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New Parametrization of Bertrand Partner D-curves in $\mathbb{E}^3$ *

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ABSTRACT: We define and study a new parametrization of a Bertrand partner D-curve $\{\alpha, \alpha^*\}$ in Euclidean 3-space by not taking Darboux frame element g^* of Bertrand partner D-curve α^* parallel to $\overrightarrow{\alpha\,\alpha^*}$. We obtain a necessary and sufficient condition for a curve to be such type of Bertrand D-curves. Also, we obtain a characterization of a new parametrization of asymptotic Bertrand D-curves and provide an example.

Key Words: Bertrand D-curves, Darboux frame, geodesic curvature, normal curvature, geodesic torsion.

Contents

1. Introduction

In the classical differential geometry of curves, J. Bertrand [4] studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve at corresponding points. Further, these curves were characterized in $\mathbb{E}^3$ with condition $ak + b\tau = 1$, where a, b are real constants and k and τ are the curvature and torsion of the curve, respectively [8]. The Bertrand pair curve was studied by many authors in Riemannian and pseudo-Riemannian metric spaces, please see [1,2,10] and references therein.

The Frenet formulas were discovered by Frenet in 1847 and independently by Serret in 1851, and by this time the theory of surfaces in $\mathbb{E}^3$ was already widely studied by many geometers. The Frenet approach of moving frames to study curves in Euclidean spaces led Darboux to introduce the Darboux frame to study surfaces. This frame is a natural moving frame to characterize curves on surfaces. Darboux frame is useful to understand the kinematics of spin-rolling motion in a nonholonomic system which expresses moving objects in terms of geometric invariants of contact curve which includes curvatures and geodesic torsion [7]. The Darboux frame model is also helpful to reduce the complexity of the problem under spin-rolling motion. Darboux frame can be used as a bridge corresponding to the Frenet frame, since functionals of surface energy are constructed by functionals of energy constructed for curves [8].

In [13], Senturk et al. characterized ruled surface with Darboux frame in $\mathbb{E}^3$. In [9], Duldul et al. extended the Darboux frame in Euclidean 4-space $\mathbb{E}^4$. In 2013, Bektas et al. [3] studied special involute-evolute partner D-curves which lie on surfaces completely in $\mathbb{E}^3$. In 2016, Kazaz et al. [11] defined the Bertrand curve lying on the surfaces and called these new associated curves as Bertrand D-curve by using the Darboux frame of the curves.

Recently, in [6], Camci et al. introduced a new relationship between Bertrand pair curves α and α^* in $\mathbb{E}^3$ by not taking the vector $\overrightarrow{\alpha^*\,\alpha}$ parallel to a normal vector of α. In view of this, we define and study a new parametrization of Bertrand partner D-curve by not taking the vector $\overrightarrow{\alpha\,\alpha^*}$ parallel to g^* of α^* in Euclidean 3-space. The paper is organized as follows: In section 2, we quote some basic notations on curves that are relevant to the rest of the paper. In section 3, we study a new parametrization of Bertrand partner D-curves in $\mathbb{E}^3$. In section 4, we provide examples of such types of curves.

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2. Preliminary

Let $\alpha = \alpha(s)$ be a regular curve with arc-length 's', fully lying on oriented surface S in $\mathbb{E}^3$. There exists a Frenet frame (T, N, B) at each point of the curve $\alpha(s)$ where T is the unit tangent vector, N is the principal normal vector and B is a binormal vector. The Frenet formula of the curve $\alpha(s)$ is given by [5,8]

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where k, τ are curvature and torsion of α, respectively. Also, there exists a Darboux frame $\{T, g, n\}$ of the curve $\alpha(s)$ on the surface S, where n is the unit normal of the surface S and a unit vector $g = n \times T$. Moreover, we have the following relation between Frenet frame and Darboux frame [5,8]:

$$\begin{pmatrix} T \\ g \\ n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.2}$$

where ϕ is the angle between g and N. Also, the derivative formula of the Darboux frame is given

$$\begin{pmatrix} T' \\ g' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ g \\ n \end{pmatrix}, \tag{2.3}$$

where k_g, k_n and τ_g are called the geodesic curvature, normal curvature, and geodesic torsion, respectively.

A regular curve α(s) lying on a surfaces S is called

- a geodesic curve if and only if $k_g = 0$.
- an asymptotic line if and only if $k_n = 0$.
- a principal line if and only if $\tau_g = 0$.

