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Certain Extended Type Hypergeometric Functions of Two and Three Variables

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ABSTRACT: The major objective of the present article is to study the new extension of hypergeometric functions of two and three variables by using the 2 parameters Mittag-Leffler function. In the present article, we mainly study the integral representations of these extended hypergeometric functions and obtain some important properties of the extended Riemann-Liouville type fractional derivative operator. We have also derived some generating functions for the generalized hypergeometric functions by using the extended Riemann-Liouville type fractional derivative operator.

Key Words: Beta function, Hypergeometric function, Mittag-Leffler function, Appell's hypergeometric functions of two variables, Lauricella's hypergeometric function of three variables, Riemann-Liouville fractional derivative operator.

Contents

1. Introduction and Preliminaries

Euler's beta, gamma functions and hypergeometric functions are the important members of the family of special functions and they play a significant role in the whole theory of special functions. These hypergeometric functions together with their extension have many applications in research fields such as engineering, chemical, statistics, fractional calculus, and physical problems. In the last decades, many generalizations and extensions of the most popular special functions have been presented by many authors [1,2,3,4]. Similarly, multi-variable hypergeometric functions such as Appell and Lauricella functions and their many extensions seem in many core areas of mathematics and their applications. In [13,6,7,8], many researchers have contributed their works on multi-variable hypergeometric functions and their properties in detail. In the past few years, several authors and researchers [9,10,11] have introduced some fascinating generalizations of the Appell and Lauricella functions by using special functions in their kernels. Inspired by the above works, here we establish new extensions of the Appell hypergeometric function of two variables and Lauricella hypergeometric function of three variables and obtained their relation with other well known special functions and their extensions.

Let us define the hypergeometric functions of two and three variables *i.e* Appell's functions $F_1(a_1; b_1, c_1; d_1; x_1, y_1)$ and $F_2(a_1; b_1, c_1; d_1, e_1; x_1, y_1)$, and Lauricella's function $F_D^3(a_1, b_1, c_1, d_1; e_1; x_1, y_1, z_1)$ respectively are defined as (see for more details, [12] and [13])

$$F_1(a_1; b_1, c_1; d_1; x_1, y_1) = \sum_{m,n=0}^{\infty} \frac{B(a_1+m+n, d_1-a_1)(b_1)_m (c_1)_n}{B(a_1, d_1-a_1)} \frac{x_1^m}{m!} \frac{y_1^n}{n!}, \tag{1.1}$$

where, $\max\{|x_1|, |y_1|\} < 1$ and $B(a_1, a_2)$ denotes the Euler's beta function defined as [12]

$$B(a_1, a_2) = \int_0^1 s^{a_1-1}(1-s)^{a_2-1} ds, \qquad \Re(a_1), \Re(a_2) > 0., \tag{1.2}$$

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and $(a_1)_l$ denotes the Pochhammer symbol defined as [14]

$$(a_1)_l := \frac{\Gamma(a_1+l)}{\Gamma(a_1)} = \begin{cases} 1 & (l=0; a_1 \in \mathbb{C}\setminus\{0\}) \\ a_1(a_1+1)\cdots(a_1+l-1) & (l\in\mathbb{N}; a_1\in\mathbb{C}). \end{cases}$$

$$F_2(a_1;b_1,c_1;d_1,e_1;x_1,y_1) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}B(b_1+m,d_1-b_1)B(c_1+n,e_1-c_1)}{B(b_1,d_1-b_1)B(c_1,e_1-c_1)} \frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{1.3}$$

where, $\mid x_1 \mid + \mid y_1 \mid < 1$.

$$F_D^3(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1) = \sum_{m,n,t=0}^{\infty} \frac{B(a_1+m+n+t,e_1-a_1)(b_1)_m(c_1)_n(d_1)_t}{B(a_1,e_1-a_1)} \frac{x_1^m}{m!}\frac{y_1^n}{n!}\frac{z_1^t}{t!}, \tag{1.4}$$

where, $\sqrt{\mid x_1 \mid} + \sqrt{\mid y_1 \mid} + \sqrt{\mid z_1 \mid} < 1$.

