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# A Double Step Size Method for Linear Complementarity Problem

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ABSTRACT: This paper proposes a novel approach to solving linear complementarity problems ($LCP$) that are known to be NP-hard research problems. We present a two-step size algorithm with an accelerated property for solving $LCP(q, M)$ problems, with $M$ a positive matrix, that reformulates the problem as a system of nonlinear equations $F_p(x) = 0$, where $p$ is large. By utilizing an inexact linear search technique and approximating the Jacobian with the acceleration parameter, we develop an efficient accelerated Newton and gradient descent method. The proposed method is shown to be convergent under specific conditions and is derivative-free, making it advantageous for solving large-scale problems. Our method offers a promising approach for solving difficult and large-scale $LCP$ problems.

Key Words: LCP, Positive matrix, system of nonlinear equations, Jacobian matrix, Newton method.

## Contents



## 1. Introduction

The term "linear complementarity problem" was coined by Richard W. Cottle [1,2] as early as 1965 and has since been abbreviated as LCP in the literature. This article aims to clarify the concept of LCP and its numerous applications by presenting some of them in detail. The supplementary section of this paper focuses on solving an LCP. The aim of the linear complementarity problem is to find two vectors $z, w \in \mathbb{R}^n$, which satisfy the following conditions:

$$w = Mz + q \tag{1.1}$$

$$w_1 z_1 + w_2 z_2 + ... + w_n z_n = 0 \tag{1.2}$$

$$w \geqslant 0,\ z \geqslant 0 \tag{1.3}$$

or to show that no such vectors exist. One writes the three conditions $(1) - (3)$ also often in one line:

$$w = Mz + q, w^t z = 0, w \geqslant 0 \text{ and } z \geqslant 0. \tag{1.4}$$

This problem is widely discussed in [3,4], and is known to have numerous equivalent formulations. For instance, it is possible to define a function whose zeros correspond exactly to the solutions of the linear complementarity problem, as explained in [6,7,8,9,10,11,12]. In practice, iterative methods for zero or fixed point search are often used to find approximate solutions, which may be subject to rounding errors. To assess the quality of an approximate solution, one can perform a solution verification. In this study, we have reformulated the linear complementarity problem as a system of nonlinear equations:

$$F(x) = 0 \tag{1.5}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map.

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Systems of nonlinear equations appear in many areas of applied science and engineering [13]. Since, in general, the roots of a function cannot be computed exactly and cannot be expressed in a closed form, this is the main reason why the most used solution methods are usually iterative. Most numerical root-finding methods use iteration and produce a sequence of numbers that hopefully converges to the root as a limit. There is a vast literature on the numerical solution of nonlinear equations. One of the basic methods for solving the nonlinear equation (5) is Newton's method [3,14]. Newton's method is given by the following iterative formula

$$x_{k+1} = x_k + \lambda_k d_k, \ k = 0, 1, ... \tag{1.6}$$

where $\lambda_k$ is a step length that can be calculated using a linear search technique [15,16,17], $x_k$ represents the previous iteration, $x_{k+1}$ is the new iterative point, and $d_k$ is the search direction that can be obtained by solving the following system of linear equations

$$J(x_k)d_k = -F(x_k) \tag{1.7}$$

where $J(x_k)$ is the Jacobian matrix of $F(x_k)$ at $x_k$. An essential requirement for the linear search is to decrease the function values sufficiently, which can be expressed as

$$\|F(x_k + \lambda_k d_k)\| \leqslant \|F(x_k)\| \tag{1.8}$$

Furthermore, problem (5) can be formulated as an unconstrained optimization problem [18]. Let $f$ be a norm function defined by

$$f(x) = \frac{1}{2}\|F(x)\| \tag{1.9}$$

where $\|\ .\|$ denotes the Euclidean norm. Then, problem (5) is equivalent to the following global optimization problem

$$\min f(x), \ x \in \mathbb{R}^n \tag{1.10}$$

and condition (8) becomes equivalent to

$$f(x_k + \lambda_k d_k) \leqslant f(x_k) \tag{1.11}$$

To ensure a descent direction, the search direction $d_k$ must satisfy the condition $\bigtriangledown f(x_k)^t d_k \leqslant 0$. Newton's method has well-known shortcomings, and therefore, a double step length approach was proposed in [11,19]. The iterative procedure is given as

$$x_{k+1} = x_k + \delta_k d_k + \delta_k^2 b_k, k = 0, 1, ..., \tag{1.12}$$

where $\delta_k$ denotes the step length and $d_k$ and $b_k$ are search directions. The ideal rule for finding the step length $\delta_k$ is the exact rule [20], which satisfies

$$f(x_k + \lambda_k d_k) = \min_{\lambda>0} f(x_k + \lambda d_k). \tag{1.13}$$

However, the exact step length is difficult to find in practice, and the inexact line search [21,22] is commonly used instead. Brown and Saad [16] suggested the line search formula