Definition 2.1. *For a given point, a set of curves such that any straight line through the point intersects all the curves in the set at the same angle, then these set of curves are called homothetic curves [12]. Otherwise, it is called non-homothetic curves.*

In [11], the authors defined Bertrand partner D-pair curve (α, α^*) completely lying on S and S^* in $\mathbb{E}^3$ where $\alpha(s)$ is given by

$$\alpha(s) = \alpha^*(s^*) + \lambda^*(s^*)g^*(s^*), \tag{2.4}$$

where λ^* is a non-zero constant and g^* is Darboux frame element of $\alpha^*(s^*)$.

Now, we define a new parametrization of mate curve α and α^* such that the vector $\overrightarrow{\alpha^*\alpha}$ does not have to be parallel to g^* which is given by

$$\alpha(s) = \alpha^*(s^*) + u^*(s^*)\,T^*(s^*) + v^*(s^*)\,g^*(s^*) + w^*(s^*)\,n^*(s^*) \tag{2.5}$$

where $u^*(s^*), v^*(s^*)$ and $w^*(s^*)$ are differentiable functions and $\{T^*(s^*), g^*(s^*), n^*(s^*)\}$ is the Darboux frame of $\alpha^*(s^*)$. If we take $u^* = w^* = 0$ in (2.5), we obtain (2.4). Hence, (2.5) is the generalization of Bertrand D-pair curves in $\mathbb{E}^3$.

3. A new parametrization of Bertrand D-pair curves in $\mathbb{E}^3$

In this section, we study a Bertrand D-pair curve satisfying (2.5).

We have:

Definition 3.1. *Let $\alpha(s) : I \to S \subset E^3$ and $\alpha^*(s^*) : I^* \to S^* \subset E^3$ be a pair curve in $\mathbb{E}^3$ with arc length parameter s and s^* and Darboux frames $\{T, g, n\}$ and $\{T^*, g^*, n^*\}$ completely lying on S and S^*, satisfying (2.5). If the Darboux frame element g of α coincides with the Darboux frame element of g^* of α^*, i.e, $g = \epsilon_1 g^*$, where $\epsilon_1 = \pm 1$ at the corresponding points of the curves, then α is called a Bertrand D-curve, α^* is called a Bertrand partner D-curve of α and the pair $\{\alpha, \alpha^*\}$ is said to be a Bertrand D-pair curve.*

The relation between the T and n of $\alpha(s)$ with the Darboux frame elements T^* and n^* of $\alpha^*(s^*)$ is given as [11]

$$T = cos\theta\, T^* - sin\theta\, n^*,\ n = sin\theta\, T^* + cos\theta\, n^*, \tag{3.1}$$

where θ is the angle between the tangent vector T and T^* at the corresponding points of α and α^*.

Now onwards, we denote k_g, k_g^*, k_n, k_n^*, τ_g, τ_g^* the geodesic curvature, normal curvature and geodesic torsion of α and α^* respectively. Also, we denote by

$$\delta = \frac{(1 - v^* k_g^* - w^* k_n^* + u^{*\prime})}{\frac{ds}{ds^*}},\quad \gamma = \frac{(u^* k_n^* + v^* \tau_g^* + w^{*\prime})}{\frac{ds}{ds^*}},\ h = \frac{\delta}{\gamma}. \tag{3.2}$$

Next, we have

Theorem 3.2. *Let $\alpha(s)$ and $\alpha^*(s^*)$ be a Bertrand D-pair curve satisfying (2.5) with arc length parameter s and s^*, completely lying on S and S^* in $\mathbb{E}^3$, respectively. Then, either*

(i) we have

$$u^* k_g^* - w^* \tau_g^* + v^{*\prime} = 0, \quad w^{*\prime} + u^* k_n^* + v^* \tau_g^* = 0, \tag{3.3}$$

and there exist a homothety map between curves α and α^* such that

$$T = \epsilon_1 T^*,\ g = \epsilon_1 g^*,\ n = \epsilon_1 n^* \tag{3.4}$$

or

(ii) we have

$$\begin{gathered} u^* k_g^* - w^* \tau_g^* + v^{*\prime} = 0, \quad w^{*\prime} + u^* k_n^* + v^* \tau_g^* \neq 0, \\ \gamma^2 = \frac{1}{1+h^2}, \quad \frac{h k_g^* - \tau_g^*}{k_g} \neq 0, \quad \frac{k_g^* + h\tau_g^*}{\tau_g} \neq 0. \end{gathered} \tag{3.5}$$

