On Cartesian product of matrices

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ABSTRACT: Bapat and Kurata [*Linear Algebra Appl.*, 562(2019), 135-153] defined the Cartesian product of two square matrices A and B as $A \oslash B = A \otimes \mathbb{J} + \mathbb{J} \otimes B$, where $\mathbb{J}$ is the matrix of all one of appropriate order and $\otimes$ is the Kronecker product. In this article, we find the expression for the trace of the Cartesian product of any finite number of square matrices in terms of the traces of the individual matrices. Also, we establish some identities involving the Cartesian product of matrices. Finally, we apply the Cartesian product to study some graph-theoretic properties.

Key Words: Cartesian product, Kronecker product, Hadamard product, Trace.

Contents

1. Introduction and terminology

By $\mathbb{M}_{m,n}$, we denote the class of all matrices of size $m \times n$. Also, by $\mathbb{M}_n$, we denote the class of all square matrices of order n. For $M \in \mathbb{M}_n$ we write m_{ij} or M_{ij} to denote the $ij-$th element of M. By $\mathbb{J}$ and $\mathbb{1}$, we mean the matrix of all one's and vector of all one's, respectively of suitable order. Similarly, 0 denotes the zero matrix or the vector. We will mention their order wherever it is necessary. Throughout this article, we denote the sum of all entries of a matrix A by S_A and the sum of the entries of $i-$th row of A by A_i. The inertia of a square matrix M with real eigenvalues is the triplet $(n_+(M), n_0(M), n_-(M))$, where $n_+(M)$ and $n_-(M)$ denote the number of positive and negative eigenvalues of M, respectively, and $n_0(M)$ is the algebraic multiplicity of 0 as an eigenvalue of M.

The *Kronecker product* of two matrices A and B of sizes $m \times n$ and $p \times q$, respectively, denoted by $A \otimes B$ is defined to be the $mp \times nq$ block matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{pmatrix}.$$

The *Hadamard product* of two matrices A and B of the same size, denoted by $A \circ B$ is defined to be the entry-wise product $A \circ B = [a_{i,j}b_{i,j}]$.

Bapat and Kurata [2] defined the *Cartesian product* of two square matrices $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$ as $A \oslash B = A \otimes \mathbb{J}_n + \mathbb{J}_m \otimes B$. The authors proved the Cartesian product to be associative. We use $A^{[k]}$ to mean $\underbrace{A \oslash A \oslash \cdots \oslash A}_{k \; times}$.

If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $A \oslash B$ can be considered as a block matrix with $i, j-$th block $a_{i,j}\mathbb{J}_n + B$, $i = 1, 2, \ldots, m$, in other words $A \oslash B$ is the matrix obtained from A replacing $a_{i,j}$ with $a_{i,j}\mathbb{J}_n + B$. It can be observed that $a_{i,j} + b_{p,q}$ is the $p, q-$th entry of the $i, j-$th block of $A \oslash B$.

All graphs considered here are finite, undirected, connected, and simple. The *distance* between two vertices $u, v \in V(G)$ is denoted by d_{uv} and is defined as the length of a shortest path between u and v in G. The *distance matrix* of G is denoted by $\mathcal{D}(G)$ and is defined by $\mathcal{D}(G) = (d_{uv})_{u,v \in V(G)}$. Since $\mathcal{D}(G)$ is a real symmetric matrix, all its eigenvalues are real. For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we have

$$x^T \mathcal{D}(G) x = \sum_{1 \leq i < j \leq n} d_{ij} x_i x_j.$$

The *Wiener index* $W(G)$ of a graph is the sum of the distances between all unordered pairs of vertices of G, in other words $W(G) = \frac{S_{\mathcal{D}(G)}}{2}$. The *distance spectral radius* $\rho^{\mathcal{D}}(G)$ of G is the largest eigenvalue of its distance matrix $\mathcal{D}(G)$. The transmission, denoted by $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G.

The *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and in which two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in G_2, or $u' = v'$ and u is adjacent to v in G_1. Let $Gu * Hv$ denote the graph obtained from two graphs G and H by identifying a vertex u from G with a vertex v from H.

The article have been organized as follows. In section 2, we discuss some existing results involving the Kronecker product of matrices and the Cartesian product of graphs. In section 3, we find trace of various compositions of matrices involving Cartesian products. Again in section 4, we obtain some identities involving Cartesian product of matrices and find some applications in graph theory.

2. Preliminaries

Kronecker product has been extensively studied in the literature. Some of the interesting properties of the Kronecker product are given below.