In the recent past, Goyal et al., [15] have studied the generalization of the Euler's beta function by using the 2-parameter Mittag-Leffler function, thus examining many basic properties and relationships of that generalized beta function. The generalized beta function is defined as

$$B^{(u)}_{(u_1,u_2)}(y_1,y_2) = \int_0^1 s^{y_1-1}(1-s)^{y_2-1}E_{u_1,u_2}\left(\frac{-u}{s(1-s)}\right)ds, \tag{1.5}$$

here, $\Re(y_1) > 0$, $\Re(y_2) > 0$ $\Re(u_1) > 0, \Re(u_2) > 0$, $u \geq 0$ and $E_{u_1,u_2}(w)$ is 2-parameter Mittag-Leffler function defined by

$$E_{y_1,y_2}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(ky_1+y_2)}, \tag{1.6}$$

where $\Re(y_1) \geq 0$, $\Re(y_2) \geq 0$ and $w \in \mathbb{C}$.

The above generalized Euler's beta function has an important role in the establishment of our new extended functions due to the 2-parameter Mittag-Leffler function used in the kernel.

Very recently inspired by the above extension, Jain et al., [16] have generalized the Gauss hypergeometric function by using the generalized Euler's beta function given in (1.5). They have also studied various properties like differentiation formulas, summation formulas, Mellin transforms, and recurrence relations of generalized Gauss hypergeometric functions. The generalized Gauss hypergeometric function is defined as

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;x) = \sum_{k=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1+k,p_2-p_1)}{B(p_1,p_2-p_1)}(p_0)_n\frac{x^k}{k!}, \tag{1.7}$$

here $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$, $\mid x \mid < 1$ and $B^{(r)}_{(r_1,r_2)}(x_1,x_2)$ extended beta function (1.5).

Also, Euler type integral representation of the above generalized Gauss hypergeometric function is defined as

$$\begin{aligned} & F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;x) \\ = \; & \frac{1}{B(p_1,p_2-p_1)}\int_0^1 s^{p_1-1}(1-s)^{p_2-p_1-1}(1-xs)^{-p_0}E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right)ds, \end{aligned} \tag{1.8}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $r \geq 0$ and $\mid x \mid < 1$.

In 2010, Özarslan et al., [9] have generalized the Appell's hypergeometric functions of two variables, $F_1(a_1;b_1,c_1;d_1;x_1,y_1;p_1)$, $F_2(a_1;b_1,c_1;d_1,e_1;x_1,y_1;p_1)$ and generalized Lauricella's hypergeometric function of three variables, $F^3_{D,p}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$ by using the generalized Euler's beta function $B_{p_1}(x_1,y_1)$ given in [2] and studied several properties of these generalized functions.

The generalized Appell's hypergeometric functions of two variables $F_1(a_1;b_1,c_1;d_1;x_1,y_1;p_1)$ and $F_2(a_1;b_1,c_1;d_1,e_1;x_1,y_1;p_1)$ defined as [9]

$$F_1(a_1;b_1,c_1;d_1;x_1,y_1;p_1) = \sum_{m,n=0}^{\infty} \frac{B_{p_1}(a_1+m+n,d_1-a_1)(b_1)_m(c_1)_n}{B(a_1,d_1-a_1)} \frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{1.9}$$

where, $\max\{|\ x_1\ |, |\ y_1\ |\} < 1$.

$$F_2(a_1;b_1,c_1;d_1,e_1;x_1,y_1;p_1) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}B_{p_1}(b_1+m,d_1-b_1)B_{p_1}(c_1+n,e_1-c_1)}{B(b_1,d_1-b_1)B(c_1,e_1-c_1)} \frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{1.10}$$

here, $|\ x_1\ | + |\ y_1\ | < 1$.

The generalized Lauricella's hypergeometric function of three variables, $F^3_D(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1;p_1)$ defined as [9]

$$F^3_{D,p_1}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1) = \sum_{m,n,t=0}^{\infty} \frac{B_{p_1}(a_1+m+n+t,d_1-a_1)(b_1)_m(c_1)_n(d_1)_t}{B(a_1,e_1-a_1)} \frac{x_1^m}{m!}\frac{y_1^n}{n!}\frac{z_1^t}{t!}, \tag{1.11}$$

where, $\sqrt{|\ x_1\ |} + \sqrt{|\ y_1\ |} + \sqrt{|\ z_1\ |} < 1$.