$$f(x_k + \lambda_k d_k) - f(x_k) \leqslant \omega\lambda_k \bigtriangledown f(x_k)^t d_k \tag{1.14}$$

where $\omega \in (0, 1)$. Similarly, different methods can be used to determine the search direction $d_k$ (see [23,24]). The Jacobian matrix is computed at each iteration according to (13), which can increase the computational complexity, particularly in the case of large-scale problems or expensive matrices. Yuan and Lu [10] proposed a new inexact backtracking technique for determining the step length $\lambda_k$, given by

$$\|F(x_k + \lambda_k d)\|^2 \leqslant \|F(x_k)\|^2 + \omega\lambda_k^2 F(x_k)^t d_k \tag{1.15}$$

where $\omega \in (0, 1)$. The global and superlinear convergence of this method has been established, and numerical results have shown that the line search method (14) outperforms normal methods.

The double direction approach was introduced by Dbaruranovic-milicic in [26] as a multi-step iterative procedure for generating new iterations. Dbaruranovic-milicic et al. [25] proposed a similar multi-step algorithm for minimizing a non-differentiable function using the dual direction approach. Petrovic and Stanimirovic [26] used an approximation to the Hessian matrix through an acceleration parameter $\gamma_k > 0$, where the gradient $\bigtriangledown f(x_k)$ is approximated by $\gamma_k I$, with $I$ being the identity matrix. Their method's unique feature is that the two proposed directions are without derivation, with the first direction using a diagonal matrix and the second being considered the steepest descent direction, making it suitable for large-scale optimization problems.

Halilu et al. [5] and Waziri et al. [14] were motivated by the rareness of derivation-free two-direction methods for solving nonlinear equations and used the scheme presented in [27] to offer a derivation-free method through a two-direction approach. They approximated the Jacobian matrix with $\gamma_k > 0$, where $J(x_k)$ is approximated by $\gamma_k I$. Habibu et al [24] improved on their work and presented two directions as the conjugate gradient and the steepest descent, respectively, to solve the symmetric nonlinear equations. While their methods showed good convergence properties in [15,24], the presence of the steepest descent direction in their schemes defined weaker performance numerically.

This paper is motivated by [10] and aims to develop a derivation-free method with a descent direction to solve the linear complementarity problem through $J(x_k) \approx \gamma_k I$. We reformulate our linear complementarity problem as a non-differentiable function $F(x) = 0$ and approximate $F$ with another sequence of differentiable functions $F_p$ that converges to $F$ as $p$ approaches infinity. Then, we use this method on $F_p$ to find an approximate solution of the solution of $F$. The proposed method has the advantage of not requiring computation of the Jacobian matrix $J$ in its iterations and has a globally convergent CPU time under appropriate conditions.

The paper is structured as follows. In Section 2, we present an equivalent reformulation of the problem as $F(x) = 0$, where $F$ represents a non-differentiable function. We introduce a differentiable function $F_p$ in this section as well, which serves as an approximation of $F$ and converges to it as $p$ approaches large values. Furthermore, we propose an iterative algorithm without derivation to solve $F_p(x) = 0$ for these large values of $p$.

In Section 3, we establish the global convergence of the algorithm under reasonable conditions. The convergence analysis provides assurance that the algorithm will yield a convergent solution given the specified conditions.

To summarize, the paper proceeds with Section 2, where we introduce the reformulation of the problem and the iterative algorithm and section 3 focuses on the global convergence analysis of our algorithm.

## 2. Main result

The main result of this paper is to solve the linear complementarity problem $LCP(q, M)$ defined by

$$\begin{cases} w = Mz + q \geqslant 0 \\ z \geqslant 0 \\ \langle w, z\rangle = 0 \end{cases} \tag{2.1}$$

where $M \in \mathbb{R}^{n\times n}$ a positive matrix and $q \in \mathbb{R}^n$ are data. We denotes by $\langle w, z\rangle$ or $z^t w$ the inner product of vectors $w, z \in \mathbb{R}^n$. It is known that (see for example, Y. El foutayeni et al. [3,5]) the linear complementarity problem $LCP(M, q)$ is equivalent to solving a system of non-linear equations $F(x) = 0$. Indeed, in (16) we transform the variables by substituting $z = |x| - x$ and $w = |x| + x$. Recall that the absolute value of $x$ means that $|x| = (|x_1|, ..., |x_n|)^t$.

We can notice that the three following conditions $w \geqslant 0$, $z \geqslant 0$ and $\langle w, z\rangle = 0$ are verified. So, we get the equation

$$(I + M)x + (I - M)|x| - q = 0$$

where $I$ is the matrix identity. Hence, we pose $M^+ = I + M$ and $M^- = I - M$. So, we obtain

$$F(x) = M^+x + M^-|x| - q = 0 \tag{2.2}$$

**Lemma 2.1** *If $M$ is a $P-$matrix, then $x = \frac{1}{2}(w - z)$ is the unique solution of the system of nonlinear equations $F(x) = 0$.*

**Proof:** On the one hand, the proof of this lemma is based on the equivalence between $LCP$ and $F(x) = 0$ to show the existence of the solution, and on the other hand, it is based on the fact that the matrix $M$ is $P$-matrix to show the uniqueness of the solution. □

We now consider a sequence of smooth functions $(F_p)_{p\geqslant 1}$ converging to $F$, to improve the convergence speed of our algorithm.