Lemma 2.1. *[5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \times \operatorname{tr}(B)$.*

Lemma 2.2. *[5] If A, B and C are matrices of suitable orders, then*

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C) \ \ and$$
$$(A + B) \otimes C = (A \otimes C) + (B \otimes C).$$

Lemma 2.3. *[5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $(A \otimes B)^T = A^T \otimes B^T$.*

Lemma 2.4. *[5] If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $(A \otimes B)^* = A^* \otimes B^*$.*

Lemma 2.5. *[5] For $A \in \mathbb{M}_m$, $B \in \mathbb{M}_n$, and $a, b \in \mathbb{C}$, $aA \otimes bB = ab A \otimes B$.*

Lemma 2.6. *[5] For matrices A, B, C and D of appropriate sizes*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Lemma 2.7. *[5] For any $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, there exists a permutation matrix P such that*

$$P^{-1}(A \otimes B)P = B \otimes A.$$

For more results on Kronecker product, we refer to [3]. The Cartesian product of two graphs have been studied by many researchers. Here we are interested in Cartesian product of two matrices because for any two connected graphs G_1 and G_2, the distance matrix of $G_1 \square G_2$ equals to the Cartesian product of the distance matrices of G_1 and G_2, i.e. $\mathcal{D}(G_1 \square G_2) = \mathcal{D}(G_1) \oslash \mathcal{D}(G_2)$. Zhang and Godsil [6] found the distance inertia of the Cartesian product of two graphs.

Theorem 2.8. *[6] If G and H are two connected graphs, where $V(G) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1, \ldots, v_n\}$, then, the inertia of distance matrix of $G \square H$ is $(n_+(Gu_m * Hu_n), (m-1)(n-1) + n_0(Gu_m * Hu_n), n_-(Gu_m * Hu_n))$.*

Corollary 2.9. *[6] Let T_1 and T_2 be two trees on m and n vertices, respectively. Then the distance inertia of $T_1 \square T_2$ is $(1, (m-1)(n-1), m+n-2)$.*

3. Trace of Cartesian product

Here we consider different compositions and products involving Cartesian products of matrices and evaluate their trace.

Lemma 3.1. *If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $\mathrm{tr}(A \oslash B) = n.\mathrm{tr}(A) + m.\mathrm{tr}(B)$.*

Proof. We have

$$\begin{aligned}
\mathrm{tr}(A \oslash B) &= \mathrm{tr}(A \otimes \mathbb{J}_n + \mathbb{J}_n \otimes B) \\
&= \mathrm{tr}(A \otimes \mathbb{J}_n) + \mathrm{tr}(\mathbb{J}_m \otimes B) \\
&= \mathrm{tr}(A) \times \mathrm{tr}(\mathbb{J}_n) + \mathrm{tr}(\mathbb{J}_m) \times \mathrm{tr}(B) \qquad \text{[using } theorem \text{ 2.1]} \\
&= n.\mathrm{tr}(A) + m.\mathrm{tr}(B).
\end{aligned}$$

$\square$

Theorem 3.2. *If $A_i \in \mathbb{M}_{n_i}$ and $k_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$, then*

$$\mathrm{tr}(k_1 A_1 \oslash k_2 A_2 \oslash \cdots \oslash k_n A_n) = \Big(\prod_{i=1}^{n} n_i \Big) \sum_{i=1}^{n} \frac{k_i \mathrm{tr}(A_i)}{n_i}.$$

Proof. We prove the result by induction on n. For $n = 1$, there is nothing to prove. For $n = 2$, the result follows from theorem 3.1. Suppose the result holds for $n = \ell \leq n - 1$. That is

$$\mathrm{tr}(k_1 A_1 \oslash k_2 A_2 \oslash \cdots \oslash k_\ell A_\ell) = \Big(\prod_{i=1}^{\ell} n_i \Big) \sum_{i=1}^{\ell} \frac{k_i \mathrm{tr}(A_i)}{n_i}. \tag{3.1}$$

Now

$$\begin{aligned}
&\mathrm{tr}(k_1 A_1 \oslash k_2 A_2 \oslash \cdots \oslash k_\ell A_\ell \oslash k_{\ell+1} A_{\ell+1}) \\
&= k_{\ell+1} n_{\ell+1} \mathrm{tr}(k_1 A_1 \oslash k_2 A_2 \oslash \cdots \oslash k_\ell A_\ell) + \Big(\prod_{i=1}^{\ell} k_i n_i \Big) \mathrm{tr}(A_{\ell+1}) \qquad \text{[using } theorem \text{ 3.1]} \\
&= k_{\ell+1} n_{\ell+1} \Big(\prod_{i=1}^{\ell} n_i \Big) \sum_{i=1}^{\ell} \frac{k_i \mathrm{tr}(A_i)}{n_i} + \Big(\prod_{i=1}^{\ell} k_i n_i \Big) \mathrm{tr}(A_{\ell+1}) \qquad \text{[using } eq. \text{ (3.1)]} \\
&= \Big(\prod_{i=1}^{\ell+1} n_i \Big) \sum_{i=1}^{\ell+1} \frac{k_i \mathrm{tr}(A_i)}{n_i}.
\end{aligned}$$

Hence the result follows by induction. $\square$

As immediate corollary of the above theorem we get the following result.