$$T = \delta T^* + \gamma n^*,\ g = \lambda g^*,\ n\,\frac{ds}{ds^*} = P(s)\,T^* + Q(s)\,n^*, \tag{3.6}$$

where prime denotes the derivative with respect to s^* and

$$\begin{cases} \lambda = \frac{(\delta k_g^* - \gamma \tau_g^*)}{k_g \frac{ds}{ds^*}} = \pm 1, \\ P(s) = \frac{-\left(h k_g^* - \tau_g^*\right)\left(u^* k_n^* + v^* \tau_g^* + w^{*\prime}\right)}{\left(\frac{ds}{ds^*}\right)^2 k_g \left(h^2+1\right)} \frac{\left(k_g^* + h\tau_g^*\right)}{\tau_g}, \\ Q(s) = \frac{\left(h k_g^* - \tau_g^*\right)\left(u^* k_n^* + v^* \tau_g^* + w^{*\prime}\right) h}{\left(\frac{ds}{ds^*}\right)^2 k_g \left(h^2+1\right)} \frac{\left(k_g^* + h\tau_g^*\right)}{\tau_g}. \end{cases} \tag{3.7}$$

Proof: Suppose α and α^* be a Bertrand partner D-curve satisfying (2.5). Then, differentiating (2.5) with respect to s^* and using (2.3), we have

$$\begin{aligned} T\frac{ds}{ds^*} = &(1 - v^* k_g^* - w^* k_n^* + u^{*\prime})T^* + (u^* k_g^* - w^* \tau_g^* + v^{*\prime})g^* + \\ &(u^* k_n^* + v^* \tau_g^* + w^{*\prime})n^*. \end{aligned} \tag{3.8}$$

By taking the scalar product of (3.8) with g, we find

$$u^* k_g^* - w^* \tau_g^* + v^{*\prime} = 0. \tag{3.9}$$

Substituting (3.9) in (3.8), we find

$$T\frac{ds}{ds^*} = (1 - v^* k_g^* - w^* k_n^* + u^{*\prime})T^* + (u^* k_n^* + v^* \tau_g^* + w^{*\prime})n^*. \tag{3.10}$$

Therefore, we have

$$T = \delta T^* + \gamma n^*. \tag{3.11}$$

Taking the inner product of (3.11) with itself and using the third relation of (3.2), we obtain

$$\gamma^2 = \frac{1}{h^2 + 1}. \tag{3.12}$$

Now, we consider the following two cases depending upon $\gamma = 0$ and $\gamma \neq 0$:

Case I. Assume that $\gamma = 0$. Then, we get

$$u^* k_n^* + v^* \tau_g^* + w^{*\prime} = 0. \tag{3.13}$$

Putting (3.9) and (3.13) in (3.8), we get

$$T\frac{ds}{ds^*} = (1 - v^* k_g^* - w^* k_n^* + u^{*\prime})T^*. \tag{3.14}$$

By taking the scalar product of (3.14) with itself, we obtain

$$\Big(\frac{ds}{ds^*}\Big) = \epsilon_1 (1 - v^* k_g^* - w^* k_n^* + u^{*\prime}), \tag{3.15}$$

where $\epsilon_1 = \pm 1$. Using (3.15) in (3.14), we have

$$T = \epsilon_1 T^*. \tag{3.16}$$

Taking inner product of first relation of (3.1) with (3.16), we obtain

$$\epsilon_1 = cos\theta \tag{3.17}$$

Using (3.17) and taking inner product of second relation of (3.1) with (3.16), we get

$$n = \epsilon_1 n^*. \tag{3.18}$$

Case II. Let $\gamma \neq 0$. Then, we have

$$(u^* k_n^* + v^* \tau_g^* + w^{*\prime}) \neq 0 \tag{3.19}$$

$$1 - v^* k_g^* - w^* k_n^* + u^{*\prime} = h(u^* k_n^* + v^* \tau_g^* + w^{*\prime}). \tag{3.20}$$