Let $F_p : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F_p(x) = M^+x + M^-(x^2 + \frac{1}{p})^{\frac{1}{2}} - q \tag{2.3}$$

where $p$ in $\mathbb{N}^*$ and $(x^2 + \frac{1}{p})^{\frac{1}{2}} := \left((x_1^2 + \frac{1}{p})^{\frac{1}{2}}, (x_2^2 + \frac{1}{p})^{\frac{1}{2}}, ..., (x_n^2 + \frac{1}{p})^{\frac{1}{2}}\right)^t$ a vector of $\mathbb{R}^n$. Let us note that the sequence of smooth functions $F_p$ is of class $C^{+\infty}$.

Our goal is summarized in these two points

1. The sequence of smooth functions $(F_p)_{p\geqslant 1}$ converges uniformly to the function $F$.
2. The zero of the sequence of smooth functions $(F_p)_{p\geqslant 1}$ is an approximation of the zero of the function $F$.

**Theorem 2.1** *If $p$ is larger, then the sequence of smooth functions $(F_p)_{p\geqslant 1}$ converges uniformly to $F$ on $\mathbb{R}^n$.*

**Proof:** We have, $\sqrt{x^2 + \frac{1}{p}} \geqslant |x|$ for all $x \in \mathbb{R}^n$ and all $p \in \mathbb{N}^*$ (The expression $\sqrt{x^2 + \frac{1}{p}} \geqslant |x|$ meaning that $\sqrt{x_i^2 + \frac{1}{p}} \geqslant |x_i|$ component by component for each $i = 1, ..., n$.).

In addition, the inequality $\sqrt{x^2 + \frac{1}{p}} \leqslant |x| + \frac{1}{p}$, then we have $\sqrt{x^2 + \frac{1}{p}} - |x| \leqslant \frac{1}{p}$. , this implying that

$$\sup_{x\in\mathbb{R}^n} \left\| \sqrt{x^2 + \frac{1}{p}} - |x| \right\| \leqslant \frac{1}{p}$$

Since the numerical sequence $(w_p)_p$ with general term $w_p = \frac{1}{p}$ converges to 0. Therefore, the sequence of smooth functions $x \in \mathbb{R}^n$ converges uniformly to $|x|$ on $\mathbb{R}^n$. Thus, the sequence of smooth functions $F_p$ converges uniformly to $F$ when $p \to +\infty$. □

**Corollary 2.1** *Let $p$ be sufficiently large. Then if $x_p^*$ is a solution of the equation $F_p(x) = 0$, hence $x_p^*$ is an approximation to the solution of the equation $F(x) = 0$.*

**Proof:** From the previous theorem we have $\forall \varepsilon > 0$, $\exists\ p_0 > 0$ such that for all $p > p_0$ we have

$$||F(x_p^*)|| = ||F(x_p^*) - F_p(x_p^*)|| \leqslant \varepsilon$$

Then, $x_p^*$ is the approximation to the solution of the equation $F(x) = 0$. □

In this section, we propose a derivative-free iterative method to solve

$$F_p(x) = 0 \tag{2.4}$$

for sufficiently large $p$.

Newton's methods have many interesting properties, but they are not ideal for solving large-scale problems because the Jacobi matrix is computed after each iteration, which requires a large number of memory consumers and computation time. Based on the deficiencies of Newton's method, the derivative-free method was proposed in [16,28] to solve a large-scale system of nonlinear equations by a two-way approach and the iterative procedure is given as follows

$$x_p^{(k+1)} = x_p^{(k)} + \delta^{(k)}b^{(k)} + (\delta^{(k)})^2c^{(k)} \tag{2.5}$$

where $\delta^{(k)}$ presents the step length, $b^{(k)}$ and $c^{(k)}$ are the search directions respectively. For $x_p \in \mathbb{R}^n$ and $k$ a positive integer, $x_p^{(k)}$ denotes the vector obtained after $k$ iterations.

Moreover, the problem (19) can be obtained from an unconstrained optimization problem [29]. Let f be a norm function defined by

$$f(x) = \frac{1}{2} \|F_p(x)\|^2 \tag{2.6}$$

The system of nonlinear equation (19) is analogous to the following global optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \tag{2.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}$.