Corollary 3.3. *For $A \in \mathbb{M}_n$, $\mathrm{tr}(A^{[k]}) = k.n^{k-1}\mathrm{tr}(A)$.*

Proposition 3.4. *If $A, B \in \mathbb{M}_n$, then*

$$\mathrm{tr}\big((A + B) \oslash (A - B)\big) = 2n.\mathrm{tr}(A).$$

Proof. From theorem 3.1, we have

$$\begin{aligned}
\mathrm{tr}\big((A + B) \oslash (A - B)\big) &= n.\mathrm{tr}(A - B) + n.\mathrm{tr}(A + B) \\
&= n[\mathrm{tr}(A) - \mathrm{tr}(B) + \mathrm{tr}(A) + \mathrm{tr}(B)] \\
&= 2n.\mathrm{tr}(A).
\end{aligned}$$

$\square$

Proposition 3.5. *For $A \in \mathbb{M}_m, B_i \in \mathbb{M}_n; i = 1, 2, \ldots, k$, then*

$$\operatorname{tr}(A \otimes (B_1 \oslash B_2 \oslash \cdots \oslash B_k)) = n^{k-1}\operatorname{tr}(A)\sum_{i=1}^{k}\operatorname{tr}(B_i).$$

Proof. We have

$$\operatorname{tr}(A \otimes (B_1 \oslash B_2 \oslash \cdots \oslash B_k))$$
$$= \operatorname{tr}(A).\operatorname{tr}(B_1 \oslash B_2 \oslash \cdots \oslash B_k) \qquad [\text{using } theorem\ 2.1]$$
$$= \operatorname{tr}(A).n^{k-1}\sum_{i=1}^{k}\operatorname{tr}(B_i) \qquad [\text{using } theorem\ 3.2]$$
$$= n^{k-1}\operatorname{tr}(A)\sum_{i=1}^{k}\operatorname{tr}(B_i).$$

$\square$

Theorem 3.6. *If $A_i \in \mathbb{M}_{n_i}$ for $i = 1, 2, \ldots, t$, then*

$$\operatorname{tr}\big[(A_1 \oslash A_2 \oslash \cdots \oslash A_\ell) \otimes (A_{\ell+1} \oslash A_{\ell+2} \oslash \cdots \oslash A_m) \otimes \cdots \otimes (A_r \oslash A_{r+1} \oslash \cdots \oslash A_t)\big]$$
$$= \prod_{p=1}^{t} n_p \sum_{i=1}^{\ell} \frac{\operatorname{tr}(A_i)}{n_i} \sum_{j=\ell+1}^{m} \frac{\operatorname{tr}(A_j)}{n_j} \cdots \sum_{k=r}^{t} \frac{\operatorname{tr}(A_k)}{n_k}.$$

Proof. By repeated application of theorem 2.1 we get

$$\operatorname{tr}\big[(A_1 \oslash A_2 \oslash \cdots \oslash A_\ell) \otimes (A_{\ell+1} \oslash A_{\ell+2} \oslash \cdots \oslash A_m) \otimes \cdots \otimes (A_r \oslash A_{r+1} \oslash \cdots \oslash A_t)\big]$$
$$= \big[\operatorname{tr}(A_1 \oslash A_2 \oslash \cdots \oslash A_\ell)\big]\big[\operatorname{tr}(A_{\ell+1} \oslash A_{\ell+2} \oslash \cdots \oslash A_m)\big]\cdots\big[\operatorname{tr}(A_r \oslash A_{r+1} \oslash \cdots \oslash A_t)\big]$$
$$= \Big(\prod_{i=1}^{\ell} n_i \sum_{i=1}^{\ell} \frac{\operatorname{tr}(A_i)}{n_i}\Big)\Big(\prod_{\ell+1}^{m} n_j \sum_{j=\ell+1}^{m} \frac{\operatorname{tr}(A_j)}{n_j}\Big)\cdots\Big(\prod_{k=r}^{t} n_k \sum_{k=r}^{t} \frac{\operatorname{tr}(A_k)}{n_k}\Big) \quad [\text{using } theorem\ 3.2]$$
$$= \prod_{p=1}^{t} n_p \sum_{i=1}^{\ell} \frac{\operatorname{tr}(A_i)}{n_i} \sum_{j=\ell+1}^{m} \frac{\operatorname{tr}(A_j)}{n_j} \cdots \sum_{k=r}^{t} \frac{\operatorname{tr}(A_k)}{n_k}.$$

