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A note on 4-self-centered graphs

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ABSTRACT: A graph G is said to be 4-self-centered if the eccentricity of each of its vertices is 4. In this paper, we discuss some properties associated to 4-self-centered graphs. We obtain its degree and girth and also the maximum number of triangles in such graphs. We also establish the existence of 8-cycles and 9-cycles in these graphs.

Key Words: Eccentricity, diameter and radius of graphs, cycle, girth and degree of graphs.

Contents

1. Introduction

Let G be a connected simple graph. The *eccentricity* of G, denoted by $e(G)$, is simply the maximum distance of one vertex from another. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum eccentricity of the vertices of a graph, while its *radius*, denoted by $Rad(G)$, is the minimum eccentricity of the vertices of the graph. A graph G having equal diameter and radius is called a *self-centered* graph. If $diam(G) = Rad(G) = k$, then G is said to be a *k-self-centered* graph. Properties associated to edge-minimal 2-self centered graphs were first studied by Shekarriz and Mirzavaziri [7], while Stanic [9] dealt with minimal self-centered graphs. One can find literature on self-centered graphs in [1-9].
The *centre* of a graph G is the collection of all those vertices of G whose eccentricity is minimum. Here the eccentricity of G is equal to the radius of G. Two vertices u and v of a graph G are said to be *adjacent* if uv is an edge in G. The *open neighborhood* a vertex u, denoted by $N(u)$, is the set of all those vertices which are adjacent to u. The neighborhood of u that also contains u is called the *closed neighborhood* of u and is denoted by $N[u]$.
The *girth* of a graph G, denoted by $gr(G)$ is the length of the smallest cycle in G. If the graph G does not contain any cycle, then $gr(G) = \infty$. The *degree* of a vertex u of a graph G is the number edges of G that are incident to u. Symbolically, it is denoted by $deg(u)$. A non-empty subset S of the set of all the vertices V of a graph is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* γ of a graph G is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called a $\gamma-$set of G.
Our emphasis through the entire length of this paper has been on connected simple graphs.

2. Main Results

This section shall introduce us to some of the basic properties of 4-self-centred graphs.

Theorem 2.1 *For any 4-self-centered connected graph G of order n and $u \in V(G)$, $2 \leq deg(u) \leq n-6$.*

Proof:
If possible, let $deg(u) = 1$ and let $v \in N(u)$. Then for each $w \in V(G)$ such that $d(u, w) = 4$, we have $d(v, w) = 3$. Consequently $Rad(G) \leq 3$, a contradiction. Therefore $deg(u) \geq 2$, $\forall\ u \in V(G)$. For the second part, for some $z \in V(G)$, if $deg(z) = n-1$, then $Rad(G) = 1$, a contradiction. Let $deg(z) = n-2$ and let $u \notin N(z)$. Then for some $x \in V(G)$, there exists a 2-path $z - x - u$ in G. Consequently $Rad(G) \leq 2$, a contradiction. Let $deg(z) = n-3$ and let $u, v \notin N(z)$. If $d(z, u) = d(z, v) = 2$, then

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$Rad(G) \leq 2$, a contradiction. If $d(z,u) = 2$ and $d(z,v) = 3$, then $d(z,v) = 3$. Consequently $Rad(G) \leq 3$, a contradiction. Let $deg(z) = n-4$ and let $u, v, w \notin N(z)$. If $d(u,w) = 2$, then for some $x \in V(G)$, there exists a 4-path $z - x - u - v - w$ in G. But since $deg(w) \geq 2$, $\exists$ some $t \in V(G)$ which is adjacent to w. If $t = u$, then $d(u,w) = 1$, a contradiction. If t is adjacent to any other vertex s, then $d(w,s) < 4$, a contradiction. Let $deg(G) = n-5$ and let $a_1, a_2, a_3, a_4 \notin N(z)$. If $d(a_1, a_4) = 3$, then $d(z, a_1) > 4$, a contradiction. If $d(a_1, a_4) = 1$, then $d(z, a_1) = 3$, contradiction. Let $d(a_1, a_4) = 2$. Then $d(z, a_1) = 4$. But since $deg(a_4) \geq 2$, a_4 must be adjacent to another vertex except z. But for each $w (\neq z) \in V(G)$, $d(a_4, w) < 4$, a contradiction. This establishes the result. □

Theorem 2.2 *Any connected 4-self-centered graph contains either an 8-cycle or a 9-cycle.*

