Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.63375

Existence and stability results for a fractional integro-differential equation with Hilfer derivatives*

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ABSTRACT: In this paper, we discuss the existence, uniqueness, and Ulam-Hyers stability of solutions for a specific type of the Hilfer fractional integro-differential equation with nonlocal Erdélyi-Kober fractional condition. First, the equivalence of this class of problem and a nonlinear Volterra integral equation is shown. Next, to guarantee the existence of a unique solution, we turn to the well-established tools of fixed-point theory, particularly Banach's and Krasnoselskii's theorems. Further, due to the Gronwall inequality, we obtain Ulam-Hyers stability of the considered problem. Finally, two examples are provided to illustrate our theory results.

Key Words: Hilfer fractional derivative, integro-differential equations, fixed point theory, Ulam stability.

Contents

1. Introduction

Fractional differential and integral equations (FDEs/FIEs) have emerged as powerful tools for modeling diverse phenomena across engineering and science, including physics, chemistry, biology, viscoelasticity, control theory, signal and image processing [8,9,15]. This has spurred extensive research on the existence and uniqueness of solutions for FDEs/FIEs, see for example [1,2,3,4,5,14,21] and references therein. More recently, Hilfer introduced a generalized form of fractional derivative, known as the Hilfer fractional derivative of order α and type $\beta \in [0,1]$ which interpolates the both Riemann–Liouville and Caputo derivatives in some sense. This allows for a more nuanced approach to modeling certain phenomena. Details and applications can be found in [10,11] and their references.

In [4], the authors proved the existence of the solutions for the fractional integro-differential equation

$$\begin{cases} {}^{C}D^{\alpha}_{a^+}y(t) &= f\left(t, ({}^{C}D^{\beta}_{a^+}y)(t), \int_a^t g(t,s,y(s))ds\right), \quad t \in (a,b], \\ y^{(k)}(a) &= y_k, \quad k = 0,1,...,m-1, \end{cases}$$

where, ${}^{C}D^{\alpha}_{a^+}$ is the Caputo fractional derivative of order α, with $m-1 < \alpha < m, n-1 < \beta < n, \beta < \alpha$ and $m,n \in \mathbb{N}$, $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions.

In [21], the authors initiated the study of non-local condition for the Hilfer implicit fractional differential equation

$$\begin{cases} D^{\alpha,\beta}_{0^+}y(t) &= f(t, y(t), D^{\alpha,\beta}_{0^+}y(t)), \quad t \in J := (0,T], \\ I^{1-\gamma}_{0^+}y(0) &= \sum_{i=1}^{m} c_i y(\tau_i), \quad \alpha \le \gamma = \alpha + \beta(1-\alpha) < 1, \tau_i \in [0,T], \end{cases}$$

* The research is partially supported by applied Mathematics Laboratory.

Submitted April 27, 2022. Published February 21, 2025
2010 *Mathematics Subject Classification:* 26A33, 45G10, 34A12,47H10.

where, $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $I_{0^+}^{1-\gamma}$ is the left-sided Riemann-Liouville fractional integral of order $1-\gamma$, c_i are real numbers, and $\tau_i, i = 1,2,...,m$ are prefixed points satisfying $0 < \tau_1 \leq \tau_2 \leq ... \leq \tau_m < T$.

Motivated by the above works, we concentrate on the following initial value problem of Hilfer fractional integro-differential equation:

$$\begin{cases} D_{0^+}^{\alpha,\beta} y(t) &= f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds\right), \quad t \in (0,T], \\ I_{0^+}^{1-\gamma} y(0) &= \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} y(\xi_i), \end{cases} \tag{1.1}$$

where, $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $I_{0^+}^{1-\gamma}$ is the left-sided Riemann-Liouville fractional integral of order $1-\gamma$, $I_{\eta_i}^{\mu_i,\delta_i}$ is the Erdélyi–Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$, and $\mu_i, \lambda_i \in \mathbb{R}$, $\xi_i \in (0,T)$ for $i = 1,...,m$, are real constants, $f : (0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions.

In the present paper, we intend to discuss the existence, uniqueness, and Ulam-Hyers stability of the nonlinear integron-differential equation involving Hilfer fractional derivative subject to the nonlocal Erdélyi–Kober fractional integral condition. We note that the nonlocal conditions are useful than the local (initial) condition to describe correctly some physics phenomena, see [6,7].

The rest of the paper is structured as follows: Section 2, contains some concepts of preliminaries about fractional calculus and auxiliary results. In Section 3, we present our main result by means of the Banach and Krasnoselskii fixed point theorems, moreover, stability analysis is discussed. In Section 4, two examples are presented to illustrate the applicability of our obtained results.

2. Preliminaries

In this section, we introduce notation, essential definitions of fractional calculus, and preliminary facts used throughout the paper. Further details can be found in [12,16,18].
For $0 \leq \gamma < 1$, we denote $C_\gamma[0,T]$ the weighted space of continuous functions on $(0,T]$, such that the function $t^\gamma f \in C[0,T]$. The norm of f in $C_\gamma[0,T]$ is defined as:

$$\|f\|_{C_\gamma[0,T]} = \max_{0 \leq t \leq T} |t^\gamma f(t)|.$$

Obviously, $C_\gamma[0,T]$ is the Banach space.
The notation $C_\gamma^n[0,T]$ defines a the Banach weighted space, for $n \in \mathbb{N}$, such that $f \in C^{n-1}[0,T]$ and $f^{(n)} \in C_\gamma[0,T]$, equipped with the norm

$$\|f\|_{C_\gamma^n[0,T]} = \sum_{k=1}^{n-1} \|f^k\|_{C[0,T]} + \|f^n\|_{C_\gamma[0,T]}.$$

In particular case, $\|f\|_{C_\gamma^0[0,T]} = \|f\|_{C_\gamma[0,T]}$.

Definition 2.1 *The left sided Riemann-Liouville fractional integral of order $0 < \alpha < 1$ for an integrable function $f : \mathbb{R}^+ \to \mathbb{R}$, is defined by*

$$I_{0^+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

where $\Gamma(.)$ is Euler's Gamma function.

Definition 2.2 *The Erdélyi–Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\mu \in \mathbb{R}$ of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$, is defined by*

$$I_\eta^{\mu,\delta} f(t) := \frac{\eta t^{-\eta(\delta+\mu)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\mu+\eta-1}}{(t^\eta - s^\eta)^{1-\delta}} f(s)ds, \quad t > 0,$$

provided the right-hand side is point-wise defined on $\mathbb{R}_+$.

In particular, if $\mu = 0$ and $\eta = 1$, the above operator is reduce to the Riemann-Liouville fractional integral with a power weight

$$I_1^{0,\delta} f(t) := \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} ds, \quad t > 0.$$

Next, we establish the following well-known formula as a lemma (see [13]).