$\square$

Theorem 3.7. *If $A_i \in \mathbb{M}_{n_i}; i = 1, 2, \ldots, t$, then*

$$\operatorname{tr}\big[(A_1 \otimes A_2 \otimes \cdots \otimes A_\ell) \oslash (A_{\ell+1} \otimes A_{\ell+2} \otimes \cdots \otimes A_m) \oslash \cdots \oslash (A_r \otimes A_{r+1} \otimes \cdots \otimes A_t)\big]$$
$$= \prod_{p=1}^{t} n_p \Big[\prod_{i=1}^{\ell} \frac{\operatorname{tr}(A_i)}{n_i} + \prod_{j=\ell+1}^{m} \frac{\operatorname{tr}(A_i)}{n_j} + \cdots + \prod_{k=r}^{t} \frac{\operatorname{tr}(A_k)}{n_k}\Big].$$

Proof. Using theorem 3.2 and then theorem 2.1, we get

$$\operatorname{tr}\big[(A_1 \otimes A_2 \otimes \cdots \otimes A_\ell) \oslash (A_{\ell+1} \otimes A_{\ell+2} \otimes \cdots \otimes A_m) \oslash \cdots \oslash (A_r \otimes A_{r+1} \otimes \cdots \otimes A_t)\big]$$
$$= \prod_{p=1}^{t} n_p \Big[\frac{\operatorname{tr}(A_1 \otimes A_2 \otimes \cdots A_\ell)}{\prod_{i=1}^{\ell} n_i} + \frac{\operatorname{tr}(A_{\ell+1} \otimes A_{\ell+2} \cdots \otimes A_m)}{\prod_{j=\ell+1}^{m} n_j} + \cdots + \frac{\operatorname{tr}(A_r \otimes \cdots \otimes A_t)}{\prod_{k=r}^{t} n_k}\Big]$$
$$= \prod_{p=1}^{t} n_p \Big[\frac{\prod_{i=1}^{\ell} \operatorname{tr}(A_i)}{\prod_{i=1}^{\ell} n_i} + \frac{\prod_{j=\ell+1}^{m} \operatorname{tr}(A_i)}{\prod_{j=1}^{m} n_j} + \cdots + \frac{\prod_{k=r}^{t} \operatorname{tr}(A_k)}{\prod_{k=r}^{t} n_k}\Big]$$
$$= \prod_{p=1}^{t} n_p \Big[\prod_{i=1}^{\ell} \frac{\operatorname{tr}(A_i)}{n_i} + \prod_{j=\ell+1}^{m} \frac{\operatorname{tr}(A_i)}{n_j} + \cdots + \prod_{k=r}^{t} \frac{\operatorname{tr}(A_k)}{n_k}\Big].$$

$\square$

4. Some identities and applications

From the definition of Cartesian product of two matrices, we get following remarks.

Remark 4.1. *If A and B are square matrices and $k \in \mathbb{C}$, then $kA \oslash kB = k(A \oslash B)$.*

Remark 4.2. *For $A \in \mathbb{M}_n$ and any $k \in \mathbb{C}$, $k \oslash A = A + k\mathbb{J}_n = A \oslash k$.*

For any square matrices A and B, from the definitions of Kronecker product and Cartesian product, it can be observed that if $a_{i,j}b_{p,q}$ is an entry of $A \otimes B$ then the corresponding entry of $A \oslash B$ is $a_{i,j} + b_{p,q}$. Thus from theorem 2.7, we see that if $P^{-1}(A \otimes B)P = B \otimes A$, then for the same P, we get $P^{-1}(A \oslash B)P = B \oslash A$. Thus we get the following result.

Remark 4.3. *If A and B are square matrices, then $A \oslash B$ is permutation similar to $B \oslash A$.*

Proposition 4.4. *For $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, $(A \oslash B)^T = A^T \oslash B^T$.*

Proof. By definition we have
$$A \oslash B = A \otimes \mathbb{J}_n + J_m \otimes B$$
which implies
$$\begin{aligned} (A \oslash B)^T &= (A \otimes \mathbb{J}_n + J_m \otimes B)^T \\ &= (A \otimes \mathbb{J}_n)^T + (\mathbb{J}_m \otimes B)^T \\ &= A^T \otimes \mathbb{J}_n^T + \mathbb{J}_m^T \otimes B^T \qquad [\text{using } lemma\ 2.3] \\ &= A^T \otimes \mathbb{J}_n + \mathbb{J}_m \otimes B^T \\ &= A^T \oslash B^T. \end{aligned}$$

Hence the result. $\square$

By repeated application of theorem 4.4, we get the following result as a corollary.