Differentiating (3.11) with respect to s^* and using (2.3) and (3.1), we obtain

$$\begin{aligned} k_g g\frac{ds}{ds^*} = \big(-k_n sin\theta\frac{ds}{ds^*} + \delta' - \gamma k_n^*\big)T^* + \big(\delta k_g^* - \gamma\tau_g^*\big)g^* + \\ \big(-k_n cos\theta\frac{ds}{ds^*} + \delta k_n^* + \gamma'\big)n^*, \end{aligned} \tag{3.21}$$

whereby comparing coefficients of T^* and n^*, we get

$$\begin{cases} -k_n sin\theta \frac{ds}{ds^*} + \delta' - \gamma k_n^* = 0, \\ -k_n cos\theta \frac{ds}{ds^*} + \delta k_n^* + \gamma' = 0. \end{cases} \tag{3.22}$$

Using (3.22) in (3.21), we find

$$k_g g \frac{ds}{ds^*} = (\delta k_g^* - \gamma \tau_g^*) g^*. \tag{3.23}$$

By taking scalar product of (3.23) with itself, we have

$$\left(\frac{ds}{ds^*}\right)^2 (k_g)^2 = \frac{(hk_g^* - \tau_g^*)^2}{h^2 + 1}, \tag{3.24}$$

this gives

$$\frac{hk_g^* - \tau_g^*}{k_g} \neq 0 \tag{3.25}$$

Therefore, we have

$$g = \lambda g^*. \tag{3.26}$$

Differentiating (3.26) with respect to s^* and using (2.3), we find

$$\tau_g \frac{ds}{ds^*} n = \lambda(-k_g^* T^* + \tau_g^* n^*) + k_g \frac{ds}{ds^*} T + \lambda' g^*. \tag{3.27}$$

Taking scalar product of (3.27) with g and using second relation of (3.1), we get

$$\lambda' = 0. \tag{3.28}$$

Using (3.28) in (3.27), we get

$$\tau_g \frac{ds}{ds^*} n = \lambda\,(-k_g^* T^* + \tau_g^* n^*) + k_g \frac{ds}{ds^*}\, T. \tag{3.29}$$

Rewriting (3.27) by using (3.10), we get

$$n \frac{ds}{ds^*} = P(s)\, T^* + Q(s)\, n^*, \tag{3.30}$$

Using (3.19), (3.25) and (3.30), we get $\frac{k_g^* + h\tau_g^*}{\tau_g} \neq 0$.

Thus proof is complete. □

Now, we have

Corollary 3.3. *Let the pair $\{\alpha, \alpha^*\}$ be a Bertrand D-pair curve satisfying (2.5).*
(i) If α and α^* are homothetic, then

$$\frac{\tau_g}{k_g} = \pm \frac{\tau_g^*}{k_g^*} \tag{3.31}$$

(ii) If α and α^* are not homothetic, then

$$\tau_g (h\, k_g^* - \tau_g^*) = \pm\, k_g (h\, \tau_g^* + k_g^*). \tag{3.32}$$

Proof: **(i)** Differentiating first, second and third relation of (3.4) with respect to s^*, and then taking inner product with itself, we get

$$\left({k_g}^2 + k_n^2\right)\left(\frac{ds}{ds^*}\right)^2 = k_g^{*2} + k_n^{*2}, \tag{3.33}$$

$$\left({k_g}^2 + \tau_g^2\right)\left(\frac{ds}{ds^*}\right)^2 = k_g^{*2} + \tau_g^{*2}, \tag{3.34}$$

$$\left({k_n}^2 + \tau_g^2\right)\left(\frac{ds}{ds^*}\right)^2 = k_n^{*2} + \tau_g^{*2}. \tag{3.35}$$

Using (3.33) in (3.35), we obtain

$$\left({\tau_g}^2 - k_g^2\right)\left(\frac{ds}{ds^*}\right)^2 = \tau_g^{*2} - k_g^{*2}. \tag{3.36}$$

Dividing (3.36) by (3.34), we obtain

$${\tau_g}^2 k_g^{*2} - {k_g}^2 \tau_g^{*2} = 0, \tag{3.37}$$

which gives (3.31).