Remark 1 *If, we substitute $p_1 = 0$ in the above equations (1.9), (1.10) and (1.11) then we get original functions given by (1.1), (1.3) and (1.4), respectively.*

In the last decades, many researchers have working in fractional calculus due to many applications in some fields of engineering and science such as electrical networks, statistics probability, and fluid dynamics, particularly fluid flows. In 2001, Srivastava and Saxena [17] studied fractional calculus and its application systematically. Also, Srivastava and Manocha [8], have described the benefits of fractional derivative in the generating function theory of special functions. In this paper, motivated by the above, we have also obtained some generating functions for the generalized hypergeometric functions $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ by using of the generalized Riemann-Liouville fractional derivative operator defined by Jain et al., [18].

In 2021, Jain et al., [18] have generalized the Riemann-Liouville type fractional derivative operator by using the 2-parameter Mittag-Leffler function in its kernel and studied various basic properties of that generalized fractional derivative operator.

Now, here recall the generalized Riemann-Liouville type fractional derivative operator defined as [18]

$$D^{u,(r)}_{x,(r_1,r_2)}[g(x)] = \begin{cases} \frac{1}{\Gamma(-u)}\int_0^x (x-s)^{-u-1}E_{r_1,r_2}\left(\frac{-rx^2}{s(x-s)}\right)g(s)ds, & (\Re(u)<0) \\ \frac{d^k}{dx^k}\{D^{u-k,(r)}_{x,(r_1,r_2)}g(x)\}, & (k-1\leq \Re(u)<k,\ k\in\mathbb{N}) \end{cases} \tag{1.12}$$

here, ($\min\{\Re(r_1), \Re(r_2)\} > 0$, $\Re(r) > 0$ and $E_{r_1,r_2}(x)$ is 2-parameter Mittag-Leffler function).

The following Theorem is very important to obtain our main results.

Theorem 1.1 *([18])The following result holds true*

$$\sum_{m=0}^{\infty} \frac{(k)_m}{m!} F^{(r)}_{(r_1,r_2)}(k+m,n,l;x)s^m = (1-s)^{-k} F^{(r)}_{(r_1,r_2)}\left(k,n,l;\frac{x}{(1-s)}\right), \tag{1.13}$$

where $|\ \frac{x}{(1-s)}\ | < 1$,$\Re(k) > 0$ *and* $\Re(n) > \Re(l) > 0$.

2. Main Results

In this section, we establish new extensions of Appell's hypergeometric functions $F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)$ and $F^{(r)}_{2,(r_1,r_2)}(a_1;b_1,c_1;d_1,e_1;x_1,y_1)$ and Lauricella's hypergeometric function
$F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$ by the using of the generalized Euler's beta function given in (1.5). We have also obtained integral representations of extended Appell's hypergeometric functions and Lauricella's hypergeometric function.

Definition 2.1 *Let $\Re(r_1)>0$, $\Re(r_2)>0$ and $\Re(r)>0$, then new extensions of Appell's hypergeometric functions, $F_1(a_1;b_1,c_1;d_1;x_1,y_1)$ and $F_2(a_1;b_1,c_1;d_1,e_1;x_1,y_1)$ are defined as*

$$F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)=\sum_{m,n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(a_1+m+n,d_1-a_1)(b_1)_m(c_1)_n}{B(a_1,d_1-a_1)}\frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{2.1}$$

where, $\max\{|x_1|,|y_1|\}<1$.

$$F^{(r)}_{2,(r_1,r_2)}(a_1;b_1,c_1;d_1,e_1;x_1,y_1)=\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}B^{(r)}_{(r_1,r_2)}(b_1+m,d_1-b_1)B(c_1+n,e_1-c_1)}{B^{(r)}_{(r_1,r_2)}(b_1,d_1-b_1)B(c_1,e_1-c_1)}\frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{2.2}$$

where, $|x_1|+|y_1|<1$.