Corollary 4.5. *For square matrices A_i for $i = 1, 2, \ldots, n$,*
$$(A_1 \oslash A_2 \oslash \cdots \oslash A_n)^T = A_1^T \oslash A_2^T \oslash \cdots \oslash A_n^T.$$

Proceeding as in theorem 4.4 and using theorem 2.4, we get the following result.

Proposition 4.6. *If $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, then $(A \oslash B)^* = A^* \oslash B^*$.*

By repeated application of theorem 4.6, we get the following result as a corollary.

Corollary 4.7. *If A_i is a square matrix for each $i = 1, 2, \ldots, n$, then*
$$(A_1 \oslash A_2 \oslash \cdots \oslash A_n)^* = A_1^* \oslash A_2^* \oslash \cdots \oslash A_n^*.$$

Theorem 4.8. *If $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, then $A \oslash B$ is symmetric if and only if A and B are both symmetric.*

Proof. If A and B are both symmetric, then $A^T = A$ and $B^T = B$. Now
$$\begin{aligned} (A \oslash B)^T &= A^T \oslash B^T \qquad [\text{by } theorem\ 4.4] \\ &= A \oslash B. \end{aligned}$$

Therefore $A \oslash B$ is symmetric.

Conversely, suppose that $A \oslash B$ is symmetric. Then $1,1$ block of $A \oslash B$ must be symmetric. But $1,1$ block of $A \oslash B$ is $a_{1,1}\mathbb{J}_n + B$ which is symmetric if and only if B is symmetric. Again since $A \oslash B$ is symmetric, the $1,1$ entry of any i,j−th block of $A \oslash B$ must be same as $1,1$ entry of j,i−th block of $A \oslash B$. That is $a_{i,j} + b_{1,1} = a_{j,i} + b_{1,1}$ for all $i,j = 1, 2 \ldots, n$. Which implies that A is symmetric. $\square$

Theorem 4.9. *If $A \in \mathbb{M}_m, B \in \mathbb{M}_n$, then $A \oslash B$ is skew-symmetric if and only if A and B are both skew-symmetric.*

Proof. If A and B are both skew-symmetric, then $A^T = -A$ and $B^T = -B$. Now

$$
\begin{aligned}
(A \oslash B)^T &= A^T \oslash B^T \qquad \text{[by \textit{theorem} 4.4]} \\
&= (-A) \oslash (-B) \\
&= (-A) \otimes \mathbb{J}_n + \mathbb{J}_m \otimes (-B) \\
&= -A \otimes \mathbb{J}_n - \mathbb{J}_n \otimes B \qquad \text{[by \textit{theorem} 2.5]} \\
&= -A \oslash B.
\end{aligned}
$$

Therefore $A \oslash B$ is skew-symmetric.

The other direction is similar to that of the proof of theorem 4.8.   □

Theorem 4.10. *If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, the $A \oslash B$ is a diagonal matrix if and only if $A = k\mathbb{J}_m$ and $B = -k\mathbb{J}_n$ for some $k \in \mathbb{C}$. Furthermore in that case $A \oslash B = 0$.*

Proof. If $A = k\mathbb{J}_m$ and $B = -k\mathbb{J}_n$ for some $k \in \mathbb{C}$, then

$$
\begin{aligned}
A \oslash B &= k\mathbb{J}_m \otimes \mathbb{J}_n + \mathbb{J}_m \otimes (-k\mathbb{J}_n) \\
&= 0. \qquad \text{[using \textit{theorem} 2.5]}
\end{aligned}
$$

Again if $A \oslash B$ is a diagonal matrix, then we must have

$$
\begin{aligned}
&a_{i,i} + b_{p,q} = 0 \text{ for } i = 1, 2, \ldots, m \text{ and } p, q = 1, 2, \ldots, n; \ p \neq q, \\
&a_{i,j} + b_{p,p} = 0 \text{ for } i, j = 1, 2, \ldots, m; \ i \neq j \text{ and } p, q = 1, 2, \ldots, n; \\
&a_{i,j} + b_{p,q} = 0 \text{ for } i = 1, 2, \ldots, m; \ i \neq j \text{ and } p = 1, 2, \ldots, n; \ p \neq q.
\end{aligned}
$$