(ii) Differentiating the second relation of (3.6) with respect to s^*, and then taking inner product with itself, we have

$$\left({k_g}^2 + \tau_g^2\right)\left(\frac{ds}{ds^*}\right)^2 = k_g^{*2} + \tau_g^{*2}. \tag{3.38}$$

Using (3.24) in (3.38), we obtain

$$(k_g^2 + \tau_g^2)\left(\frac{\delta\, k_g^* - \gamma\, \tau_g^*}{k_g}\right)^2 = k_g^{*2} + \tau_g^{*2}. \tag{3.39}$$

Using (3.12) in (3.39), we obtain

$${\tau_g}^2\left(h^2 k_g^{*2} + \tau_g^{*2} - 2hk_g^*\tau_g^*\right) = {k_g}^2\left(h^2\tau_g^{*2} + k_g^{*2} + 2hk_g^*\tau_g^*\right), \tag{3.40}$$

which gives (3.32). Whereby, proof is complete. □

If we take $u^* = w^* = 0$ in (2.5), we get the old parametrization of Bertrand partner D-curve given by [11]

$$\alpha(s) = \alpha^*(s^*) + v^*(s^*)g^*(s^*). \tag{3.41}$$

Now, we have

Corollary 3.4. *Let the pair $\{\alpha, \alpha^*\}$ be a Bertrand D-pair curve satisfying (3.41), then $v^* = 0$ or $v^* = \frac{1}{k_g^* + h\tau_g^*}$.*

Proof: Putting $u^* = w^* = 0$ in (3.3), we get

$$v^* = 0. \tag{3.42}$$

Putting $u^* = w^* = 0$ in third relation of (3.2), we find

$$v^* = \frac{1}{k_g^* + h\tau_g^*}. \tag{3.43}$$

Thus proof is complete. □

Theorem 3.5. *Let $\alpha(s)$ and $\alpha^*(s^*)$ be a pair curve with arc length parameter s and s^* completely lying on S and S^* in $\mathbb{E}^3$, respectively. If pair curve $\{\alpha,\alpha^*\}$ satisfying*

$$u^* k_g^* - w^* \tau_g^* + v^{*\prime} = 0, \quad -k_n\Big(\frac{ds}{ds^*}\Big) = \frac{\big(h' - k_n^*(h^2+1)\big)}{(h^2+1)}, \tag{3.44}$$

and (2.5). Then $\{\alpha,\alpha^*\}$ is a Bertrand D-pair curve.

Proof: Let the assumptions of Theorem 3.3 hold.

Differentiating (2.5) with respect to s^* and using (2.3), (3.2) and first equation of (3.44), we get

$$T\frac{ds}{ds^*} = A\Big(hT^* + n^*\Big), \tag{3.45}$$

where $A = u^* k_n^* + v^* \tau_g^* + w^{*\prime}$.

Taking inner product of (3.45) with itself, we obtain

$$\frac{ds}{ds^*} = \epsilon_1\, A\sqrt{h^2+1}, \tag{3.46}$$

where $\epsilon_1 = \pm 1$.

Differentiating (3.45) with respect to s^* and using (2.3), we get

$$\Big(k_g\, g + k_n\, n\Big)\Big(\frac{ds}{ds^*}\Big)^2 + T\frac{d^2 s}{ds^{*2}} = a_1\, T^* + a_2\, g^* + a_3\, n^*,$$

where,

$$a_1 = hA' + A\Big(h' - k_n^*\Big), \quad a_2 \ = A\Big(hk_g^* - \tau_g^*\Big), \quad a_3 = A' + h\,A\,k_n^*.$$

Taking cross product of (3.45) with (3.47), we find

$$\Big(k_g\, n - k_n\, g\Big)\Big(\frac{ds}{ds^*}\Big)^3 = A^2\Big\{\big(hk_g^* - \tau_g^*\big)\big(h\, n^* - T^*\big) + \big(h' - k_n^*(1+h^2)\big)g^*\Big\}. \tag{3.47}$$

Using (3.46) and the second condition of (3.44) in (3.47), we have

$$\Big(k_g\, n - k_n\, g\Big)\Big(\frac{ds}{ds^*}\Big)^3 = A^2\big(hk_g^* - \tau_g^*\big)\big(h\, n^* - T^*\big) - k_n\Big(\frac{ds}{ds^*}\Big)^3 g^*. \tag{3.48}$$

Taking cross product of (3.45) with (3.48), we get

$$-\Big(k_g\, g + k_n\, n\Big)\Big(\frac{ds}{ds^*}\Big)^4 = -A\Big\{A^2\big(hk_g^* - \tau_g^*\big)\,\big(h^2+1\big)\, g^* + k_n\Big(\frac{ds}{ds^*}\Big)^3\big(hn^* - T^*\big)\Big\}. \tag{3.49}$$