Proof: For any 4-self-centered graph G and $u \in V(G)$, let $v \in N(u)$. Since G is 4-self-centered, $\exists$ some $x \in V(G)$ such that $d(u,x) = 4$. From theorem 2.1, since $deg(u) \geq 2$, let $w \in N(u)$ such that $w \neq v$. Since G is 4-self-centered, $\exists$ some $y \in V(G)$ such that $d(u,y) = 4$. If $y = x$, then we get an 8-cycle. Let $y \neq x$. Since $deg(y) \geq 2$, y must be adjacent to another vertex of G. This vertex is x. This gives us a cycle of length 9. This completes the proof. □

Theorem 2.3 *For any connected 4-self-centered graph G,* $3 \leq gr(G) \leq 9$.

Proof: Since the graph G is connected and $deg(u) \geq 2 \ \forall \ u \in G$, G contains a cycle. Clearly $gr(G) \geq 3$ (Since a triangle is the smallest cycle). We now show that $gr(G) \leq 8$. If possible, let $gr(G) > 9$. Then for each vertex u in an $m-$cycle of G, where $m > 9$, $\exists$ another vertex v such that $d(u,v) > 4$, a contradiction. Hence $3 \leq gr(G) \leq 9$. □

Theorem 2.4 *The only 4-self-centered graph of order 8 is the cycle* C_8.

Proof: The fact that C_8 is a 4-self-centered graph is obvious. To establish its uniqueness, let $\{v_1, v_2, ..., v_8\}$ be the vertex set of the 4-self-centered graph G. Then $\exists$ a 4-path $v_1 - v_2, v_3 - v_4, v_5$ in G. Since G is 4-self-centered, there exists another vertex, say v_6 such that v_5 is adjacent to v_6. If v_6 is adjacent to v_1, then then $d(v_1, v_5) < 4$, a contradiction. G being 4-self-centered, there exists another vertex, say v_7 such that v_6 is adjacent to v_7. If v_7 is adjacent to v_1, then again $d(v_1, v_5) < 4$, a contradiction. So there exists another vertex, say v_8 such that v_7 is adjacent to v_8. Since $deg(v_1) \geq 2$, so v_1 must be adjacent to some vertex besides v_2. This vertex is v_8. Consequently the vertex set forms the cycle C_8. □

Theorem 2.5 *The only 4-self-centered cycles are* C_8 *and* C_9.

Proof: The fact that C_8 and C_9 are 4-self-centered cycles can be easily seen from Figure 1 and Figure 2. The uniqueness of C_8 as a 4-self-centered graph can be seen from theorem 2.4. Now let G be a 4-self-centered cycle of order greater than 9 and let $\{v_1, v_2, ..., v_k\}$ be the vertex set of G, where $k > 9$. Then $d(v_1, v_{\lfloor \frac{k}{2} \rfloor}) = 4$, but $d(v_{\lfloor \frac{k}{2} \rfloor + 1}, v_1) > 4$, $\forall \ k > 9$, a contradiction. This establishes the result. □

Theorem 2.6 *For any connected 4-self-centered graph of order n such that* $n \geq 8$*, the maximum number of triangles in G is* $\frac{(n-3)(n-7)(n-8)}{6}$.

Proof: We first partition the vertex set of G into two subsets:

$A = \{v_1, v_2, ..., v_7\}$ and $B = \{v_8, v_9, ..., v_n\}$.

Let the vertices of A form the path $v_1 - v_2 - v_3 - ... - v_7$ and that of B form the complete graph K_{n-7}. Also let v_1 and v_7 be adjacent to all the vertices of B. Then $d(v_i, v_{i+1}) = 4$, $\forall \ i = 1, 2, 3, 4$ and $d(x, y) \leq 4$, for each $x \in A$ and $y \in B$. Clearly the eccentricity of each vertex of G is 4. Since v_1 is not adjacent to v_7, so the number of triangles in G are ${}^{n-5}C_3 - (n-7)$, *i.e.* $\frac{(n-3)(n-7)(n-8)}{6}$. Since the existence of any other triangle in G would contradict theorem 2.2, this is the maximum number of possible triangles in G. □