Definition 2.2 *Let $\Re(r_1)>0$, $\Re(r_2)>0$ and $\Re(r)>0$. Then new extension of Lauricella's hypergeometric function $F^3_D(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$ is defined as*

$$F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)=\sum_{m,n,t=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(a_1+m+n+t,e_1-a_1)(b_1)_m(c_1)_n(d_1)_t}{B(a_1,e_1-a_1)}\frac{x_1^m}{m!}\frac{y_1^n}{n!}\frac{z_1^t}{t!}, \tag{2.3}$$

where, $\sqrt{|x_1|}+\sqrt{|y_1|}+\sqrt{|z_1|}<1$

Remark 2 *(i)If we consider $r_2=r_1=1$, then new extended Appell's hypergeometric functions defined in (2.1) and (2.2) and new extended Lauricella's hypergeometric function defined in (2.3) reduces to extended Appell's hypergeometric functions given by (1.9) and (1.10) and extended Lauricella's hypergeometric function given by (1.11) respectively.*

(ii)If we substitute $r_2=r_1=1$ and $r=0$ then new extended Appell's hypergeometric functions defined in (2.1) and(2.2) and new extended Lauricella's hypergeometric function defined in (2.3) reduces to original Appell's hypergeometric functions given by (1.1) and (1.3) and Lauricella's hypergeometric function given by (1.4) respectively.

Then, we carry on to obtain the integral representations of the functions $F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)$, $F^{(r)}_{2,(r_1,r_2)}(a_1;b_1,c_1;d_1,e_1;x_1,y_1)$ and $F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$.

Theorem 2.3 *For $F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)$, the following integral representation hold true*

$$F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1) = \frac{\Gamma(d_1)}{\Gamma(a_1)\Gamma(d_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c_1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt, \tag{2.4}$$

where, $\Re(r_1)>0$, $\Re(r_2)>0$ and $\Re(r)>0$, $\Re(d_1)>\Re(a_1)>0$ with $|x_1|<1$ and $|y_1|<1$.

Proof 1 *From (2.1), we have*

$$F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)=\sum_{m,n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(a_1+m+n,d_1-a_1)(b_1)_m(c_1)_n}{B(a_1,d_1-a_1)}\frac{x_1^m}{m!}\frac{y_1^n}{n!}, \tag{2.5}$$

Now, using the definition given in (1.5) in above equation, we get

$$F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1) \\ =\frac{1}{B(a_1,d_1-a_1)}\sum_{m,n=0}^{\infty}\left\{\int_0^1 t^{a_1+m+n-1}(1-t)^{d_1-a_1-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt\right\}(b_1)_m(c_1)_n\frac{x_1^m}{m!}\frac{y_1^n}{n!}. \tag{2.6}$$

On interchanging summation and integration sign in above equation and manipulate some terms, we have

$$F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1) \\ =\frac{1}{B(a_1,d_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)\left\{\sum_{m,n=0}^{\infty}(b_1)_m(c_1)_n\frac{(x_1t)^m}{m!}\frac{(y_1t)^n}{n!}\right\}dt \tag{2.7}$$

Now using the identities in the above equation,

$$(1-zt)^{-p}=\sum_{k=0}^{\infty}\frac{(p)_k}{k!}(zt)^k, \mid z\mid<1, \tag{2.8}$$

and

$$B(x_1,y_1)=\frac{\Gamma(x_1)\Gamma(y_1)}{\Gamma(x_1+y_1)} \tag{2.9}$$

Then, we get our desired result

$$F^{(r)}_{1,(r_1,r_2)}(a_1,b_1,c_1;d_1;x_1,y_1) \\ =\frac{\Gamma(d_1)}{\Gamma(a_1)\Gamma(d_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{d_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c_1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt \tag{2.10}$$

Theorem 2.4 *For $F^{(r)}_{2,(r_1,r_2)}(a_1;b_1,c_1;d_1,e_1;x_1,y_1)$, the following integral representation hold true*

$$F^{(r)}_{2,(r_1,r_2)}(a_1;b_1,c_1;d_1,e_1;x_1,y_1)=\frac{\Gamma(d_1)\Gamma(e_1)}{\Gamma(b_1)\Gamma(c_1)\Gamma(d_1-b_1)\Gamma(e_1-c_1)}\times \\ \int_0^1\int_0^1 t^{b_1-1}(1-t)^{d_1-b_1-1}s^{c_1-1}(1-s)^{e_1-c_1-1}(1-xt-ys)^{-a_1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right)dtds \tag{2.11}$$

where, $\Re(r_1)>0$, $\Re(r_2)>0$ and $\Re(r)>0$, $\Re(d_1)>\Re(b_1)>0$ and $\Re(e_1)>\Re(c_1)>0$ with $\mid x\mid+\mid y\mid<1$.