Solving (3.48) and (3.49) for n, we find

$$\Big(k_g^2 + k_n^2\Big)\Big(\frac{ds}{ds^*}\Big)^4 n = a_4\,\big(h\, n^* - T^*\big) + a_5\, g^*, \tag{3.50}$$

where

$$\begin{aligned} a_4 &= A\frac{ds}{ds^*}\Big(Ak_g(hk_g^* - \tau_g^*) + k_n^2\Big(\frac{ds}{ds^*}\Big)^2\Big), \\ a_5 &= \Big(A^3 k_n\big(hk_g^* - \tau_g^*\big)\big(h^2+1\big) - k_n\, k_g\,\Big(\frac{ds}{ds^*}\Big)^4\Big). \end{aligned}$$

From (3.48), we get

$$\left(\frac{ds}{ds^*}\right)^3 = \epsilon_1 \, \frac{A^2\left(hk_g^* - \tau_g^*\right)\sqrt{h^2+1}}{k_g}. \tag{3.51}$$

Using (3.46) and (3.51) in (3.50), we obtain

$$a_6 \; n = a_7 \left(h \, n^* - T^*\right), \tag{3.52}$$

where

$$a_6 = \epsilon_1 \left(k_g^2 + k_n^2\right) \frac{\left(hk_g^* - \tau_g^*\right)\sqrt{h^2+1}}{k_g}, \quad a_7 = k_g(hk_g^* - \tau_g^*) + Ak_n^2(h^2+1).$$

Putting the value of $\left(h \, n^* - T^*\right)$ from (3.52) in (3.48), we get

$$k_n{}^2 \left[k_g \left(\frac{ds}{ds^*}\right)^2 - A(hk_g^* - \tau_g^*)\right] n - b_1 \, g = -b_1 \, g^*, \tag{3.53}$$

where $b_1 = k_n \left[Ak_g(hk_g^* - \tau_g^*) + k_n{}^2 \left(\frac{ds}{ds^*}\right)^2\right]$.

Using (3.46) and (3.51) in (3.53), we obtain

$$\left(\frac{ds}{ds^*}\right)^2 = A \frac{(hk_g^* - \tau_g^*)}{k_g}. \tag{3.54}$$

Putting (3.54) in (3.53), we obtain $g = g^*$. So, the Darboux frame element g of α coincides with g^* of α^* at the corresponding points of the curves, which complete the proof of Theorem. □

In [6], the new relation between Bertrand pair curve is defined as

$$\alpha^*(s^*) = \alpha(s) + u(s)T(s) + v(s)N(s) + w(s)B(s), \tag{3.55}$$

where $u(s)$, $v(s)$ and $w(s)$ are differentiable functions.

Now, using (2.1), (2.3), (3.44) of Theorem 3.3 and (3.55), we have

Corollary 3.6. *Let $\alpha(s)$ and $\alpha^*(s^*)$ be two asymptotic Bertrand D-partner curves satisfying (2.5). Then, we have*

$$v^{*\prime} + u^* k_g^* - w^* \tau_g^* = 0, \;\; h' = 0, \tag{3.56}$$

and the Frenet frames $(T, \, N, \, B)$ and $(T^*, \, N^*, \, B^*)$ of the curve $\alpha(s)$ and $\alpha^*(s^*)$ coincides with its Darboux frames $(T, \, g, \, n)$ and $(T^*, \, g^*, \, n^*)$, respectively, and $k_g^* = \pm \, k^*$, $\tau_g^* = \pm \, \tau^*$, and $k_g = \pm \, k$, $\tau_g = \pm \, \tau$. Then new parametrization of Bertrand partner D-curve becomes the Bertrand partner curve in Euclidean 3-space, satisfying (3.55).

Proof: Let the curve α and α^* be asymtotic, then $k_n = k_n^* = 0$. Using this in (2.1) and (2.3), we find

$$kN = k_g \, g, \quad k^* N^* = k_g^* \, g^*. \tag{3.57}$$

From (3.57), we obtain

$$k = \pm \, k_g, \quad k^* = \pm \, k_g^*, \tag{3.58}$$

$$N = \pm \, g, \quad N^* = \pm, g^*. \tag{3.59}$$

Taking cross product of first and second relation of (3.59) with T and T^*, respectively, we get

$$n = \pm \, B, \quad n^* = \pm \, B^*. \tag{3.60}$$

Now, again using the third relations of (2.1) and (2.3), we obtain

$$\tau = \pm \, \tau_g, \quad \tau^* = \pm \, \tau_g^*. \tag{3.61}$$

Also, putting $k_n = k_n^* = 0$ in (3.44), we get (3.56).