Solving all those equations we see that all entries of A are equal (say k) and all entries of B are also equal $(-k)$. Thus we get our required result.   □

Corollary 4.11. *There exist no square matrices A, B such that $A \oslash B = I$.*

Theorem 4.12. *If $A, C \in \mathbb{M}_m$, $B, D \in \mathbb{M}_n$, then $A \oslash B = C \oslash D$ if and only if $C = A - k\mathbb{J}_m$ and $D = B + k\mathbb{J}_n$ for some $k \in \mathbb{C}$.*

Proof. If $C = A - k\mathbb{J}_m$ and $D = B + k\mathbb{J}_n$ for some $k \in \mathbb{C}$, then

$$
\begin{aligned}
C \oslash D &= (A - k\mathbb{J}_m) \oslash (B + k\mathbb{J}_n) \\
&= (A - k\mathbb{J}_m) \otimes \mathbb{J}_n + \mathbb{J}_m \otimes (B + k\mathbb{J}_n) \\
&= A \otimes \mathbb{J}_m - k\mathbb{J}_m \otimes \mathbb{J}_n + \mathbb{J}_m \otimes B + \mathbb{J}_m \otimes k\mathbb{J}_n \\
&= A \oslash B.
\end{aligned}
$$

Conversely, suppose that $A \oslash B = C \oslash D$. Then every block of $A \oslash B$ equals to the corresponding block of $C \oslash D$,

$$
\text{i.e. } a_{i,j}\mathbb{J}_n + B = c_{i,j}\mathbb{J}_n + D \text{ for } i, j = 1, 2, \ldots, m.
$$

Which implies that $a_{i,j} + b_{p,q} = c_{i,j} + d_{p,q}$ for any $i, j = 1, 2, \ldots, m$ and $p, q = 1, 2, \ldots, n$. That is $a_{i,j} - c_{i,j} = d_{p,q} - b_{p,q}$ for any $i, j = 1, 2, \ldots, m$ and $p, q = 1, 2, \ldots, n$. Therefore we must have $A - C = \lambda\mathbb{J}_m$ and $D - B = \lambda\mathbb{J}_n$ for some $\lambda \in \mathbb{C}$. Hence the theorem follows.   □

Theorem 4.13. *If $A, B \in \mathbb{M}_n$, then $A \oslash B = B \oslash A$ if and only if $B = A + k\mathbb{J}_n$ for some $k \in \mathbb{C}$.*

Proof. If $B = A + k\mathbb{J}_n$, then by direct calculation we have

$$A \oslash B = B \oslash A = A \oslash A + k\mathbb{J}_{n^2}.$$

Now suppose $A \oslash B = B \oslash A$. Then $a_{i,j} + b_{p,q} = b_{i,j} + a_{p,q}$ for all $i, j, p, q = 1, 2 \dots, n$. Therefore

$$\sum_{p,q=1}^{n} (a_{i,j} + b_{p,q}) = \sum_{p,q=1}^{n} (b_{i,j} + a_{p,q})$$

which gives $n^2 a_{i,j} + S_B = n^2 b_{i,j} + S_A$

i.e. $b_{i,j} = a_{i,j} + \dfrac{S_B - S_A}{n^2}$ for all $i, j = 1, 2, \dots, n.$

Thus $B = A + k\mathbb{J}_n$ for $k = \frac{S_B - S_A}{n^2}.$ $\qquad\square$

Theorem 4.14. *If $A, B, C, D \in \mathbb{M}_n$, then*

(i) $(A \oslash B)(C \oslash D) = AC \oslash BD + A\mathbb{J}_n \otimes \mathbb{J}_n B + \mathbb{J}_n C \otimes D\mathbb{J}_n.$

(ii) $(A \oslash B) \circ (C \oslash D) = (A \circ C) \oslash (B \circ D) + A \otimes D + C \otimes B.$

Proof. (i) We have

$$\begin{aligned}
(A \oslash B)(C \oslash D) &= (A \otimes \mathbb{J}_n + \mathbb{J}_n \otimes B)(C \otimes \mathbb{J}_n + \mathbb{J}_n \otimes D) \\
&= (A \otimes \mathbb{J}_n)(C \otimes \mathbb{J}_n) + (A \otimes \mathbb{J}_n)(\mathbb{J}_n \otimes B) + (\mathbb{J}_n \otimes B)(C \otimes \mathbb{J}_n) \\
&\quad + (\mathbb{J}_n \otimes B)(\mathbb{J}_n \otimes D) \\
&= AC \otimes \mathbb{J}_{n^2} + A\mathbb{J}_n \otimes \mathbb{J}_n B + \mathbb{J}_n C \otimes B\mathbb{J}_n + \mathbb{J}_{n^2} \otimes BD \\
&= AC \oslash BD + A\mathbb{J}_n \otimes \mathbb{J}_n B + \mathbb{J}_n C \otimes D\mathbb{J}_n.
\end{aligned}$$