Proof 2 *Similarly, on following the same steps as the proof of Theorem 2.3, and using (1.5) in (2.2) with the identity*

$$\sum_{n=0}^{\infty}f(n)\frac{(a+b)^n}{n!}=\sum_{k,l=0}^{\infty}f(k+l)\frac{a^k}{k!}\frac{b^l}{l!}, \tag{2.12}$$

we, get our desired result.

$$F^{(r)}_{2,(r_1,r_2)}(a_1,b_1,c_1;d_1,e_1;x_1,y_1)=\frac{\Gamma(d_1)\Gamma(e_1)}{\Gamma(b_1)\Gamma(c_1)\Gamma(d_1-b_1)\Gamma(e_1-c_1)}\times \\ \int_0^1\int_0^1 t^{b_1-1}(1-t)^{d_1-b_1-1}s^{c_1-1}(1-s)^{e_1-c_1-1}(1-xt-ys)^{-a_1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)E_{r_1,r_2}\left(\frac{-r}{s(1-s)}\right)dtds. \tag{2.13}$$

Theorem 2.5 *For* $F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1)$, *the following integral representation hold true*

$$\begin{aligned} &F^{3,(r)}_{D,(r_1,r_2)}(a_1,b_1,c_1,d_1;e_1;x_1,y_1,z_1) \\ &= \frac{\Gamma(e_1)}{\Gamma(a_1)\Gamma(e_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{e_1-a_1-1}(1-x_1t)^{-b_1}(1-y_1t)^{-c}(1-z_1t)^{-d_1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt, \end{aligned} \tag{2.14}$$

where, $\Re(r_1)>0$, $\Re(r_2)>0$ *and* $\Re(r)>0$, $\Re(e_1)>\Re(a_1)>0$ *with* $\mid x_1\mid<1$, $\mid y_1\mid<1$ *and* $\mid z_1\mid<1$.

Proof 3 *By the following same parallel line of proof as Theorem 2.3, we get our desired result.*

Now, here we obtain some results of extended Riemann-Liouville type fractional derivative operator and some generating functions of extended hypergeometric function by the use of the extended Riemann-Liouville type fractional derivative operator.

Theorem 2.6 *Consider* $\Re(u)<0$, $\Re(k)>0$, $\Re(l)>0$, $\Re(m)>0$ $\mid az\mid<1$ *and* $\mid bz\mid<1$ *then*

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \frac{\Gamma(k)}{\Gamma(u)}z^{u-1}F^{(r)}_{1,(r_1,r_2)}(k;l,m;u;az,bz). \tag{2.15}$$

Proof 4 *From* (1.12), *we get*

$$\begin{aligned} &D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \\ &\frac{1}{\Gamma(u-k)}\int_0^z (z-t)^{u-k-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-1}(1-at)^{-l}(1-bt)^{-m}dt. \end{aligned} \tag{2.16}$$

On taking out z *from integral, we have*

$$\begin{aligned} &D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \\ &\frac{z^{u-k-1}}{\Gamma(u-k)}\int_0^z \left(1-\frac{t}{z}\right)^{u-k-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-1}(1-at)^{-l}(1-bt)^{-m}dt. \end{aligned} \tag{2.17}$$

Then set the value of $t=xz$ *in above equation and changing the limit from* $t=0, t=z$ *to* $x=0, x=1$, $dt=zdx$ *with some re-arranging the terms, we get*

$$\begin{aligned} &D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \\ &\frac{z^{u-1}}{\Gamma(u-k)}\int_0^1 (1-x)^{u-k-1}E_{r_1,r_2}\left(\frac{-r}{x(1-x)}\right)x^{k-1}(1-axz)^{-l}(1-bxz)^{-m}dx. \end{aligned} \tag{2.18}$$

From the integral formula of the extended Appell's hypergeometric function $F^{(r)}_{1,(r_1,r_2)}(a_1;b_1,c_1;d_1;x_1,y_1)$ (1.9), *we get our desired result.*

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}] = \frac{\Gamma(k)}{\Gamma(u)}z^{u-1}F^{(r)}_{1,(r_1,r_2)}(k;l,m;u;az,bz). \tag{2.19}$$

Theorem 2.7 *Consider* $\Re(u)<0$, $\Re(k)>0$, $\Re(l)>0$, $\Re(m)>0$, $\Re(n)>0$, $\mid az\mid<1$, $\mid bz\mid<1$ *and* $\mid cz\mid<1$ *then*