Lemma 2.1 *Let $\delta, \eta > 0, \mu, q \in \mathbb{R}$, then we have*

$$I_\eta^{\mu,\delta} t^q = \frac{\Gamma(\mu + \frac{q}{\eta} + 1)}{\Gamma(\mu + \frac{q}{\eta} + \delta + 1)} t^q.$$

Definition 2.3 *The left sided Caputo fractional derivative of order $0 < \alpha < 1$ of a continuous function f, is defined by*

$$^C D_{0^+}^\alpha f(t) := I_{0^+}^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad t > 0.$$

Definition 2.4 *The left sided Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of an integrable function f, is defined by*

$$D_{0^+}^\alpha f(t) := \frac{d}{dt} I_{0^+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad t > 0.$$

Then, we present the following lemma of power functions:

Lemma 2.2 *Let $t > 0$, then we have*

$$I_{0^+}^\alpha t^{q-1} = \frac{\Gamma(q)}{\Gamma(\alpha + q)} t^{\alpha + q - 1}, \quad \alpha, q > 0.$$
$$D_{0^+}^\alpha t^{\alpha - 1} = 0, \quad 0 < \alpha < 1.$$

Definition 2.5 *The left sided Hilfer fractional derivative of order α and parameter β for an integrable function f defined on the interval $]0, \infty[$ as follows:*

$$D_{0^+}^{\alpha,\beta} f(t) := I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0^+}^{(1-\beta)(1-\alpha)} f(t),$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

Remark 2.1 Based on the established definition of the Hilfer fractional derivative, it becomes evident that:

R_1. The operator $D_{0^+}^{\alpha,\beta}$ can be written as

$$D_{0^+}^{\alpha,\beta} f(t) := I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0^+}^{1-\gamma} f(t) = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

R_2. The parameter γ satisfies

$$0 < \gamma \leq 1, \quad 1 - \gamma < 1 - \beta(1-\alpha).$$

R_3. The Hilfer derivative $D_{0^+}^{\alpha,\beta}$ interpolates between Riemann-Liouville (for $\beta = 0$) and Caputo (for $\beta = 1$) derivatives, since

$$D_{0^+}^{\alpha,0} = \frac{d}{dt} I_{0^+}^{1-\alpha} = D_{0^+}^\alpha, \quad D_{0^+}^{\alpha,1} = I_{0^+}^{1-\alpha} \frac{d}{dt} = {}^C D_{0^+}^\alpha.$$

The fundamental properties of the fractional integral operator I_{0+}^{α} are summarized in the following lemmas, along with proofs that can be found in [12,18].

Lemma 2.3 *Let $\alpha, \beta > 0$ and $f \in L(0,T)$. Then, the semi group property holds:*

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t), \quad t \in (0,T].$$

Lemma 2.4 *Let $0 < \alpha < 1, 0 \leq \gamma < 1$. If $f \in C_\gamma[0,T]$ and $I_{0+}^{1-\gamma} f \in C_\gamma^1[0,T]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) - \frac{I_{0+}^{1-\alpha} f(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \forall t \in (0,T]$$

and for $f \in C_\gamma[0,T]$, we have

$$D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t), \quad \forall t \in (0,T].$$

Lemma 2.5 *Let $\alpha > 0$ and $0 \leq \gamma < 1$. If $\gamma \leq \alpha$, then the fractional integrals operator I_{0+}^{α} is bounded from $C_\gamma[0,T]$ to $C[0,T]$.*

Lemma 2.6 *Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then the Riemann-Liouville fractional integral operator I_{0+}^{α} is bounded from $C_\gamma[0,T]$ to $C_\gamma[0,T]$.*

Due to the semi group property, we have the following lemma:

Lemma 2.7 *For $0 < \alpha < 1, 0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}^{\gamma}[0,T]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha,\beta} f = I_{0+}^{\gamma} D_{0+}^{\gamma} f$$

and

$$D_{0+}^{\gamma} I_{0+}^{\alpha} f = D_{0+}^{\beta(1-\alpha)} f.$$

Later on, we will need to use following lemma:

Lemma 2.8 *Let $0 \leq \gamma < 1$ and $f \in C_\gamma[0,T]$. Then*

$$I_{0+}^{\alpha} f(0) = \lim_{t \to 0} I_{0+}^{\alpha} f(t) = 0, \quad 0 \leq \gamma < \alpha.$$

Next, we continue with the following fixed point theorems:

Theorem 2.1 *(Banach's theorem [20])*
For a Banach space X and a contraction mapping $\Phi : X \to X$ with a contraction constant L, there exists a unique fixed point of Φ in X.

Theorem 2.2 *(Krasnosel'skii's fixed point theorem [19])*
Let M be a non-empty, closed, bounded, and convex subset of a Banach space. If A and B are operators satisfying the following conditions:

(i) $Ax + By \in M$, for all $x, y \in M$ (Closure under the operation);

(ii) A is compact and continuous (Ensures existence of a limit);

(iii) B is contraction mapping (Guarantees a "pulling towards" a fixed point).

Then, there exists at least one element z within M such that $z = Az + Bz$ (i.e., z is a fixed point of the combined operator).

We rely on the definition of Ulam-Hyers stability in Rus [17] to define the Ulam-Hyers stability of the fractional integron-differential equation (1.1) as follows:

Definition 2.6 *We say the equation (1.1) is Ulam-Hyers stable, if there exists a positive real number* c_f *such that for any* $\epsilon > 0$ *and any function* z *in the space* $C^{\gamma}_{1-\gamma}[0,T]$ *solution of inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f\left(t, D^{\alpha,\beta}_{0^+} z(t), \int_0^t g(t,s,z(s))ds\right)\right| \leq \epsilon,\ \textit{for } t \in [0,T], \tag{2.1}$$

there exists a solution $y \in C^{\gamma}_{1-\gamma}[0,T]$ *of equation (1.1) with*

$$|z(t) - y(t)| \leq c_f \varepsilon,\ \textit{for } t \in [0,T].$$

Remark 2.2 A function $z \in C^{\gamma}_{1-\gamma}[0,T]$ is a solution of inequality (2.1) if and only if there exists a function $h \in C^{\gamma}_{1-\gamma}[0,T]$ satisfying the following conditions:

(i) $|h(t)| \leq \varepsilon$, for all $t \in [0,T]$;

(ii) $D^{\alpha,\beta}_{0^+} z(t) = f\left(t, D^{\alpha,\beta}_{0^+} z(t), \int_0^t g(t,s,z(s))ds\right) + h(t),$ for $t \in [0,T]$.

At the end, we collect the following generalization of Gronwall's Lemma for singlular kernals (see [22]):

Lemma 2.9 *Given positive constants* β, k, *and positive functions locally integrable* y *and* ω *on* $[0,T)$, *with* ω *is non-decreasing, if the inequality*

$$y(t) \leq \omega(t) + k\int_0^t (t-s)^{\alpha-1} y(s)ds,$$

holds for all $t \in [0,T)$, *then the following inequality is also valid*

$$y(t) \leq \omega(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1}\omega(s)ds, \quad t \in [0,T).$$

If the weight ω remains constant (denoted by a), then the previously mentioned inequality simplifies to

$$y(t) \leq aE_{\alpha}(k\Gamma(\alpha)t^{\alpha}), \quad t \in [0,T)$$

here, E_{α} denotes the Mittag-Leffler function, which you can learn more about in [16].