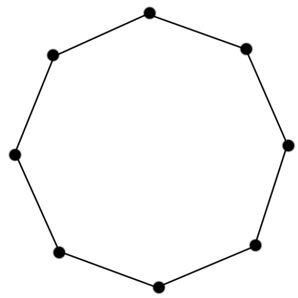

Figure 1: This is the only 4-self-centered graph on 8 vertices

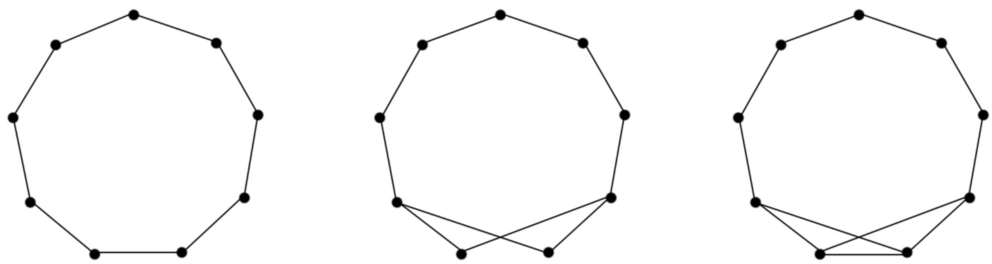

Figure 2: 4-self-centered graphs on 9 vertices

Theorem 2.7 *For any connected 4-self-centred graph G, $\gamma(G) = 3$.*

Proof: From theorem 2.6, the vertex set $\{v_1, v_4, v_7\}$ is a dominating set of G and, thus, $\gamma(G) \leq 3$. Let $\gamma(G) = 2$ and let $\{u, v\}$ be a dominating set of G. Let $\{u_1, u_2, ..., u_k\} \in N(u)$ and $\{v_1, v_2, ..., v_k \in N(v)\}$. If $u_i = v_j$ for some $i, j \in \mathbb{N}$, then $d(u, v) \leq 3$. Let $u_i \neq v_j$ for all $i, j \in \mathbb{N}$. Then $d(u, v) = 3$. Also for each $x, y \in N(u)$ and $z, w \in N(v)$, $d(x, y) = d(z, w) = 2$. In each of these cases, $Rad(G) \leq 3$, a contradiction. Again, if $\gamma(G) = 1$, then $Rad(G) = 1$, a contradiction. Therefore $\gamma(G) = 3$. □

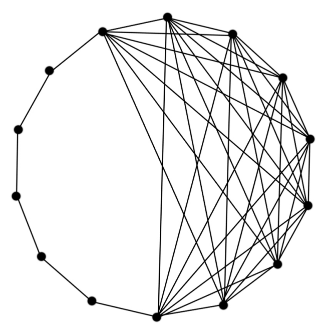

Figure 3: A 4-self-centered graph on 14 vertices containing 77 triangles

For the following results, we shall use the notation $N_1(u)$ to denote the neighbors of a vertex u of G, the notation $N_2(u)$ to denote the neighbors of $N_1(u)$, the notation $N_3(u)$ to denote the neighbors of $N_2(u)$ and so on.

Theorem 2.8 *Let G be a connected $k-$regular 4-self-centered graph with $gr(G) = 8$. Then $|V(G)| = (k-1)(2k^2 - 2k + 1) + k + 1$.*

Proof: Let $u \in V(G)$. Since G is $k-$regular so $|N_1(u)| = k$, $|N_2(u)| = k(k-1)$, $|N_3(u)| = k(k-1)^2$ and $|N_4(u)| = k(k-1)^3$. Let $w \in N_2(u)$. Since $gr(G) = 8$, therefore $|N_3(w) \cap N_4(u)| = (k-1)^3$. So the total number of vertices of G are $(k-1)^3 + k(k-1)^2 + k(k-1) + k + 1$, *i.e* $(k-1)(2k^2 - 2k + 1) + k + 1$. □

Theorem 2.9 *For any connected $k-$regular 4-self-centered graph G with $gr(G) = 9$, $|V(G)| = k(k-1)(k^2 - k + 1) + k + 1$.*

Proof: For any $u \in V(G)$, since G is $k-$regular so $|N_1(u)| = k$, $|N_2(u)| = k(k-1)$, $|N_3(u)| = k(k-1)^2$ and $|N_4(u)| = k(k-1)^3$. Let $w \in N_2(u)$. Since $gr(G) = 9$, so $|N_3(w) \cap N_4(u)| = k(k-1)^3$. So the total number of vertices of G are $k(k-1)^3 + k(k-1)^2 + k(k-1) + k + 1$, *i.e* $k(k-1)(k^2 - k + 1) + k + 1$. □

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