(ii) Here

$$\begin{aligned}
(A \oslash B) \circ (C \oslash D) &= (A \otimes \mathbb{J}_n + \mathbb{J}_n \otimes B) \circ (C \otimes \mathbb{J}_n + \mathbb{J}_n \otimes D) \\
&= (A \otimes \mathbb{J}_n) \circ (C \otimes \mathbb{J}_n) + (A \otimes \mathbb{J}_n) \circ (\mathbb{J}_n \otimes D) + (\mathbb{J}_n \otimes B) \circ (C \otimes \mathbb{J}_n) \\
&\quad + (\mathbb{J}_n \otimes B) \circ (\mathbb{J}_n \otimes D) \\
&= (A \circ C) \otimes \mathbb{J}_n + A \otimes D + C \otimes B + \mathbb{J}_n \otimes (B \circ D) \\
&= (A \circ C) \oslash (B \circ D) + A \otimes D + C \otimes B.
\end{aligned}$$

$\qquad\square$

Proposition 4.15. *For matrices A, B, C of suitable orders,*

$$(A + B) \oslash C = \frac{1}{2}[A \oslash C + B \oslash C + (A + B) \otimes \mathbb{J}_n]$$

and

$$A \oslash (B + C) = \frac{1}{2}[A \oslash B + A \oslash C + \mathbb{J} \otimes (B + C)].$$

Proof. We prove only the first result as the second one can be proved similarly. If A and B are matrices of same order (say m) and matrix C is of order n, then

$$\begin{aligned}
(A + B) \oslash C &= (A + B) \otimes \mathbb{J}_n + \mathbb{J}_m \otimes C \\
&= A \otimes \mathbb{J}_n + B \otimes \mathbb{J}_n + \mathbb{J}_m \otimes C. \qquad \text{[since $\otimes$ is distributive]}
\end{aligned} \tag{4.1}$$

From eq. (4.1), we have

$$(A + B) \oslash C = A \otimes \mathbb{J}_n + B \oslash C \tag{4.2}$$

and

$$(A + B) \oslash C = A \oslash C + B \otimes \mathbb{J}_n. \tag{4.3}$$

Now adding eq. (4.2) and eq. (4.3), we get

$$2((A + B) \oslash C) = A \oslash C + B \oslash C + (A + B) \otimes \mathbb{J}_n.$$

Hence the result follows. □

Theorem 4.16. *If $A_i \in \mathbb{M}_m$, $B_i \in \mathbb{M}_n$ for $i = 1, 2, \ldots, k$, then*

$$\Big(\sum_{i=1}^{k} A_i \Big) \oslash \Big(\sum_{i=1}^{k} B_i \Big) = \sum_{i=1}^{k} (A_i \oslash B_i).$$

Proof. We prove the result by induction on k. For $k = 1$, the result is trivial. For $k = 2$, we have

$$\begin{aligned}
(A_1 + A_2) \oslash (B_1 + B_2) &= (A_1 + A_2) \otimes \mathbb{J}_n + \mathbb{J}_m \otimes (B_1 + B_2) \\
&= A_1 \otimes \mathbb{J}_n + A_2 \otimes \mathbb{J}_n + \mathbb{J}_m \otimes B_1 + \mathbb{J}_m \otimes B_2 \quad [\text{since } \otimes \text{ is distributive}] \\
&= A_1 \oslash B_1 + A_2 \oslash B_2. \tag{4.4}
\end{aligned}$$

Thus the result holds for $k = 2$. Suppose the identity holds for $k = 1, 2, \ldots, \ell < k$, then

$$\begin{aligned}
\Big(\sum_{i=1}^{\ell+1} A_i \Big) \oslash \Big(\sum_{i=1}^{\ell+1} B_i \Big) &= \Big(\sum_{i=1}^{\ell} A_i \Big) \oslash \Big(\sum_{i=1}^{\ell} B_i \Big) + A_{\ell+1} \oslash B_{\ell+1} \quad [\text{by } eq. \text{ (4.4)}] \\
&= \sum_{i=1}^{\ell} (A_i \oslash B_i) + A_{\ell+1} \oslash B_{\ell+1} \quad [\text{by induction hypothesis}] \\
&= \sum_{i=1}^{\ell+1} (A_i \oslash B_i).
\end{aligned}$$

Hence the result follows. □

Using theorem 4.16 repeatedly, we get the following general result.