The idea of the derivative-free method is on the one hand to reduce the two directions of (20) into one, and on the other hand to approximate the Jacobian matrix via the acceleration parameter $\gamma^{(k)}$ i.e., $J_p \approx \gamma^{(k)} I$, where $I$ is a matrix identity and $\gamma^{(k)} > 0$. To improve the correct direction to the solution, we propose to define the directions $b^{(k)}$ and $c^{(k)}$ in (20) as follows

$$b^{(k)} = c^{(k)} = -(\gamma^{(k)})^{-1} F_p(x_p^{(k)}) \tag{2.8}$$

then we obtain

$$d^{(k)} = -(\gamma^{(k)})^{-1} \left[ M^+ x_p^{(k)} + M^- \left( \left( x_p^{(k)} \right)^2 + \frac{1}{p} \right)^{\frac{1}{2}} - q \right] \tag{2.9}$$

where $d^{(k)}$ is a new direction obtained by (23).

By replacing (23) in (20) we get

$$x_p^{(k+1)} = x_p^{(k)} + (\delta^{(k)} + (\delta^{(k)})^2) d^{(k)} \tag{2.10}$$

Now applying the first order Taylor expansion to obtain the formula for the acceleration parameter $\gamma^{(k)}$

$$F_p(x_p^{(k+1)}) = F_p(x_p^{(k)}) + J_p(\theta)(x_p^{(k+1)} - x_p^{(k)}) \tag{2.11}$$

where $\theta \in [x_p^{(k)}, x_p^{(k+1)}]$, with

$$\theta = x_p^{(k)} + \alpha(x_p^{(k+1)} - x_p^{(k)}) = x_p^{(k)} + \alpha(\delta^{(k)} + (\delta^{(k)})^2) d^{(k)}, \ \alpha \in [0, 1] \tag{2.12}$$

Keeping in mind that the distance from $x_p^{(k)}$ and $x_p^{(k+1)}$ is sufficiently small. Taking $\alpha = 1$ in (27), we get $\theta = x_p^{(k+1)}$. We are therefore interested in approximating the Jacobian with

$$J_p(\theta) \approx \gamma^{(k+1)} I \tag{2.13}$$

From (26) and (28) we verify that

$$F_p(x_p^{(k+1)}) - F_p(x_p^{(k)}) = \gamma^{(k+1)}(x_p^{(k+1)} - x_p^{(k)})$$

we pose $y_p^{(k)} = F_p(x_p^{(k+1)}) - F_p(x_p^{(k)})$ and $s_p^{(k)} = x_p^{(k+1)} - x_p^{(k)}$, we obtain

$$y_p^{(k)} = \gamma^{(k+1)} s_p^{(k)} \tag{2.14}$$

then,

$$M^+ s_p^{(k)} + M^- diag \left( \frac{x_p^{(k+1)} + x_p^{(k)}}{\sqrt{\left( x_p^{(k+1)} \right)^2 + \frac{1}{p}} + \sqrt{\left( x_p^{(k)} \right)^2 + \frac{1}{p}}} \right) s_p^{(k)} = \gamma^{(k+1)} s_p^{(k)}$$

$$M^+d^{(k)} + M^-diag\left(\frac{x_p^{(k+1)} + x_p^{(k)}}{\sqrt{\left(x_p^{(k+1)}\right)^2 + \frac{1}{p}} + \sqrt{\left(x_p^{(k)}\right)^2 + \frac{1}{p}}}\right) d^{(k)} = \gamma^{(k+1)}d^{(k)}$$

we pose $X_p^k = diag\left(\frac{x_p^{(k+1)}+x_p^{(k)}}{\sqrt{\left(x_p^{(k+1)}\right)^2+\frac{1}{p}}+\sqrt{\left(x_p^{(k)}\right)^2+\frac{1}{p}}}\right)$, (recall that the expression $\frac{a}{b}$ meaning that $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n}\right)$ component by component for each $i = 1...n$; and for each vector $v \in \mathbb{R}^n$, $diag(v)$ denotes the diagonal matrix of $v$, and it is defined as $diag(v) = \begin{pmatrix} v_1 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & v_n \end{pmatrix}$).

Then we obtain

$$M^+d^{(k)} + M^-X_p^k d^{(k)} = \gamma^{(k+1)}d^{(k)} \tag{2.15}$$

Multiplying both sides by $(d^{(k)})^t$, we obtain the formula for the acceleration parameter as follows

$$\gamma^{(k+1)} = \frac{\left(d^{(k)}\right)^t M^+d^{(k)} + \left(d^{(k)}\right)^t M^-X_p^k d^{(k)}}{\left(d^{(k)}\right)^t d^{(k)}} \tag{2.16}$$

To calculate the step length $\delta^{(k)}$, we use the derivative-free line formula employed in [29].
Let $\sigma_1 > 0$, $\sigma_2 > 0$ and $r \in (0,1)$ are constants and let $\{\beta_k\}$ is a given positive sequence where :

$$\sum_{k=0}^{\infty} \beta_k < \beta < \infty \tag{2.17}$$

Let $\delta^{(k)} = r^{i_k}$ , $i_k$ is the smallest positive integer $i$ that satisfies

$$\begin{aligned} f\left(x_p^{(k)} + (\delta^{(k)} + (\delta^{(k)})^2)d^{(k)}\right) - f(x_p^{(k)}) \quad \leqslant \quad & -\sigma_1 \left\|(\delta^{(k)} + (\delta^{(k)})^2)F_p(x_p^{(k)})\right\|^2 \\ & -\sigma_2 \left\|(\delta^{(k)} + (\delta^{(k)})^2)d^{(k)}\right\|^2 + \beta_k f(x_p^{(k)}) \end{aligned} \tag{2.18}$$