3. Main results

To establish the existence and uniqueness of solutions for problem (1.1), we introduce the following weighted spaces:

$$C^{\alpha,\beta}_{1-\gamma}[0,T] = \{f \in C_{1-\gamma}[0,T], D^{\alpha,\beta}_{0^+} f \in C_{1-\gamma}[0,T], \quad 0 < \gamma \leq 1\};$$
$$C^{\gamma}_{1-\gamma}[0,T] = \{f \in C_{1-\gamma}[0,T], D^{\gamma}_{0^+} f \in C_{1-\gamma}[0,T], \quad 0 < \gamma \leq 1\}.$$

From Remark 2.1-R_1 and Lemma 2.6, we infer that

$$C^{\gamma}_{1-\gamma}[0,T] \subset C^{\alpha,\beta}_{1-\gamma}[0,T].$$

Theorem 3.1 *Let* $\alpha, \beta, \delta_i, \eta_i$ *be positive real numbers with* $0 < \alpha < 1, 0 \leq \beta \leq 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$, μ_i, λ_i *real numbers, and* ξ_i *in* $(0,T)$ *for* $i = 1,..,m$. *For a function* $f : (0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfying* $f(.,u(.),v(.)) \in C_{1-\gamma}[0,T]$ *for any* $u, v \in C_{1-\gamma}[0,T]$. *A function* y *in the space* $C^{\gamma}_{1-\gamma}[0,T]$ *is a solution to problem* (1.1) *if and only if it satisfies the integral equation*

$$\begin{aligned} y(t) = {}& \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I^{\mu_i,\delta_i}_{\eta_i} I^{\alpha}_{0^+}\left[f\left(\xi_i, D^{\alpha,\beta}_{0^+} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds\right)\right] t^{\gamma-1} \\ &+ I^{\alpha}_{0^+}\left[f\left(t, D^{\alpha,\beta}_{0^+} y(t), \int_0^t g(t,s,y(s))ds\right)\right], \end{aligned} \tag{3.1}$$

where

$$\Lambda = 1 - \frac{1}{\Gamma(\gamma)} \sum_{i=1}^{m} \lambda_i \frac{\Gamma(\mu_i + \frac{\gamma-1}{\eta_i} + 1)}{\Gamma(\mu_i + \frac{\gamma-1}{\eta_i} + \delta_i + 1)} \xi_i^{\gamma-1} \neq 0. \tag{3.2}$$

Proof: As a first step, we demonstrate the necessity: Let $y \in C_{1-\gamma}^{\gamma}[0,T]$ be a solution of (1.1), we will prove that y satisfies the integral equation (3.1). $y \in C_{1-\gamma}^{\gamma}[0,T]$ means that

$$y \in C_{1-\gamma}[0,T], D_{0+}^{\gamma} y = \frac{d}{dt} I_{0+}^{1-\gamma} y \in C_{1-\gamma}[0,T], \tag{3.3}$$

from Lemma 2.5, we get $I_{0+}^{1-\gamma} y \in C[0,T]$, therefor, we deduce that

$$I_{0+}^{1-\gamma} y \in C_{1-\gamma}^{1}[0,T]. \tag{3.4}$$

Performing the integral operator I_{0+}^{α} on the first equation of problem (1.1) and by using Lemma 2.7, we obtain

$$\begin{aligned} I_{0+}^{\alpha} D_{0+}^{\alpha,\beta} y(t) &= I_{0+}^{\alpha} \left[f \left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds \right) \right] \\ I_{0+}^{\gamma} D_{0+}^{\gamma} y(t) &= I_{0+}^{\alpha} \left[f \left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds \right) \right], \end{aligned}$$

from (3.3) and (3.4), we apply Lemma 2.4 to find

$$y(t) = \frac{I_{0+}^{1-\gamma} y(0)}{\Gamma(\gamma)} t^{\gamma-1} + I_{0+}^{\alpha} \left[f \left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds \right) \right]. \tag{3.5}$$

Next, substituting $t = \xi_i$ and applying the Erdélyi-Kober fractional integral operator $I_{\eta_i}^{\mu_i,\delta_i}$ on both sides of equation (3.5), we get

$$\begin{aligned} I_{\eta_i}^{\mu_i,\delta_i} y(\xi_i) &= \frac{I_{0+}^{1-\gamma} y(0)}{\Gamma(\gamma)} I_{\eta_i}^{\mu_i,\delta_i} (\xi_i^{\gamma-1}) \\ &\quad + I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[f \left(\xi_i, D_{0+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds \right) \right]. \end{aligned}$$

From the initial condition and Lemma 2.1, we find

$$\begin{aligned} I_{0+}^{1-\gamma} y(0) &= \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} y(\xi_i) \\ &= \frac{I_{0+}^{1-\gamma} y(0)}{\Gamma(\gamma)} \sum_{i=1}^{m} \lambda_i \frac{\Gamma(\mu_i + \frac{\gamma-1}{\eta_i} + 1)}{\Gamma(\mu_i + \frac{\gamma-1}{\eta_i} + \delta_i + 1)} \xi_i^{\gamma-1} \\ &\quad + \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[f \left(\xi_i, D_{0+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds \right) \right], \end{aligned}$$

which implies

$$I_{0+}^{1-\gamma} y(0) = \frac{1}{\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[f \left(\xi_i, D_{0+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds \right) \right], \tag{3.6}$$

where Λ is given by (3.2). Submitting (3.6) to (3.5), we derive that (3.1). The necessity established.
Now, we turn to the proof of sufficiency: Suppose that $y \in C_{1-\gamma}^{\gamma}[0,T]$ satisfies the integral equation (3.1), then we will show that y is a solution of problem (1.1). Applying the operator $D_{0+}^{\alpha,\beta}$ on both sides of (3.1), we obtain

$$\begin{aligned} D_{0+}^{\alpha,\beta} y(t) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[f \left(\xi_i, D_{0+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds \right) \right] D_{0+}^{\alpha,\beta} t^{\gamma-1} \\ &\quad + D_{0+}^{\alpha,\beta} I_{0+}^{\alpha} \left[f \left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds \right) \right]. \end{aligned}$$

From Remark 2.1-R_1, Lemma 2.2 and Lemma 2.7, we have

$$\begin{aligned}
D_{0^+}^{\alpha,\beta} y(t) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} \\
&\times I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} t^{\gamma-1} \\
&+ I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} I_{0^+}^{\alpha} \left[f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^{t} g(t, s, y(s)) ds \right) \right] \\
&= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] (I_{0^+}^{\beta(1-\alpha)} \times 0) \\
&+ I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\beta(1-\alpha)} \left[f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^{t} g(t, s, y(s)) ds \right) \right].
\end{aligned}$$

Therefore, in view of Lemma 2.4, Remark 2.1-R_2 and Lemma 2.8, we get

$$D_{0^+}^{\alpha,\beta} y(t) = f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^{t} g(t, s, y(s)) ds \right).$$