Thus, the proof is complete by combining (3.56) and (3.58)∼ (3.61). □

4. Examples

Example 4.1. *Consider the curve $\alpha^*(s^*)$ defined by*

$$\alpha^*(s^*) = \Big(r\,\cos\Big(\frac{s^*}{r}\Big),\, r\,\sin\Big(\frac{s^*}{r}\Big),\, \sqrt{R^2-r^2}\Big), \tag{4.1}$$

lies on the surface

$$S^*(s^*,v^*) = \Big(R\sin(v^*)\cos\Big(\frac{s^*}{r}\Big), R\sin(v^*)\sin\Big(\frac{s^*}{r}\Big), R\cos(v^*)\Big), \tag{4.2}$$

with Darboux frame $(T^*,\, g^*,\, n^*)$ given as

$$\begin{cases} T^* = \Big(-\sin\Big(\frac{s^*}{r}\Big),\, \cos\Big(\frac{s^*}{r}\Big), 0\Big),\\ g^* = \Big(\frac{\sqrt{R^2-r^2}}{R}\,\cos\Big(\frac{s^*}{r}\Big),\, \frac{\sqrt{R^2-r^2}}{R}\,\sin\Big(\frac{s^*}{r}\Big),\, \frac{-r}{R}\Big),\\ n^* = \Big(\frac{-r}{R}\,\cos\Big(\frac{s^*}{r}\Big),\, \frac{-r}{R}\,\sin\Big(\frac{s^*}{r}\Big),\, \frac{-\sqrt{R^2-r^2}}{R}\Big), \end{cases} \tag{4.3}$$

where R and r is a constant.

Using (4.1), (4.3) and taking $u^*=1$, $v^*=\frac{\sqrt{R^2-r^2}}{rR}s^*$ and $w^*=0$ in (2.5), we obtain

$$\alpha(s) = \big(\alpha_1,\, \alpha_2,\, \alpha_3\big), \tag{4.4}$$

where

$$\alpha_1 = \Big(r + \frac{(R^2-r^2)\,s^*}{r\,R^2}\Big)\,\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big),$$
$$\alpha_2 = \Big(r + \frac{(R^2-r^2)\,s^*}{r\,R^2}\Big)\,\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big),$$
$$\alpha_3 = \sqrt{R^2-r^2}\Big(1 - \frac{s^*}{R^2}\Big).$$

The curve $\alpha(s)$ lies on the surface

$$S(s^*,v^*) = \Big(x(s^*,v^*),\, y(s^*,v^*),\, z(s^*,v^*)\Big), \tag{4.5}$$

where

$$x(s^*,v^*) = r\,\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big) + \Big(\frac{\sqrt{R^2-r^2}s^*}{rR}\Big)\,\cos v^*\,\cos\Big(\frac{s^*}{r}\Big),$$
$$y(s^*,v^*) = r\,\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big) + \Big(\frac{\sqrt{R^2-r^2}s^*}{rR}\Big)\,\cos v^*\,\sin\Big(\frac{s^*}{r}\Big),$$
$$z(s^*,v^*) = \sqrt{R^2-r^2} - \frac{s^*}{R}\cos v^*,$$

The Darboux frame $(T,\, g,\, n)$ of $\alpha(s)$ is given by

$$T = \frac{1}{\sqrt{U^2+1}}\Big(-U\,\sin\Big(\frac{s^*}{r}\Big) - \frac{r}{R}\,\cos\Big(\frac{s^*}{r}\Big),\, U\,\cos\Big(\frac{s^*}{r}\Big) - \frac{r}{R}\,\sin\Big(\frac{s^*}{r}\Big),\, -\frac{\sqrt{R^2-r^2}}{R}\Big),$$
$$g = \Big(-\frac{\sqrt{R^2-r^2}}{R}\,\cos\Big(\frac{s^*}{r}\Big),\, -\frac{\sqrt{R^2-r^2}}{R}\,\sin\Big(\frac{s^*}{r}\Big),\, \frac{r}{R}\Big),$$
$$n = \frac{1}{\sqrt{U^2+1}}\Big(\frac{rU}{R}\,\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big),\, \frac{rU}{R}\,\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big),\, \frac{U\sqrt{R^2-r^2}}{R}\Big),$$

where $U = \Big(R + \frac{(R^2-r^2)\,s^*}{r^2\,R}\Big)$.

Then, the curve pair $(\alpha,\, \alpha^*)$ is a Bertrand D-pair curve with $g=-g^*$.