**Algorithme**

**Algorithm 2.1** *Step 0: Determine* $x_p^{(0)}, q$, $\gamma^{(0)} = 1$, $\varepsilon = 1/10^4$, *set* $k = 0$, $\beta_k = \frac{1}{(k+1)^2}$

*Step 1: Calculate* $F_p(x_p^{(k)}) = M^+x_p^{(k)} + M^-\left(\left(x_p^{(k)}\right)^2 + \frac{1}{p}\right)^{\frac{1}{2}} - q$

*Step 2: If* $\left\|F_p(x_p^{(k)})\right\| \leqslant \varepsilon$ *then stop, else go to Step 3*

*Step 3: Calculate the search direction*

$$d^{(k)} = -(\gamma^{(k)})^{-1}\left[M^+x_p^{(k)} + M^-\left(\left(x_p^{(k)}\right)^2 + \frac{1}{p}\right)^{\frac{1}{2}} - q\right]$$

*Step 4: Calculate the step length* $\delta^{(k)}$ *(using (33)).*

*Step 5: Set* $x_p^{(k+1)} = x_p^{(k)} + (\delta^{(k)} + (\delta^{(k)})^2)d^{(k)}$ *and compute* $F_p(x_p^{(k+1)})$

*Step 6: Update* $X_p^k = diag\left(\frac{x_p^{(k+1)}+x_p^{(k)}}{\sqrt{\left(x_p^{(k+1)}\right)^2+\frac{1}{p}}+\sqrt{\left(x_p^{(k)}\right)^2+\frac{1}{p}}}\right)$ *and*

$$\gamma^{(k+1)} = \frac{\left(d^{(k)}\right)^t M^+d^{(k)} + \left(d^{(k)}\right)^t M^-X_p^k d^{(k)}}{\left(d^{(k)}\right)^t d^{(k)}}.$$

*Step 7: Set* $k = k + 1$, *and go to Step 1.*

## 3. The analysis of convergence

For this section, we show the overall convergence of our method. To start, let us defined the set of levels

$$\digamma = \{x \in \mathbb{R}^n / \|F_p(x)\| \leqslant \|F_p(x_0)\|\} \tag{3.1}$$

To analyze the convergence of our algorithm, we have the following assumptions are verified

1. The sequence of smooth functions $F_p$ is of class $C^{+\infty}$ in some neighborhood $H$ of $x$ containing $\digamma$.
2. There exists $x_p^* \in \mathbb{R}^n$ from which $F_p(x_p^*) = 0$.
3. The Jacobian matrix $J_p(x)$ is bounded on $H$, i.e., there is a positive constant $m > 0$ such that

$$\| F_p'(x) \| \leqslant m, \ \forall x \in H \tag{3.2}$$

4. The Jacobian matrix $J_p(x)$ is positive definite on $H$. We have

$$\begin{aligned} J_p(x) &= (I + M) + (I - M)D_x \\ &= (I + D_x) + M(I - D_x) \end{aligned}$$

where $D_x = diag\left(\frac{x}{\sqrt{x^2 + \frac{1}{p}}}\right)$, $x \in \mathbb{R}^n$. Then the matrices $I + D_x$ and $I - D_x$ are positive definite, and as $M$ is a positive matrix, hence there exists a positive constant $m_0 > 0$ such that

$$\langle J_p(x)y, y\rangle \geqslant m_0 \|y\|^2, \forall x \in H, y \in \mathbb{R}^n \tag{3.3}$$

**Remark 3.1** *Assumptions* $1 - 4$ *imply that there are two constants* $M > m > 0$ *such that*

$$m\|h\| \leqslant \| F_p'(x)h \| \leqslant M\|h\|, \ \forall x \in H, \ h \in \mathbb{R}^n \tag{3.4}$$

$$m\|x - y\| \leqslant \| F_p(x) - F_p(y) \| \leqslant M\|x - y\|, \ \forall x, y \in H \tag{3.5}$$

Since $\gamma^{(k)}I$ approximates $J_p(x)$ along the direction $d_k$, we can consider another assumption

**5.** $\gamma^{(k)}I$ provides a good approximation to $J_p(x)$, i.e.

$$\| (J_p(x_p^{(k)}) - \gamma^{(k)}I)d_k \| \leqslant \epsilon \| F_p(x_p^{(k)}) \| . \tag{3.6}$$

with $\epsilon \in (0, 1)$ being a small quantity.