The initial condition in (1.1) can be obtained without difficulty, by applying the operator $I_{0^+}^{1-\gamma}$ on both sides of (3.1), we find

$$\begin{aligned}
I_{0^+}^{1-\gamma} y(t) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] I_{0^+}^{1-\gamma} t^{\gamma-1} \\
&+ I_{0^+}^{1-\gamma} I_{0^+}^{\alpha} f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^{t} g(t, s, y(s)) ds \right),
\end{aligned}$$

using Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
I_{0^+}^{1-\gamma} y(t) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] \frac{\Gamma(\gamma)}{1} \\
&+ I_{0^+}^{1-\beta(1-\alpha)} f\left(t, D_{0^+}^{\alpha,\beta} y(t), \int_0^{t} g(t, s, y(s)) ds \right).
\end{aligned}$$

Passing limit as $t \to 0$, from Remark 2.1-R_2 and Lemma 2.8, we obtain

$$I_{0^+}^{1-\gamma} y(0) = \frac{1}{\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right]. \tag{3.7}$$

On other hand, replace $t = \xi_i$ into equation (3.1), we get

$$\begin{aligned}
y(\xi_i) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] \xi_i^{\gamma-1} \\
&+ I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right].
\end{aligned}$$

Applying $\sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i}$ on both sides of the last equality, we find

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} y(\xi_i) &= \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right] \\
&\times \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} \xi_i^{\gamma-1} \\
&+ \sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha} \left[f\left(\xi_i, D_{0^+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i, s, y(s)) ds \right) \right],
\end{aligned}$$

in light of Lemma 2.1, we have

$$\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}y(\xi_i)=\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0+}^{\alpha}\left[f\left(\xi_i,D_{0+}^{\alpha,\beta}y(\xi_i),\int_0^{\xi_i}g(\xi_i,s,y(s))ds\right)\right]$$
$$\times\left(1+\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i\frac{\Gamma(\mu_i+\frac{\gamma-1}{\eta_i}+1)}{\Gamma(\mu_i+\frac{\gamma-1}{\eta_i}+\delta+1)}\xi_i^{\gamma-1}\right)$$
$$=\frac{1}{\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0+}^{\alpha}\left[f\left(\xi_i,D_{0+}^{\alpha,\beta}y(\xi_i),\int_0^{\xi_i}g(\xi_i,s,y(s))ds\right)\right]. \tag{3.8}$$

Upon comparison of equations (3.7) and (3.8), it follows that

$$I_{0+}^{1-\gamma}y(0)=\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}y(\xi_i).$$

□

Throughout this paper, we assume that the following hypotheses hold:

(H_1) There exists a constant $0<M_1<1$ such that
$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le M_1\left(|u-\bar{u}|+|v-\bar{v}|\right)\quad \forall u,\bar{u},v,\bar{v}\in\mathbb{R},t\in(0,T]$.

(H_2) There exists a constant $M_2>0$ such that
$|g(t,s,u)-g(t,s,\bar{u})|\le M_2|u-\bar{u}|,\quad \forall u,\bar{u}\in\mathbb{R},t,s\in[0,T]$.

For simplicity, we denote

$$\sup_{t\in[0,T]}|f(t,0,0)|=M_f<\infty;$$
$$\sup_{t,s\in[0,T]}|g(t,s,0)|=M_g<\infty.$$

The first result is based on Banach's fixed point theorem.

Theorem 3.2 *Let $f(t,u,v):(0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(.,u(.),v(.))\in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$ for any $u,v\in C_{1-\gamma}[0,T]$. Suppose that $(H_1)-(H_2)$ are satisfied. Then the problem (1.1) has a unique solution in $C_{1-\gamma}^{\gamma}[0,T]$. Provided that*

$$\sigma_1=\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i L_i\xi_i^{\alpha+\gamma}+\frac{M_1M_2\Gamma(\gamma)}{(1-M_1)\Gamma(\alpha+\gamma+1)}T^{\alpha+1}<1, \tag{3.9}$$

where

$$L_i=\frac{M_1M_2\Gamma(\gamma)\Gamma(\mu_i+\frac{\alpha+\gamma}{\eta_i}+1)}{(1-M_1)\Gamma(\alpha+\gamma+1)\Gamma(\mu_i+\frac{\alpha+\gamma}{\eta_i}+\delta_i+1)}. \tag{3.10}$$

Proof: According to Theorem 3.1, we introduce an operator $\Phi:C_{1-\gamma}[0,T]\to C_{1-\gamma}[0,T]$ associated with the problem (1.1) as follows:

$$\Phi y(t)=\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0+}^{\alpha}\left[f\left(\xi_i,D_{0+}^{\alpha,\beta}y(\xi_i),\int_0^{\xi_i}g(\xi_i,s,y(s))ds\right)\right]t^{\gamma-1}$$
$$+I_{0+}^{\alpha}\left[f\left(t,D_{0+}^{\alpha,\beta}y(t),\int_0^{t}g(t,s,y(s))ds\right)\right]. \tag{3.11}$$

Firstly, we demonstrate that the operator Φ is well-defined on $C_{1-\gamma}[0,T]$, i.e. for any $y \in C_{1-\gamma}[0,T]$, the operator $\Phi \in C_{1-\gamma}[0,T]$. Let $y \in C_{1-\gamma}[0,T]$, we have

$$|t^{1-\gamma}\Phi y(t)| \le \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha}\left[\left|f\left(\xi_i, D_{0+}^{\alpha,\beta}y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds\right)\right|\right] + t^{1-\gamma} I_{0+}^{\alpha}\left[\left|f\left(t, D_{0+}^{\alpha,\beta}y(t), \int_0^{t} g(t,s,y(s))ds\right)\right|\right]. \tag{3.12}$$

Taking (H_1) into consideration, we get

$$\begin{aligned}
&\left|f\left(t, D_{0+}^{\alpha,\beta}y(t), \int_0^t g(t,s,y(s))ds\right)\right| \\
&\le \left|f\left(t, D_{0+}^{\alpha,\beta}y(t), \int_0^t g(t,s,y(s))ds\right) - f(t,0,0)\right| + |f(t,0,0)| \\
&\le M_1\left[|D_{0+}^{\alpha,\beta}y(t)| + \int_0^t |g(t,s,y(s)) - g(t,s,0)|ds + \int_0^t |g(t,s,0)|ds\right] + M_f \\
&\le M_1\left[|D_{0+}^{\alpha,\beta}y(t)| + M_2\int_0^t |y(s)|ds + \int_0^t |g(t,s,0)|ds\right] + M_f,
\end{aligned}$$

considering the first equation in (1.1), we have

$$\left|f\left(t, D_{0+}^{\alpha,\beta}y(t), \int_0^t g(t,s,y(s))ds\right)\right| \le \frac{M_1M_2}{1-M_1}\int_0^t |y(s)|ds + M, \tag{3.13}$$

where

$$M = \frac{1}{1-M_1}\left(M_1M_gT + M_f\right).$$

Then, by substituting (3.13) into inequality (3.12), we derive

$$\begin{aligned}
|t^{1-\gamma}\Phi y(t)| &\le \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha}\left[\frac{M_1M_2}{(1-M_1)}\int_0^{\xi_i} s^{\gamma-1}|s^{1-\gamma}y(s)|ds + M\right] \\
&\quad + t^{1-\gamma} I_{0+}^{\alpha}\left[\frac{M_1M_2}{(1-M_1)}\int_0^{t} s^{\gamma-1}|s^{1-\gamma}y(s)|ds + M\right] \\
&\le \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha}\left[\frac{M_1M_2\xi_i^{\gamma}}{(1-M_1)\gamma}\|y\|_{C_{1-\gamma}[0,T]} + M\right] \\
&\quad + t^{1-\gamma} I_{0+}^{\alpha}\left[\frac{M_1M_2t^{\gamma}}{(1-M_1)\gamma}\|y\|_{C_{1-\gamma}[0,T]} + M\right].
\end{aligned}$$