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-az)^{-l}(1-bz)^{-m}(1-cz)^{-n}] = \frac{\Gamma(k)}{\Gamma(u)}z^{u-1}F^{3,(r)}_{D,(r_1,r_2)}(k,l,m,n;u;az,bz,cz). \tag{2.20}$$

Proof 5 *By the similar steps as proof of Theorem 2.6, we can get our desired result.*

Theorem 2.8 *Consider* $\Re(u)<0$, $\Re(k)>0$, $\Re(l)>0$, *and* $\mid \frac{x}{1-z}\mid<1$ *then*

$$\begin{aligned} &D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}(1-z)^{-l}F^{(r)}_{(r_1,r_2)}\left(l,m,p,\frac{x}{1-z}\right)\right\} \\ &\qquad = \frac{1}{B(m,p-m)\Gamma(u-k)}z^{u-1}F^{(r)}_{2,(r_1,r_2)}(l;m,k;p,u;x,z). \end{aligned} \tag{2.21}$$

Proof 6 *By using the (1.7), in the definition of the extended Riemann-Liouville fractional derivative operator (1.12), we have*

$$\begin{aligned}
&D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}(1-z)^{-l}F^{(r)}_{(r_1,r_2)}\left(l,m,p,\frac{x}{1-z}\right)\right\}= \\
&\quad D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}(1-z)^{-l}\left\{\frac{1}{B(m,p-m)}\sum_{n=0}^{\infty}\frac{(l)_n B^{(r)}_{(r_1,r_2)}(m+n,p-m)}{n!}\left(\frac{x}{1-z}\right)^n\right\}\right\} \\
&\quad = \frac{1}{B(m,p-m)}D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}\sum_{n=0}^{\infty}\frac{(l)_n B^{(r)}_{(r_1,r_2)}(m+n,p-m)x^n(1-z)^{-l-n}}{n!}\right\} \\
&\quad = \frac{1}{B(m,p-m)}D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{m+k-1}\sum_{m,n=0}^{\infty}\frac{(l)_n(l+n)_m B^{(r)}_{(r_1,r_2)}(m+n,p-m)x^n}{n!m!}\right\} \\
&\quad = \frac{1}{B(m,p-m)}\sum_{m,n=0}^{\infty}B^{(r)}_{(r_1,r_2)}(m+n,p-m)\frac{x^n}{n!}\frac{(l)_n(l+n)_m}{m!}D^{k-u,(r)}_{z,(r_1,r_2)}\{z^{m+k-1}\}
\end{aligned} \tag{2.22}$$

Now using the result given in [18]

$$D^{u,(r)}_{z,(r_1,r_2)}[z^k]=\frac{B^{(r)}_{(r_1,r_2)}(k+1,-u)}{\Gamma(-u)}z^{k-u}. \tag{2.23}$$

Then, we have

$$\begin{aligned}
&D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}(1-z)^{-l}F^{(r)}_{(r_1,r_2)}\left(l,m,p,\frac{x}{1-z}\right)\right\}= \\
&\quad \frac{1}{B(m,p-m)}\sum_{m,n=0}^{\infty}B^{(r)}_{(r_1,r_2)}(m+n,p-m)\frac{x^n}{n!}\frac{(l)_{m+n}}{m!}\frac{B^{(r)}_{(r_1,r_2)}(k+m,u-k)}{\Gamma(u-k)}z^{u+m-1}
\end{aligned} \tag{2.24}$$

Then using the (2.2), we get our desired result.

$$\begin{aligned}
&D^{k-u,(r)}_{z,(r_1,r_2)}\left\{z^{k-1}(1-z)^{-l}F^{(r)}_{(r_1,r_2)}\left(l,m,p,\frac{x}{1-z}\right)\right\} \\
&\qquad\qquad = \frac{1}{B(m,p-m)\Gamma(u-k))}z^{u-1}F^{(r)}_{2,(r_1,r_2)}(l;m,k;p,u;x,z).
\end{aligned} \tag{2.25}$$

Theorem 2.9 *For generalized hypergeometric function $F^{(r)}_{(r_1,r_2)}(a,b,c,z)$ the following generating relation holds true*

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}F^{(r)}_{(r_1,r_2)}(d-m,l,m;x)t^m=(1-t)^{-\lambda}F^{(r)}_{1,(r_1,r_2)}\left(l;d,\lambda;m;x,\frac{-xt}{(1-t)}\right), \tag{2.26}$$

where $\mid\frac{x}{(1-t)}\mid<1$,$\Re(\lambda)>0$ *and* $\Re(m)>\Re(l)>0$.