**Lemma 3.1** *Assume that assumption* 5 *holds and that* $\{x_p^{(k)}\}$ *be generated by our algorithm. So,* $d_k$ *is a descent direction for* $f(x_p^{(k)})$ *in* $x_p^{(k)}$*, i.e.,*

$$\nabla f(x_p^{(k)})d_k < 0 \tag{3.7}$$

**Proof:** We have

$$\nabla f(x_p^{(k)})d_k = \frac{1}{2}\nabla \left\|F_p(x_p^{(k)})\right\| d_k = F_p(x_p^{(k)})^t J_p(x_p^{(k)})d_k$$

therefore, from (23) and (39) we obtain

$$\begin{aligned} \nabla f(x_p^{(k)})d_k &= F_p(x_p^{(k)})^t \left[(J_p(x_p^{(k)}) - \gamma^{(k)}I)d_k - F_p(x_p^{(k)})\right] \\ &= F_p(x_p^{(k)})^t (J_p(x_p^{(k)}) - \gamma^{(k)}I)d_k - F_p(x_p^{(k)})^t F_p(x_p^{(k)}) \\ &\leqslant \left\|F_p(x_p^{(k)})^t\right\| \left\|(J_p(x_p^{(k)}) - \gamma^{(k)}I)d_k\right\| - \left\|F_p(x_p^{(k)})\right\|^2 \\ &\leqslant (\epsilon - 1)\left\|F_p(x_p^{(k)})^t\right\|^2 \end{aligned}$$

for $\epsilon \in (0,1)$. Then, the proof is complete. □

Based on Lemma 3.1, it can be inferred that the norm function $f(x^k)$ exhibits a descent property along $d_k$, indicating that the inequality $\| F(x^{k+1}) \| \leqslant \| F(x^k) \|$ holds true.

**Lemma 3.2** *Assume that assumption* 5 *holds and that* $\{x_p^{(k)}\}$ *is generated by our algorithm. Hence,* $\{x_p^{(k)}\} \subset \digamma$.

**Proof:** Using Lemma 3.1, we can deduce that $\| F_p(x_p^{(k+1)}) \| \leqslant \| F_p(x_p^{(k)}) \|$. In addition, we have for any $k$

$$\| F_p(x_p^{(k+1)}) \| \leqslant \| F_p(x_p^{(k)}) \| \leqslant \| F_p(x_p^{(k-1)}) \| \leqslant \cdots \leqslant \| F_p(x_p^{(0)}) \|$$

Which implies that $\{x_p^{(k)}\} \subset \digamma$. □

**Lemma 3.3** *Suppose that the sequence* $\{x_p^{(k)}\}$ *is generated by our algorithm. So there exists a constant* $\eta > 0$ *such that for each* $k$,

$$\left(F_p\left(x_p^{(k+1)}\right) - F_p\left(x_p^{(k)}\right)\right)^t \left(x_p^{(k+1)} - x_p^{(k)}\right) \geqslant \eta \left\|x_p^{(k+1)} - x_p^{(k)}\right\|^2 \tag{3.8}$$

**Proof:** From (26) we have

$$\left(F_p\left(x_p^{(k+1)}\right) - F_p\left(x_p^{(k)}\right)\right)^t \left(x_p^{(k+1)} - x_p^{(k)}\right) = \left(x_p^{(k+1)} - x_p^{(k)}\right)^t J_p(\theta) \left(x_p^{(k+1)} - x_p^{(k)}\right)$$

where $\theta = x_p^{(k)} + \alpha\left(x_p^{(k+1)} - x_p^{(k)}\right)$ and $\alpha \in [0,1]$.

however, the Jacobian matrix $J_p$ is positive definite, so according to assumption 4 we have

$$\left(x_p^{(k+1)} - x_p^{(k)}\right)^t J_p(\theta) \left(x_p^{(k+1)} - x_p^{(k)}\right) \geqslant m_0 \left\|x_p^{(k+1)} - x_p^{(k)}\right\|^2$$

□

**Lemma 3.4** *Suppose that the sequence* $\{x_p^{(k)}\}$ *is generated by our algorithm. So, we have*

$$\lim_{k \to +\infty} \left\|\delta^{(k)} d^{(k)}\right\| = 0 \tag{3.9}$$

*and*

$$\lim_{k \to +\infty} \left\|\delta^{(k)} F_p\left(x_p^{(k)}\right)\right\| = 0 \quad for \quad p \ large \tag{3.10}$$