Thus, from Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}
|t^{1-\gamma}\Phi y(t)| &\le \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i\left[L_i\xi_i^{\alpha+\gamma}\|y\|_{C_{1-\gamma}[0,T]} + N_iM\xi_i^{\alpha}\right] \\
&\quad + \frac{M_1M_2\Gamma(\gamma)t^{\alpha+1}}{(1-M_1)\Gamma(\alpha+\gamma+1)}\|y\|_{C_{1-\gamma}[0,T]} + \frac{t^{1-\gamma+\alpha}M}{\Gamma(\alpha+1)},
\end{aligned}$$

where L_i is defined by (3.10) and

$$N_i = \frac{\Gamma(\mu_i + \frac{\alpha}{\eta_i} + 1)}{\Gamma(\alpha+1)\Gamma(\mu_i + \frac{\alpha}{\eta_i} + \delta_i + 1)}. \tag{3.14}$$

Consequently, we deduce that $\Phi y \in C_{1-\gamma}[0,T]$.

Next, let $y, z \in C_{1-\gamma}[0,T]$, firstly, from $(H_1)-(H_2)$ and in view of the first equation of problem (1.1), we have

$$\begin{aligned}
&\left| f\left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds\right) - f\left(t, D_{0+}^{\alpha,\beta} z(t), \int_0^t g(t,s,z(s))ds\right)\right| \\
&\leq M_1 \left[|D_{0+}^{\alpha,\beta} y(t) - D_{0+}^{\alpha,\beta} z(t)| + \int_0^t |g(t,s,y(s)) - g(t,s,z(s))|\, ds \right] \\
&\leq M_1 \left[|D_{0+}^{\alpha,\beta} y(t) - D_{0+}^{\alpha,\beta} z(t)| + M_2 \int_0^t |y(s) - z(s)| ds \right] \\
&\leq \frac{M_1 M_2}{1-M_1} \int_0^t |y(s) - z(s)| ds. \qquad (3.15)
\end{aligned}$$

Then, equation (3.15) implies

$$\begin{aligned}
&|t^{1-\gamma}\left[\Phi y(t) - \Phi z(t)\right]| \\
&\leq \frac{M_1 M_2}{(1-M_1)\Gamma(\gamma)\Lambda} \sum_{i=1}^m \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[\int_0^{\xi_i} s^{\gamma-1} |s^{1-\gamma}[y(s)-z(s)]| ds \right] \\
&+ \frac{M_1 M_2 t^{1-\gamma}}{(1-M_1)} I_{0+}^{\alpha} \left[\int_0^t s^{\gamma-1} |s^{1-\gamma}[y(s)-z(s)]| ds \right] \\
&\leq \frac{M_1 M_2}{(1-M_1)\Gamma(\gamma)\Lambda} \|y-z\|_{C_{1-\gamma}[0,T]} \sum_{i=1}^m \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[\frac{\xi_i^{\gamma}}{\gamma} \right] \\
&+ \frac{M_1 M_2 t^{1-\gamma}}{(1-M_1)} \|y-z\|_{C_{1-\gamma}[0,T]} I_{0+}^{\alpha} \left[\frac{t^{\gamma}}{\gamma} \right].
\end{aligned}$$

It follows from Lemmas 2.1 and 2.2 that:

$$\begin{aligned}
&\|\Phi y(t) - \Phi z(t)\|_{C_{1-\gamma}[0,T]} \\
&\leq \left[\frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^m \lambda_i L_i \xi_i^{\alpha+\gamma} + \frac{M_1 M_2 \Gamma(\gamma)}{(1-M_1)\Gamma(\alpha+\gamma+1)} T^{\alpha+1} \right] \|y-z\|_{C_{1-\gamma}[0,T]}.
\end{aligned}$$

Condition (3.9) ensures that the operator Φ admits a unique fixed point in $C_{1-\gamma}[0,T]$, which is solution of (3.1). To establish the desired regularity of this solution, we proceed by applying the operator D_{0+}^{γ} to both sides of the integral equation (3.1), we get

$$\begin{aligned}
D_{0+}^{\gamma} y(t) = &\frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^m \lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0+}^{\alpha} \left[f\left(\xi_i, D_{0+}^{\alpha,\beta} y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds\right)\right] D_{0+}^{\gamma} t^{\gamma-1} \\
&+ D_{0+}^{\gamma} I_{0+}^{\alpha} \left[f\left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds\right)\right].
\end{aligned}$$

By Lemma 2.7 and the condition $\gamma \geq \alpha$, we have

$$\begin{aligned}
D_{0+}^{\gamma} y(t) &= D_{0+}^{\gamma-\alpha} \left[f\left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds\right)\right] \\
&= D_{0+}^{\beta(1-\alpha)} \left[f\left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s))ds\right)\right],
\end{aligned}$$

since $f \in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$, we deduce that $D_{0+}^{\gamma} y \in C_{1-\gamma}[0,T]$, and thus $y \in C_{1-\gamma}^{\gamma}[0,T] \subset C_{1-\gamma}^{\alpha,\beta}[0,T]$ is the unique solution of the problem (1.1). □

The second result is based on Krasnoselskii's fixed point theorem.

Theorem 3.3 *Let $f(t,u,v):(0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function, such that $f(.,u(.),v(.))\in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$ for any $u,v\in C_{1-\gamma}[0,T]$. Suppose that $(H_1)-(H_2)$ are satisfied. Then the problem (1.1) has at least one solution in the space $C_{1-\gamma}^{\gamma}[0,T]$, provided*

$$\sigma_2=\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i L_i\xi_i^{\alpha+\gamma}<1, \tag{3.16}$$

where Λ and L_i are defined by (3.2) and (3.10), respectively.