Example 4.2. *Similar to Example 4.1, for* $u^* = 1$, $v^* = \frac{\sqrt{R^2-r^2}}{rR}s^*$, $w^* = \frac{1}{R}$, *the curve pair* $\{\alpha, \alpha^*\}$ *can be shown a Bertrand D-pair with* $g = -g^*$, *lying on* S *and* S^*, *where*

$$\begin{aligned}
\alpha(s) &= \big(\alpha_1,\, \alpha_2,\, \alpha_3\big),\\
\alpha_1 &= \left(r\Big(1-\frac{1}{R^2}\Big) + \frac{(R^2-r^2)\,s^*}{r\,R^2}\right)\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big),\\
\alpha_2 &= \left(r\Big(1-\frac{1}{R^2}\Big) + \frac{(R^2-r^2)\,s^*}{r\,R^2}\right)\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big),\\
\alpha_3 &= \sqrt{R^2-r^2}\Big(1-\frac{1}{R^2}(1+s^*)\Big),
\end{aligned}$$

$$S(s^*, v^*) = \Big(x(s^*, v^*),\, y(s^*, v^*),\, z(s^*, v^*)\Big), \tag{4.6}$$

$$\begin{aligned}
x(s^*, v^*) &= \Big(1-\frac{r}{R^2}\Big)r\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big) + \Big(\frac{\sqrt{R^2-r^2}s^*}{rR}\Big)\cos v^*\cos\Big(\frac{s^*}{r}\Big),\\
y(s^*, v^*) &= \Big(1-\frac{r}{R^2}\Big)r\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big) + \Big(\frac{\sqrt{R^2-r^2}s^*}{rR}\Big)\cos v^*\sin\Big(\frac{s^*}{r}\Big),\\
z(s^*, v^*) &= \sqrt{R^2-r^2}\Big(1-\frac{1}{R^2}\Big) - \frac{s^*}{R}\cos v^*.
\end{aligned}$$

The Darboux frame $(T,\, g,\, n)$ *of* $\alpha(s)$ *can be obtained as*

$$\begin{aligned}
T &= \frac{1}{\sqrt{A^2+1}}\Big(A\sin\Big(\frac{s^*}{r}\Big) - \frac{r}{R}\cos\Big(\frac{s^*}{r}\Big),\, -A\cos\Big(\frac{s^*}{r}\Big) - \frac{r}{R}\sin\Big(\frac{s^*}{r}\Big),\, -\frac{\sqrt{R^2-r^2}}{R}\Big),\\
g &= \Big(-\frac{\sqrt{R^2-r^2}}{R}\cos\Big(\frac{s^*}{r}\Big),\, -\frac{\sqrt{R^2-r^2}}{R}\sin\Big(\frac{s^*}{r}\Big),\, \frac{r}{R}\Big),\\
n &= \frac{1}{\sqrt{A^2+1}}\Big(-\frac{rA}{R}\cos\Big(\frac{s^*}{r}\Big) - \sin\Big(\frac{s^*}{r}\Big),\, -\frac{rA}{R}\sin\Big(\frac{s^*}{r}\Big) + \cos\Big(\frac{s^*}{r}\Big),\, -\frac{A\sqrt{R^2-r^2}}{R}\Big),
\end{aligned}$$

where $A = \Big(-R\Big(1+\frac{s^*}{r^2}\Big) + \frac{1}{R}(s^*+1)\Big)$.

Remark 4.3. *The surface* S *in above examples can be obtained from*

$$S(s^*, v^*) = \alpha^*(s^*) + u^*(s^*, v^*)\,T^*(s^*) + v^*(s^*, v^*)\,g^*(s^*) + w^*(s^*, v^*)\,n^*(s^*), \tag{4.7}$$

In Example 4.1, by putting $u^*(s^*, v^*) = 1$, $v^*(s^*, v^*) = \frac{s^*\cos v^*}{r}$, $w^*(s^*, v^*) = 0$ *in (4.7), we get (4.5).*

In Example 4.2, by putting $u^*(s^*, v^*) = 1$, $v^*(s^*, v^*) = \frac{s^*\cos v^*}{r}$, $w^*(s^*, v^*) = \frac{1}{R}$ *in (4.7), we get (4.6).*

Figure 1: The black graphic is α^* and the red graphic is α of Example 4.1

Figure 2: Curve α^* on S^* and the curve α on S of Example 4.1

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