Proof 7 *Let the series identity*
$[1-(1-x)t]^{-\lambda}=(1-t)^{-\lambda}\left(1+\frac{xt}{(1-t)}\right)^{-\lambda}$.
Then by using the binomial expansion of $(1-(1-x)t)^{-\lambda}$ *on LHS of the above equation, we get*

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m(1-x)^m t^m}{m!}=(1-t)^{-\lambda}\left(1-\frac{-xt}{(1-t)}\right)^{-\lambda}. \tag{2.27}$$

Now, multiply both hand side by $x^{l-1}(1-x)^{-d}$ *and operate extended Riemann-Liouville fractional derivative operator* $D^{l-m,(r)}_{x,(r_1,r_2)}$, *then we have*

$$D^{l-m,(r)}_{x,(r_1,r_2)}\left[\sum_{m=0}^{\infty}\frac{(\lambda)_m(1-x)^{m-d}x^{l-1}t^m}{m!}\right]= (1-t)^{-\lambda}D^{l-m,(r)}_{x,(r_1,r_2)}\left[x^{l-1}(1-x)^{-d}\left(1-\frac{-xt}{(1-t)}\right)^{-\lambda}\right]. \tag{2.28}$$

After some calculations, we have

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}D^{l-m,(r)}_{x,(r_1,r_2)}\left[x^{l-1}(1-x)^{-(-m+d)}\right]t^m= (1-t)^{-\lambda}D^{l-m,(r)}_{x,(r_1,r_2)}\left[x^{l-1}(1-x)^{-d}\left(1-\frac{-xt}{(1-t)}\right)^{-\lambda}\right]. \tag{2.29}$$

Then using the Theorem 1.1 proved in [18] and Theorem 2.3, we get our desired result.

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}F^{(r)}_{(r_1,r_2)}(d-m,l,m;x)t^m=(1-t)^{-\lambda}F^{(r)}_{1,(r_1,r_2)}\left(l;d,\lambda;m;x,\frac{-xt}{(1-t)}\right), \tag{2.30}$$

Theorem 2.10 *For the generalized hypergeometric function* $F^{(r)}_{(r_1,r_2)}(a,b,c;z)$ *the following result holds true*

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(p,-n,q;y)F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n = (1-t)^{-k}\frac{B(p,q-p)}{B(l,m-l)}F^{(r)}_{2,(r_1,r_2)}\left(k;l,p;m,q;\frac{x}{(1-t)},\frac{-yt}{(1-t)}\right), \tag{2.31}$$

where $\mid\frac{x}{(1-t)}\mid<1$, $\mid\frac{y}{(1-t)}\mid<1$,$\Re(k)>0$ *and* $\Re(q)>\Re(p)>0$.

Proof 8 *By the using the result, proved in [18],*

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n=(1-t)^{-k}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-t)}\right), \tag{2.32}$$

now, changing the value t *by* $(1-y)t$ *in the above equation, and multiplying the resulting equation by* y^{p-1}, *we have*

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)(1-y)^ny^{p-1}t^n=(1-(1-y)t)^{-k}y^{p-1}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-(1-y)t)}\right), \tag{2.33}$$

Then applying the extended fractional derivative operator $D^{p-q,(r)}_{y,(r_1,r_2)}$ *both the side, we get*

$$D^{p-q,(r)}_{y,(r_1,r_2)}\left\{\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)(1-y)^ny^{p-1}t^n\right\} = D^{p-q,(r)}_{y,(r_1,r_2)}\left\{(1-(1-y)t)^{-k}y^{p-1}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{x}{(1-(1-y)t)}\right)\right\} \tag{2.34}$$

After some calculations, we have

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}D^{p-q,(r)}_{y,(r_1,r_2)}\left\{(1-y)^ny^{p-1}\right\}F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n = (1-t)^{-k}D^{p-q,(r)}_{y,(r_1,r_2)}\left\{y^{p-1}\left(1-\frac{-yt}{1-t}\right)^{-k}F^{(r)}_{(r_1,r_2)}\left(k,l,m;\frac{\frac{x}{1-t}}{1-\frac{-yt}{1-t}}\right)\right\} \tag{2.35}$$