Proof: The proof is divided into five steps. First, we consider

$$D_r=\{y\in C_{1-\gamma}[0,T], \|y\|_{C_{1-\gamma}[0,T]}\le r\},$$

where r satisfies the following inequality

$$r>\frac{\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i N_i M\xi_i^{\alpha}+\frac{T^{1-\gamma+\alpha}M}{\Gamma(\alpha+1)}}{1-\left(\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i L_i\xi_i^{\alpha+\gamma}+\frac{M_1M_2\Gamma(\gamma)}{(1-M_1)\Gamma(\alpha+\gamma+1)}T^{\alpha+1}\right)}.$$

Next, we define the operators Q_1 and Q_2 on D_r as follows:

$$Q_1y(t)=\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0^+}^{\alpha}\left[f\left(\xi_i,D_{0^+}^{\alpha,\beta}y(\xi_i),\int_0^{\xi_i}g(\xi_i,s,y(s))ds\right)\right]t^{\gamma-1},\quad t\in[0,T]$$
$$Q_2y(t)=I_{0^+}^{\alpha}\left[f\left(t,D_{0^+}^{\alpha,\beta}y(t),\int_0^{t}g(t,s,y(s))ds\right)\right],\quad t\in[0,T].$$

Step 1: We show $Q_1+Q_2\in D_r$. In fact, for $y,z\in D_r$, we have

$$\begin{aligned}|t^{1-\gamma}(Q_1y(t)+Q_2z(t))|\le{}&\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0^+}^{\alpha}\left[\left|f\left(\xi_i,D_{0^+}^{\alpha,\beta}y(\xi_i),\int_0^{\xi_i}g(\xi_i,s,y(s))ds\right)\right|\right]\\&+t^{1-\gamma}I_{0^+}^{\alpha}\left[\left|f\left(t,D_{0^+}^{\alpha,\beta}z(t),\int_0^{t}g(t,s,z(s))ds\right)\right|\right],\end{aligned}$$

then, according (3.13), we get

$$\begin{aligned}|t^{1-\gamma}(Q_1y(t)+Q_2z(t))|\le{}&\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0^+}^{\alpha}\left[\frac{M_1M_2}{(1-M_1)}\int_0^{\xi_i}s^{\gamma-1}|s^{1-\gamma}y(s)|ds+M\right]\\&+t^{1-\gamma}I_{0^+}^{\alpha}\left[\frac{M_1M_2}{(1-M_1)}\int_0^{t}s^{\gamma-1}|s^{1-\gamma}z(s)|ds+M\right].\end{aligned}$$

Thus, from Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}|t^{1-\gamma}(Q_1y(t)+Q_2z(t))|\le{}&\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i\left[L_i\xi_i^{\alpha+\gamma}\|y\|_{C_{1-\gamma}[0,T]}+N_iM\xi_i^{\alpha}\right]\\&+\frac{M_1M_2\Gamma(\gamma)}{(1-M_1)\Gamma(\alpha+\gamma+1)}t^{\alpha+1}\|z\|_{C_{1-\gamma}[0,T]}+\frac{t^{1-\gamma+\alpha}M}{\Gamma(\alpha+1)}<r,\end{aligned}$$

where L_i and N_i are defined by (3.10) and (3.14), respectively. This implies that $Q_1+Q_2\in D_r$.

Step 2: The mapping Q_1 is a contraction on D_r. Indeed, let $y,z\in D_r$, from (3.15) we find

$$\begin{aligned}|t^{1-\gamma}[Q_1y(t)-Q_1z(t)]|\le{}&\frac{M_1M_2}{(1-M_1)\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0^+}^{\alpha}\left[\int_0^{\xi_i}s^{\gamma-1}|s^{1-\gamma}[y(s)-z(t)]|ds\right]\\\le{}&\frac{M_1M_2}{(1-M_1)\Gamma(\gamma)\Lambda}\|y-z\|_{C_{1-\gamma}[0,T]}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}I_{0^+}^{\alpha}\left[\frac{\xi_i^{\gamma}}{\gamma}\right].\end{aligned}$$

Thus, from Lemma 2.1 we get

$$\|Q_1 y(t) - Q_1 z(t)\|_{C_{1-\gamma}[0,T]} \leq \frac{1}{\Gamma(\gamma)\Lambda} \sum_{i=1}^{m} \lambda_i L_i \xi_i^{\alpha+\gamma} \|y - z\|_{C_{1-\gamma}[0,T]},$$

using (3.16) to conclude that the operator Q_1 is a contraction.

Step 3: The mapping Q_2 is continuous. Let $\{y_n\}$ be sequence, such that $y_n \to y$ in $C_{1-\gamma}[0,T]$, when $n \to \infty$. Then for each $t \in [0,T]$ and from (3.15), we have

$$\begin{aligned} \left|t^{1-\gamma}\left[Q_2 y_n(t) - Q_2 y(t)\right]\right| &\leq \frac{M_1 M_2 t^{1-\gamma}}{(1-M_1)} I_{0+}^{\alpha} \left[\int_0^t s^{\gamma-1} |s^{1-\gamma}[y_n(s) - y(s)]| ds\right] \\ &\leq \frac{M_1 M_2 t^{1-\gamma}}{(1-M_1)} \|y_n - y\|_{C_{1-\gamma}[0,T]} I_{0+}^{\alpha} \left[\frac{t^{\gamma}}{\gamma}\right], \end{aligned}$$

since $y_n \to y$ as $n \to \infty$, then we deduce that $Q_2 y_n \to Q_2 y$ in $C_{1-\gamma}[0,T]$ as $n \to \infty$, which implies the continuity of the operator Q_2.

Step 4: The operator Q_2 is uniformly bounded on D_r. Let $y \in D_r$, then from (3.13) we find

$$\begin{aligned} |t^{1-\gamma} Q_2 y(t)| &\leq t^{1-\gamma} I_{0+}^{\alpha} \left[\left|f\left(t, D_{0+}^{\alpha,\beta} y(t), \int_0^t g(t,s,y(s)) ds\right)\right|\right] \\ &\leq t^{1-\gamma} I_{0+}^{\alpha} \left[\frac{M_1 M_2}{(1-M_1)} \int_0^t s^{\gamma-1} |s^{1-\gamma} y(s)| ds + M\right] \\ &\leq t^{1-\gamma} I_{0+}^{\alpha} \left[\frac{M_1 M_2 t^{\gamma}}{(1-M_1)\gamma} \|y\|_{C_{1-\gamma}[0,T]} + M\right], \end{aligned}$$

using Lemma 2.2 to get

$$\begin{aligned} |t^{1-\gamma} Q_2 y(t)| &\leq \frac{M_1 M_2 \Gamma(\gamma) t^{\alpha+1}}{(1-M_1)\Gamma(\alpha+\gamma+1)} \|y\|_{C_{1-\gamma}[0,T]} + \frac{t^{1-\gamma+\alpha} M}{\Gamma(\alpha+1)} \\ &\leq \frac{M_1 M_2 \Gamma(\gamma) T^{\alpha+1}}{(1-M_1)\Gamma(\alpha+\gamma+1)} \|y\|_{C_{1-\gamma}[0,T]} + \frac{T^{1-\gamma+\alpha} M}{\Gamma(\alpha+1)} < \infty. \end{aligned}$$