**Lemma 3.5** *Suppose that the sequence* $\{x_p^{(k)}\}$ *is generated by our algorithm. Therefore, there exists an* $C \geqslant 0$ *so that for every* $k \geqslant 1$,

$$\left\|d^{(k)}\right\| \leqslant C \tag{3.11}$$

**Proof:** According to (23), (29), (38) and (41), we have

$$\begin{aligned}
\left\|d^{(k)}\right\| &= \left\|-(\gamma^{(k)})^{-1} F_p(x_p^{(k)})\right\| \\
&= \left\|-\frac{(y_p^{(k-1)})^t . s_p^{(k-1)}}{(y_p^{(k-1)})^t y_p^{(k-1)}} F_p(x_p^{(k)})\right\| \\
&\leqslant \frac{\left\|(y_p^{(k-1)})\right\| \left\|s_p^{(k-1)}\right\| \left\|F_p(x_p^{(k)})\right\|}{\left\|(y_p^{(k-1)})\right\|^2} \\
&\leqslant \frac{\left\|(y_p^{(k-1)})\right\| \left\|F_p(x_p^{(k)})\right\|}{\eta^2 \left\|(s_p^{(k-1)})\right\|} \\
&\leqslant \frac{M \|F_p(x_0)\|}{\eta^2}
\end{aligned}$$

If we take $C = \frac{M\|F_p(x_0)\|}{\eta^2}$ , we have (44). □

**Theorem 3.1** *Suppose that the sequence $\{x_p^{(k)}\}$ is generated by our algorithm and that for all $k \geqslant 1$,*

$$\delta^{(k)} \geqslant C_0 \frac{|F_p(x_p^{(k)})^t d^{(k)}|}{\left\|d^{(k)}\right\|} \tag{3.12}$$

*where $C_0$ is a non-negative constant. Therefore,*

$$\lim_{k\to+\infty} \left\|F_p(x_p^{(k)})\right\| = 0 \tag{3.13}$$

**Proof:** From lemma 3.4, lemma 3.5, and the boundedness of $\left\{\left\|d^{(k)}\right\|\right\}$, we have

$$\lim_{k\to+\infty} \delta^{(k)} \left\|d^{(k)}\right\|^2 = 0 \tag{3.14}$$

Then, according to (45), we obtain

$$\lim_{k\to+\infty} |F_p(x_p^{(k)})^t d^{(k)}| = 0$$

However, we have

$$F_p(x_p^{(k)})^t d^{(k)} = -(\gamma^{(k)})^{-1} F_p(x_p^{(k)})^t F_p(x_p^{(k)}),$$

then,

$$\begin{aligned} \left\|F_p(x_p^{(k)})\right\|^2 &= \left\|-\gamma^{(k)} F_p(x_p^{(k)})^t d^{(k)}\right\| \\ &\leqslant \left|\gamma^{(k)}\right| \left|F_p(x_p^{(k)})^t d^{(k)}\right| \end{aligned}$$

and from (31) and (38), we have

$$\left|\gamma^{(k)}\right| \leqslant \frac{M^2}{m}$$

then,

$$\left\|F_p(x_p^{(k)})\right\|^2 \leqslant \frac{M^2}{m} \left|F_p(x_p^{(k)})^t d^{(k)}\right| \to 0$$

therefore,

$$\lim_{k\to+\infty} \left\|F_p(x_p^{(k)})\right\| = 0$$

□

## 4. Conclusion

In conclusion, the proposed algorithm in this study aims to solve the linear complementarity problem using a novel and non-derivative approach. By reformulating the $LCP(M, q)$ problem as a non-differentiable function $F(x) = 0$ and approximating it with a sequence of differentiable functions $F_p$ that converge to $F$ for large values of $p$, we have achieved global convergence.

This innovative approach offers several advantages. Firstly, by eliminating the need for complex matrix calculations through the reformulation into a non-differentiable function, we have simplified the resolution process. Additionally, by gradually approximating the function $F$ using differentiable functions, we have achieved global convergence of the algorithm.

Our method proves to be computationally efficient by avoiding the computational burden of intensive matrix calculations through the use of an acceleration parameter to approximate the Jacobian matrix $J_p$.

It is important to note that the global convergence of the algorithm is conditional on specific assumptions. This means that while the algorithm guarantees a convergent solution within predefined circumstances, it may not be applicable in all situations. Therefore, it is crucial to verify these conditions before applying the algorithm.

In conclusion, the proposed algorithm offers an innovative and effective approach to solving the linear complementarity problem. By reformulating the problem as a non-differentiable function and utilizing a sequence of differentiable functions to approximate it, we simplify the resolution process. With the incorporation of an acceleration parameter, the algorithm avoids intensive matrix calculations while ensuring global convergence. However, it is essential to consider the convergence conditions when applying this algorithm.