Theorem 4.17. *For $A_i \in \mathbb{M}_m, B_i \in \mathbb{M}_n, \ldots, C_i \in \mathbb{M}_\ell$, for $i = 1, 2 \ldots, k$, then*

$$\Big(\sum_{i=1}^{k} A_i \Big) \oslash \Big(\sum_{i=1}^{k} B_i \Big) \oslash \cdots \oslash \Big(\sum_{i=1}^{k} C_i \Big) = \sum_{i=1}^{k} (A_i \oslash B_i \oslash \cdots \oslash C_i).$$

Lemma 4.18. *If A and B are any square matrices, then*

$$S_{A \otimes B} = S_A S_B.$$

Proof. If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then the i, j−th block of $A \otimes B$ is $a_{i,j} B$ and $S_{a_{i,j}B} = a_{i,j} S_B$. Therefore we get

$$S_{A \oslash B} = S_B \sum_{i,j=1}^{m} a_{i.j} = S_B S_A.$$

Hence the result follows. □

Theorem 4.19. *If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then*

$$S_{A \oslash B} = n^2 S_A + m^2 S_B.$$

Proof. We have

$$
\begin{aligned}
S_{A \oslash B} &= S_{A \otimes \mathbb{J}_n + \mathbb{J}_m \otimes B} \\
&= S_{A \otimes \mathbb{J}_n} + S_{\mathbb{J}_m \otimes B} \\
&= S_A \times n^2 + m^2 \times S_B. \qquad [\text{using } theorem\ 4.18].
\end{aligned}
$$

Hence the theorem holds. $\qquad\qquad\square$

As a corollary of theorem 4.19, we get the expression for the Wiener index of Cartesian product of two connected graphs.

Corollary 4.20. *If G_1 and G_2 are two connected graphs of order m and n respectively, then*

$$
W(G_1 \square G_2) = n^2 W(G_1) + m^2 W(G_2).
$$

As an application of the above corollary we get the following result.

Corollary 4.21. *If H is any fixed connected graph and G_1, G_2 are connected graphs of same order with $W(G_1) \geq W(G_2)$, then*

$$
W(H \square G_1) \geq W(H \square G_2),
$$

with equality if and only if $W(G_1) = W(G_2)$.

Theorem 4.22. *If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $A \oslash B$ has a constant row sum if and only if A and B both have constant row sums.*

Proof. Let us consider any arbitrary row of $A \oslash B$. If the first entry of that row is $a_{i,1} + b_{j,i}$, then the row sum of that row of $A \oslash B$ equals to

$$
(na_{i,1} + B_j) + (na_{i,2} + B_j) + \cdots + (na_{i,m} + B_j) = nA_j + mB_j. \qquad (4.5)
$$

Now if A and B have constant row sums, then $A_i = \frac{S_A}{m}$ and $B_i = \frac{S_B}{n}$. Therefore, by eq. (4.5), $A \oslash B$ has constant row sum equal to $\frac{n}{m} S_A + \frac{m}{n} S_B$.

Again if $A \oslash B$ has constant row sum (say k), then from eq. (4.5) we get

$$
nA_i + mB_j = k \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.
$$

Keeping i fixed, we see that B_j is constant for $j = 1, 2, \ldots, n$. Similarly, keeping j fixed we get A_i is constant for $i = 1, 2, \ldots, m$. Hence, the theorem holds. $\qquad\square$

The following result is a reformulation of theorem 4.22. Therefore, the proof is omitted.

Theorem 4.23. *If $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$, then $\mathbb{1}_{mn}$ is an eigenvector of $A \oslash B$ if and only if $\mathbb{1}_m$ and $\mathbb{1}_n$ are eigenvectors of A and B respectively.*

As an application of theorem 4.22, we get the following result as a corollary.

Corollary 4.24. *The Cartesian product $G_1 \square G_2$ of two connected graphs G_1 and G_2 is transmission regular if and only if G_1 and G_2 are both transmission regular.*

From the proof of theorem 4.22, we get a lower bound for the distance spectral radius of the Cartesian product of two connected graphs.

Corollary 4.25. *If G_1 and G_2 are two connected graphs of order m and n respectively, then*

$$
\rho^{\mathcal{D}}(G_1 \square G_2) \geq \frac{n}{m} W(G_1) + \frac{m}{n} W(G_2),
$$

with equality if and only if G_1 and G_2 are both transmission regular.

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