Step 5: The operator Q_2 is equicontinuous. In fact, for any $0 < t_1 < t_2 < T$ and $y \in D_r$, from Definition 2.1 we have

$$\begin{aligned} &|t_2^{1-\gamma} Q_2 y(t_2) - t_1^{1-\gamma} Q_2 y(t_1)| \\ &\leq \frac{1}{\Gamma(\gamma)} \left| \int_0^{t_1} t_2^{1-\gamma} (t_2 - r)^{\alpha-1} \left[f\left(r, D_{0+}^{\alpha,\beta} y(r), \int_0^r g(r,s,y(s)) ds\right)\right] dr \right. \\ &+ \int_{t_1}^{t_2} t_2^{1-\gamma} (t_2 - r)^{\alpha-1} \left[f\left(r, D_{0+}^{\alpha,\beta} y(r), \int_0^r g(r,s,y(s)) ds\right)\right] dr \\ &- \left. \int_0^{t_1} t_1^{1-\gamma} (t_1 - r)^{\alpha-1} \left[f\left(r, D_{0+}^{\alpha,\beta} y(r), \int_0^r g(r,s,y(s)) ds\right)\right] dr \right|, \end{aligned}$$

from (3.13), we get

$$\begin{aligned} &|t_2^{1-\gamma} Q_2 y(t_2) - t_1^{1-\gamma} Q_2 y(t_1)| \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^{t_1} \left|t_2^{1-\gamma}(t_2 - r)^{\alpha-1} - t_1^{1-\gamma}(t_1 - r)^{\alpha-1}\right| \left[\frac{M_1 M_2}{1-M_1} \int_0^r s^{\gamma-1} |s^{1-\gamma} y(s)| ds + M\right] dr \\ &+ \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} t_2^{1-\gamma} (t_2 - r)^{\alpha-1} \left[\frac{M_1 M_2}{1-M_1} \int_0^r s^{\gamma-1} |s^{1-\gamma} y(s)| ds + M\right] dr \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^{t_1} \left|t_2^{1-\gamma}(t_2 - r)^{\alpha-1} - t_1^{1-\gamma}(t_1 - s)^{\alpha-1}\right| \left[\frac{M_1 M_2}{(1-M_1)\gamma} r^{\gamma+1} + M\right] dr \\ &+ \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} t_2^{1-\gamma} (t_2 - r)^{\alpha-1} \left[\frac{M_1 M_2}{(1-M_1)\gamma} r^{\gamma+1} + M\right] dr, \end{aligned}$$

which is independent of y and tend to zero as $t_1 \to t_2$. By virtue of the Arzelà-Ascoli theorem, the operator Q_2 is completely continuous. Consequently, an application of Krasnoselskii's fixed point theorem yields a fixed point y of Φ on D_r . Moreover, since $f \in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$, we deduce that $y \in C_{1-\gamma}^{\gamma}[0,T]$ is the solution of problem (1.1). □

Finally, we state with the following Ulam–Hyers stable result.

Theorem 3.4 *Under the hypotheses of Theorem 3.2, the problem (1.1) is Ulam–Hyers stable on $[0,T]$.*

Proof: Let $z \in C_{1-\gamma}^{\gamma}[0,T]$ be a solution of inequality (2.1) and $y \in C_{1-\gamma}^{\gamma}[0,T]$ is a unique solution of Cauchy problem (1.1). By using the Theorem 3.1, we have

$$y(t) = K_y + I_{0^+}^{\alpha}\left[f\left(t, D_{0^+}^{\alpha,\beta}y(t), \int_0^t g(t,s,y(s))ds\right)\right],$$

where

$$K_y = \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha}\left[f\left(\xi_i, D_{0^+}^{\alpha,\beta}y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds\right)\right]t^{\gamma-1}.$$

Since, we assumed that z is a solution of the inequality (2.1), then by Remark 2.2, we have

$$\begin{cases} D_{0^+}^{\alpha,\beta}z(t) &= f\left(t, D^{\alpha,\beta}0^+z(t), \int_0^t g(t,s,z(s))\right) + h(t), \text{ for } t \in (0,T] \\ I_{0^+}^{1-\gamma}z(0) &= \sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}z(\xi_i). \end{cases} \tag{3.17}$$

Again by Theorem 3.1, the solution of problem (3.17) is given by

$$z(t) = \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha}\left[f\left(\xi_i, D_{0^+}^{\alpha,\beta}z(\xi_i), \int_0^{\xi_i} g(\xi_i,s,z(s))ds\right) + h(\xi)\right]t^{\gamma-1}$$
$$+ I_{0^+}^{\alpha}\left[f\left(t, D_{0^+}^{\alpha,\beta}z(t), \int_0^t g(t,s,z(s))ds\right) + h(t)\right].$$

In light of this, we have

$$\begin{aligned} &\left|z(t) - K_z - I_{0^+}^{\alpha}\left[f\left(t, D_{0^+}^{\alpha,\beta}z(t), \int_0^t g(t,s,z(s))ds\right)\right]\right| \\ &= \left|\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha}\left[h(\xi)\right]t^{\gamma-1} + I_{0^+}^{\alpha}h(t)\right| \\ &\leq \left(\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\frac{\Gamma(\mu_i + \frac{\alpha}{\eta_i} + 1)\lambda_i}{\Gamma(\alpha+1)\Gamma(\mu_i + \frac{\alpha}{\eta_i} + \delta_i + 1)}T^{\gamma+\alpha-1} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)\varepsilon. \end{aligned} \tag{3.18}$$

For every $t \in (0,T]$, we have

$$|z(t) - y(t)| \leq \left|z(t) - K_z - I_{0^+}^{\alpha}\left[f\left(t, D_{0^+}^{\alpha,\beta}z(t), \int_0^t g(t,s,z(s))ds\right)\right]\right| + |K_z - K_y|$$
$$+ \left|I_{0^+}^{\alpha}\left[f\left(t, D_{0^+}^{\alpha,\beta}z(t), \int_0^t g(t,s,z(s))ds\right) - f\left(t, D_{0^+}^{\alpha,\beta}y(t), \int_0^t g(t,s,y(s))ds\right)\right]\right|.$$

On the other hand, from (3.15) we get

$$\begin{aligned} |K_y - K_z| &\leq \frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} \\ &\times I_{0^+}^{\alpha}\left[f\left(\xi_i, D_{0^+}^{\alpha,\beta}y(\xi_i), \int_0^{\xi_i} g(\xi_i,s,y(s))ds\right) - f\left(\xi_i, D_{0^+}^{\alpha,\beta}z(\xi_i), \int_0^{\xi_i} g(\xi_i,s,z(s))ds\right)\right]t^{\gamma-1} \\ &\leq \frac{M_1M_2}{(1-M_1)\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i} I_{0^+}^{\alpha}\left[\int_0^{\xi_i}|y(s) - z(s)|ds\right], \end{aligned}$$

taking into account that

$$
\begin{aligned}
I_{0+}^{\alpha}\left(\int_0^{\xi_i}|y(s)-z(s)|ds\right) &= \frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}(\xi_i-r)^{\alpha-1}\left(\int_0^{r}|y(s)-z(s)|ds\right)dr\\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}\int_0^{r}(\xi_i-r)^{\alpha-1}|y(s)-z(s)|dsdr\\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}|y(s)-z(s)|\int_s^{\xi_i}(\xi_i-r)^{\alpha-1}drds\\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^{\xi_i}(\xi_i-s)^{\alpha}|y(s)-z(s)|ds\\
&= I_{0+}^{\alpha+1}|y(\xi_i)-z(\xi_i)|, \qquad (3.19)
\end{aligned}
$$

if $y(\xi_i)=z(\xi_i)$, we deduce that $K_y=K_z$. Therefore, from (3.15) and (3.18), we find