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## References

1. R.W. Cottle, J.S. Pang, R.E. Stone,*The Linear Complementarity Problem*, Academic Press, New York, 1992.
2. K. Murty,*Linear Complementarity, Linear and Nonlinear Programming*, Heldermann-Verlag, 1988.
3. Y. El foutayeni, M. Khaladi,*Using vector divisions in solving the linear complementarity problem*, Journal of Computational and Applied Mathematics, 236 (2012) 1919-1925.
4. Y. El foutayeni, H. El bouanani, M. Khaladi,*An (m+1)-step iterative method of convergence order (m+2) for linear complementarity problems*, Journal of Applied Mathematics and Computing, 54 (2017) 229-242.
5. A. Halilu , M. Waziri, Y. Musa, Inexact Double Step Length Method for Solving Systems of Nonlinear Equations, Statistics, Optimization and Information Computing, Vol. 8, March 2020, pp 165-174.
6. A. Halilu, M. Waziri,*Solving System of Nonlinear Equations using Improved Double Direction Method*, Journal of the Vol. 39, Issue 2, pp. 287-301, 2020.
7. P.S. Stanimirović, M.B. Miladinović, *Accelerated gradient descent methods with line search*, Springer Science + Business Media, LLC 2009, DOI 10.1007/s11075-009-9350-8, Numer Algor (2010) 54:503-520.
8. J.E. Dennis, J. Moré,*A Characterization of Superlinear Convergence and Its Application to Quasi-Newton Methods*, Mathematics of Computation, Vol. 28, No. 126 (Apr., 1974), pp. 549-560.
9. G. Yuan , X. Lu,*A new backtracking inexact BFGS method for symmetric nonlinear equations*, Computers and Mathematics with Applications 55 (2008) 116-129.
10. M.J. Petrovic,*An Accelerated Double Step Size model in unconstrained optimization*, Applied Mathematics and Computation 250 (2015) 309-319.
11. E. Polak, G. Riere,*Note sur la convergence de méthodes de directions conjuguées*, Revue française d'informatique et de recherche opérationnelle. Série rouge, tome 3, n R1 (1969), p. 35-43.
12. A. Naseem, M.A. Rehman, T. Abdeljawad,*Numerical Algorithms for Finding Zeros of Nonlinear Equations and Their Dynamical Aspects*, Journal of Mathematics vol 2020.
13. J.E Dennis, R.B Schnabel,*Numerical method for unconstrained optimization and non-linear equations*, practice Hall, Englewood cliffs, NJ, USA, 1983.
14. M.Y Waziri, A.S. Halilu,*An improved derivative-free method via double direction approach for solving systems of nonlinear equations*, Journal of the Ramanujan Mathematical Society, vol. 33, no. 1, pp. 75-89, 2018.
15. P.N. Brown, Y. Saad,*Convergence theory of nonlinear Newton-Krylov algorithms search*, SIAM Journal on Optimization, vol.4, no.2, pp.297-330, 1994.
16. A.S Halilu, M.Y Waziri,*A transformed double step length method for solving large-scale systems of nonlinear equations*, Journal of Numerical Mathematics and Stochastics, vol.9, no. 1, pp. 20–23, 2017.

17. D.Li, M. Fukushima,*A global and superlinear convergent Gauss-Newton based BFGS method for symmetric nonlinear equations*, SIAM Journal on numerical Analysis, vol.37, pp. 152–172, 1999.

18. A.S Halilu, M.Y Waziri,*Enhanced matrix-free method via double step length approach for solving systems of nonlinear equations*, International Journal of Applied Mathematical Research, vol.6, no. 4, pp. 147-156, 2017.

19. Z. Shi, J. Shen,*Convergence of nonmonotone line search method*, Journal of Computational and Applied Mathematics, vol.193, pp. 397-412, 2006.

20. I.A. Osinuga, S.O. Yusuff,*Quadrature based Broyden-like method for systems of nonlinear equations*, Statistics, Optimization and Information Computing, vol.6, pp. 130-138, 2018.

21. W. Cheng, Q. Hu,*A Nonmonotone Spectral Gradient Method for l -regularized Least Squares*, Statistics, Optimization and Information Computing, vol.4, pp.265-277, 2016.

22. M.Y Waziri, J. Sabiu,*A derivative-free conjugate gradient method and its global convergence for symmetric nonlinear equation*, Journal of Mathematics and Mathematical Sciences, vol. 2015, pp. 8, 2015.

23. A.S Halilu, M.Y Waziri,*A transformed double step length method for solving large-scale systems of nonlinear equations*, Journal of Numerical Mathematics and Stochastics, vol.9, no. 1, pp. 20–23, 2017.

24. N.I. Dbaruranovic-milicic, M. Gardasevic-Filipovic,*A multistep curve search algorithm in nonlinear convex case*, Facta universitatis series: Mathematics and informatics 25/ 11-24, 2010.

25. N.I. Dbaruranovic-milicic,*A multi step curve search algorithm in nonlinear optimization*, Yugoslav Journal of operation research 18/ 47-52, 2008.

26. M.J. petrovic, P.S. Stanimirovic, *Accelerated double direction method for solving unconstrained optimization problems*, Mathematical problem Eng., 2014.

27. N.I Dbaruranovic-milicic, *A multi step curve search algorithm in nonlinear opti-mization*, Yugoslav Journal of operation research 18 47-52, 2008.

28. G. Yuan , X. Lu, *A new backtracking inexact BFGS method for symmetric nonlinear equations*, Journal of Computers and Mathematics with Applications, vol.55, pp. 116-129, 2008.

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