. Now, to prove T is surjective, we have to prove that $T^{-1}x$ exists. Since $\lambda_k T^{n_k}x\to x$ then it is enough to prove that $T^{-1}\lambda_k T^{n_k}x$ converges. We have

$$\begin{aligned}\left\|\lambda_k T^{n_k-1}x-\lambda_r T^{n_r-1}x\right\| &= \left\|T^{n_k-1}\left(\lambda_k I-\lambda_r T^{n_r-n_k}\right)x\right\| \\ &= \left\|T^{n_k-1}x\right\|\left\|\lambda_k I-\lambda_r T^{n_r-n_k}\right\| \\ &= \|x\|\left\|\lambda_k I-\lambda_r T^{n_r-n_k}\right\| \text{ since for all } n\in\mathbb{N}, \|T^n x\|=\|x\| \\ &= \|T^{n_k}x\|\left\|\lambda_k I-\lambda_r T^{n_r-n_k}\right\| \\ &= \|\lambda_k T^{n_k}x-\lambda_k T^{n_r}x\|\to 0 \text{ since } \lambda_k T^{n_k}x \text{ is convergent so it is cauchy}\end{aligned}$$

It follows that $\lambda_k T^{n_k-1}x$ is also a cauchy sequence and so converges. □

Proposition 2.7 *If T is power bounded and $\mathbb{D}$-recurrent then T^{-1} is also power bounded and $\mathbb{D}$-recurrent.*

Proof: Let $\|\cdot\|_1$ be an equivalent norm to $\|\cdot\|$ defined by $\|x\|_1=\sup_{n\ge 1}\|T^n x\|$. Then, $\|T\|_1\le 1$ and T is $\mathbb{D}$-recurrent in $(\mathcal{X},\|\cdot\|_1)$. By proposition 2.6, T is a surjective isometry (and so invertible) in $(\mathcal{X},\|\cdot\|_1)$. Thus, T is invertible in $(\mathcal{X},\|\cdot\|)$ too. Now, there exists a positive real number c such that

$$\begin{aligned}\left\|T^{-n}x\right\| &\le C\left\|T^{-n}x\right\|_1 \\ &\le C\sup_{n\ge 1}\left\|T^n T^{-n}x\right\|\end{aligned}$$

It follows that $\|T^{-n}\|\le C$ and so T^{-1} is power bounded. By [6, proposition 3.2], T^{-1} is $\mathbb{D}$-recurrent. □

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