$$
\begin{aligned}
|z(t)-y(t)| \leq{}& \left(\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\frac{\Gamma(\mu_i+\frac{\alpha}{\eta_i}+1)\lambda_i}{\Gamma(\alpha+1)\Gamma(\mu_i+\frac{\alpha}{\eta_i}+\delta_i+1)}T^{\gamma+\alpha-1}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)\varepsilon\\
&+\frac{M_1M_2}{1-M_1}I_{0+}^{\alpha}\left(\int_0^{t}|y(s)-z(s)|ds\right),
\end{aligned}
$$

in view of (3.19), we obtain

$$
\begin{aligned}
|z(t)-y(t)| \leq{}& \left(\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\frac{\Gamma(\mu_i+\frac{\alpha}{\eta_i}+1)\lambda_i}{\Gamma(\alpha+1)\Gamma(\mu_i+\frac{\alpha}{\eta_i}+\delta_i+1)}T^{\gamma+\alpha-1}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)\varepsilon\\
&+\frac{M_1M_2}{(1-M_1)\Gamma(\alpha+1)}\int_0^{t}(t-r)^{\alpha}|y(r)-z(r)|dr,
\end{aligned}
$$

using Lemma 2.9, we infer

$$
\begin{aligned}
&|z(t)-y(t)|\\
&\leq \left(\frac{1}{\Gamma(\gamma)\Lambda}\sum_{i=1}^{m}\frac{\Gamma(\mu_i+\frac{\alpha}{\eta_i}+1)\lambda_i}{\Gamma(\alpha+1)\Gamma(\mu_i+\frac{\alpha}{\eta_i}+\delta_i+1)}T^{\gamma+\alpha-1}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha}\left(\frac{M_1M_2}{1-M_1}T^{\alpha+1}\right)\varepsilon\\
&:= c_f\varepsilon.
\end{aligned}
$$

□

4. Examples

To illustrate the applicability of our results, we present the following examples:

Example 4.1 Consider the following Hilfer-type fractional integro-differential problem subject to non-local initial condition

$$
\begin{cases}
D_{0^+}^{\alpha,\beta}y(t) &= \dfrac{1}{5(2+t)}\left[\sin(D_{0^+}^{\alpha,\beta}y(t))+\dfrac{1}{10}\displaystyle\int_0^{t}\frac{s}{1+e^{t}}y(s)ds\right], \quad t\in(0,1]\\
I_{0^+}^{1-\gamma}y(0) &= \displaystyle\sum_{i=1}^{m}\lambda_i I_{\eta_i}^{\mu_i,\delta_i}y(\xi_i),
\end{cases} \qquad (4.1)
$$

where

$$
\begin{aligned}
&m=3, \alpha=1/2, \beta=2/3, \gamma=5/6, \lambda_1=\pi, \lambda_2=\sqrt{7}/5, \lambda_3=ln(2)/3,\\
&\mu_1=e^2/2, \mu_2=2/5, \mu_3=1/3, \delta_1=1/7, \delta_2=9/2, \delta_3=5/2,\\
&\eta_1=3/8, \eta_2=e/2, \eta_3=\sqrt{\pi}/6, \xi_1=3/4, \xi_2=2/3, \xi_3=2/7.
\end{aligned}
$$

Define the functions f, g by

$$f(t, y(t), z(t)) = \frac{1}{5(2+t)}\left[\sin(y(t)) + z(t)\right], \quad t \in (0,1], y, z \in \mathbb{R}$$
$$g(t, s, y(t)) = \frac{s}{1+e^t} y(t), \quad s, t \in [0,1], y \in \mathbb{R}.$$

For each $y, \bar{y}, z, \bar{z} \in \mathbb{R}$ and $t \in (0, T]$, we have

$$|f(t, y, z) - f(t, \bar{y}, \bar{z})| \leq \frac{1}{10}\left(|y - \bar{y}| + |z - \bar{z}|\right);$$
$$|g(t, s, y) - g(t, s, \bar{y})| \leq \frac{1}{2}|y - \bar{y}|.$$

The Hypotheses (H_1) and (H_2) hold, with $M_1 = \frac{1}{10}$ and $M_2 = \frac{1}{2}$ respectively. By using the Matlab program, we can find that $\Lambda = -1.5320$, then $\sigma_1 = 0.0040 < 1$. As all hypotheses of Theorem 3.2 are satisfied, problem (4.1) admits a unique solution on $[0, 1]$. Moreover, Theorem 3.4 guarantees that problem (4.1) is Ulam–Hyers stable.

Example 4.2 Consider the following Hilfer fractional integro-differential equation with nonlocal initial condition:

$$\begin{cases} D_{0^+}^{\alpha,\beta} y(t) &= \dfrac{1}{2(1+|y(t)|)} + \dfrac{1}{2}\displaystyle\int_0^t ts^2(1 + arctg(y(s))ds, \quad t \in (0,2], \\ I_{0^+}^{1-\gamma} y(0) &= \displaystyle\sum_{i=1}^{m} \lambda_i I_{\eta_i}^{\mu_i,\delta_i} y(\xi_i), \end{cases} \tag{4.2}$$

where

$$m = 3, \alpha = 1/2, \beta = 1/2, \gamma = 3/4, \lambda_1 = -e, \lambda_2 = \pi/\sqrt{2}, \lambda_3 = e^3/2,$$
$$\mu_1 = 1/8, \mu_2 = -5/6, \mu_3 = \sqrt{e}, \delta_1 = 9/4, \delta_2 = 7/4, \delta_3 = 11/6,$$
$$\eta_1 = 1/ln(2), \eta_2 = 3/5, \eta_3 = 1/3, \xi_1 = 1/2, \xi_2 = 4/3, \xi_3 = 1.$$

Define the functions f, g by

$$f(t, y(t), z(t)) = \frac{1}{2(1+|y(t))|} + \frac{z(t)}{2}, \quad t \in (0,1], y, z \in \mathbb{R}$$
$$g(t, s, y(t)) = ts^2(1 + arctg(y(t)), \quad s, t \in [0,1], y \in \mathbb{R}.$$

For each $y, \bar{y}, z, \bar{z} \in \mathbb{R}$ and $t \in (0, T]$, we have

$$|f(t, y, z) - f(t, \bar{y}, \bar{z})| \leq \frac{1}{2}\left(|y - \bar{y}| + |z - \bar{z}|\right);$$
$$|g(t, s, y) - g(t, s, \bar{y})| \leq |y - \bar{y}|.$$

This means that the assumptions (H_1) and (H_2) hold, with $M_1 = \frac{1}{2}$ and $M_2 = 1$ respectively. By using the Matlab program, we can find that $\Lambda = 9.6307$, then $\sigma_2 = 0.0697 < 1$. Thus, by using Theorem 3.2, the problem (4.2) has at least one solution on $[0, 2]$.

Conclusion

This paper delved into the analysis of a Hilfer fractional integro-differential equation equipped with a non-local initial condition. By establishing an equivalence between the given problem and a Volterra integral equation, we employed fixed point theory to guarantee the existence and uniqueness of solutions. Furthermore, the Ulam-Hyers stability of the problem was demonstrated. Future research will focus on extending these results to coupled systems of fractional differential inclusions involving various fractional derivative types and boundary conditions.

Acknowledgments

We thank the referee by your suggestions.

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