Now using the Theorem 1.1 and 2.4, we get our desired result.

$$\sum_{n=0}^{\infty}\frac{(k)_n}{n!}F^{(r)}_{(r_1,r_2)}(p,-n,q;y)F^{(r)}_{(r_1,r_2)}(k+n,l,m;x)t^n = (1-t)^{-k}\frac{B(p,q-p)}{B(l,m-l)}F^{(r)}_{2,(r_1,r_2)}\left(k;l,p;m,q;\frac{x}{(1-t)},\frac{-yt}{(1-t)}\right), \tag{2.36}$$

3. Conclusion

In this work, we have introduced new extensions of hypergeometric functions of two and three variables i.e. Appell and Lauricella functions respectively. Then, we have derived integral representations of all these extensions. After that by using an extended Riemann-Liouville type fractional derivative operator, we derived some relations between generalized Gauss hypergeometric function with extended Appell's hypergeometric functions and Lauricella's hypergeometric function, and then using these relations, we obtained some generating functions for generalized hypergeometric functions. Finally, we conclude our research by mentioning that all the results obtained in the present article are new and important. Moreover, the general massive one-loop Feynman integral can be represented as a meromorphic function of space-time dimensions using the extended type Appell and Lauricella functions for self-energy, vertex, and box integrals, respectively

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References

1. Chaudhry, M. A., Zubair, S. M., *Generalized incomplete gamma functions with Applications*, J. Comput. Appl. Math, 55, 99-124, (1994).
2. Chaudhry, M. A., Qadir, A., Rafique, M., Zubair, S. M., *Extension of Euler's Beta function*, J. Comput. Appl. Math., 78, 19-32, (1997).
3. Chaudhry, M. A., Qadir, A., Srivastava, H. M., Paris, R. B., *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput, 159, 589-602, (2004).
4. Özergin, E., Özarslan, M. A., Altın, A., *Extension Gamma, Beta and Hypergeometric Functions*, J. Comput. Appl. Math., 235, 4601-4610, (2011).
5. Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York- ´ Toronto-London, (1951).
6. Exton, H., *Multiple hypergeometric functions and applications*, Ellis Horwood, (1976).
7. Srivastava, H. M., Karlsson, P.W.,*Multiple Gaussian Hypergeometric Series*, Halsted, New York, (1985).
8. Srivastava, H. M., Manocha,H.L., *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, (1984).
9. Özarslan, M. A., Özergin, E., *Some generating relations for extended hypergeometric functions via generalized fractional ¨ derivative operator*, Math. Comput. Model. 52, 1825–1833, (2010).
10. Şahin, R., *An extension of some Lauricella hypergeometric functions*, AIP Conf. Proc. 1558, 1140–1143, (2013).
11. Srivastava, H. M., Parmar, R. K., Chopra, P., . *A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions*, Axioms-I 238–258, (2012).
12. Rainville, E. D., *Special Functions*, Macmillan: New York, NY, USA, (1960).
13. Exton, H., *Multiple Hypergeometric Functions and Applications,* Ellis Horwood, Chichester; Halsted Press (John Wiley & Sons), New York, (1976).
14. Olver, W. J. F., Lozier, W. D., Boisvert, F. R., Clark, W. C., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, USA, (2010).
15. Goyal, R., Agarwal, P., Momami, S., Rassias, M. T., *An Extension of Beta Function by Using wiman's function*, Axioms, 10, 187, (2021).
16. Jain, S., Goyal, R., Agarwal, P., Lupica, A., Cesarano, C., *Some results of extended beta function and hypergeometric functions by using Wiman's function*, Mathematics, 9(22), 2944, (2021).

17. Srivastava, H. M., Saxena R. K., *Operators of fractional integration and their applications*, Appl. Math. Comput. 118, 1-52, (2011).

18. Jain, S., Goyal, R., Agarwal, R.P., Agarwal, P., *Certain Extension of Riemann Liouville Fractional Derivative Operator by Using Wiman's Function*, book chapter edited by Pardalos, P. M., Rassias, T. M., Mathematical Ananysis, Optimization, Approximation and Applications, 2021.

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