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On the Stabilization of Some Degenerate Vibrating Equation by Fractional Damping*

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ABSTRACT: We study the stabilization of the degenerate wave equation $u_{tt} - (\sqrt{1-x^2}u_x)_x = 0$ with $x \in (-1,1)$, by a fractional boundary damping acting at $x = 1$. Thus, using semigroup theory and method inspired from Rozendaal, stahn and Seifertr. We prove the logarithm decays of its total energy with $(\ln t)^{-2}$ decay rate where $0 < \alpha \leq 1$.

Key Words: Weakly degenerate wave equation, Boundary dissipation of fractional derivative type, Polynomial stability, Airy functions.

Contents

1. Introduction

In this work, we are concerned with the boundary stabilization of fractional type for degenerate wave equation of the form

$$\begin{cases} u_{tt}(x,t) - (\sqrt{1-x^2}u_x(x,t))_x = 0 & \text{in}(-1,1)\times(0,+\infty),\\ u(-1,t) = 0 & \text{in } (0,+\infty),\\ (\sqrt{1-x^2}u_x)(1,t) = -\rho\partial_t^{\alpha,\eta}u(1,t) & \text{in}(-1,1),\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } (-1,1), \end{cases} \tag{P}$$

where $\rho > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha < 1)$, with respect to the time variable (see [8]). It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \begin{cases} w_t & \text{for } \alpha = 1, \ \eta \geq 0,\\ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)ds & \text{for } \ 0 < \alpha < 1, \eta \geq 0. \end{cases}$$

Here, the coefficient $a(x) = \sqrt{1-x^2}$ vanishes at the boundary and the problem is weakly degenerate, in the sense that $\frac{1}{a} \in L^1(-1,1)$.

The degenerate wave equation (P) can describe the vibration problem of an elastic string. Indeed a mathematical model that describes transverse vibration of an elastic string is given by

$$u_{tt}(x,t) - \left(\frac{T(x)}{\rho(x)}u_x(x,t)\right)_x + \text{lower terms} = 0,$$

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where T is the tension of a string and ρ is the density of the string. The elasticity of the string can stretch proportionally to a variation in tension. Hence, the wave equation becomes degenerate when $T(x) \to 0$ as $x \to 0$ or $\rho(x) \to +\infty$ as $x \to 0$.

Moreover, these form of diffusion coefficient and on the boundary were used before in the context of study of controlability of degenerate parabolic equation wich are related to several applied models (see [4], [7], [5] and [6]). For instance, in climatology, the so-called Budyko-Sellers model studies the role played by continental and oceanic areas of ice on climate change.

Recently, controllability issues for degenerate hyperbolic equations have been a mainstream topic over the past several years, and numerous developments have been pursued (see for example [10], [20] and the references therein). In [20], for any $\alpha > 0$, $T > 0$ and $L > 0$, the null controllability of the following degenerate wave equation was considered:

$$\begin{cases} u_{tt}(x,t) - (x^\alpha u_x(x,t))_x = 0, & \text{on } (0,L) \times (0,T), \\ u(0,t) = 0, \ 0 < \alpha < 1 & \\ & \text{on } (0,T), \\ (x^\alpha u_x)(0,t) = 0, \ \alpha \geq 1 & \text{on } (0,T) \\ u(L,t) = \theta(t), & \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{on } [0,T], \end{cases} \tag{PA}$$

where $\theta(t)$ is the control variable and it acts on the degenerate boundary. Recently, in [20], the authors studied the null controllability problems of one-dimensional degenerate wave equations as in [10] but the control acts on the nondegenerate boundary. They proved that any initial value in state space is controllable. Also, an explicit expression for the controllability time is given.
In [15] B. Mbodje investigates the decay rate of the energy of the wave equation with a fractional boundary damping, that is,

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & \text{on } (0,1) \times (0,+\infty), \\ u(0,t) = 0, & \text{on } (0,+\infty) \\ , & \\ u_x(1,t) = -\gamma \partial_t^{\alpha,\eta} u(1,t), & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{on } (0,+\infty). \end{cases}$$

Recently in [3], Benaissa and Aichi considered the stabilization for the following degenerate wave equation with fractional damping acting on the nondegenerate boundary $x = 1$, that is,

$$\begin{cases} u_{tt}(x,t) - (\tilde{a}(x) u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,+\infty), \\ \begin{cases} u(0,t) = 0 & 0 \leq \mu_{\tilde{a}} < 1 \\ (\tilde{a}(x) u_x)(0,t) = 0 & 1 \leq \mu_{\tilde{a}} < 2 \end{cases} & t \in (0,+\infty), \\ \beta u(1,t) + u_x(1,t) = -\varrho \partial_t^{\alpha,\eta} u(1,t) & t \in (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases} \tag{PB}$$

where the coefficient $\tilde{a}$ is a positive function on $]0,1]$ but vanishes at zero. The degeneracy of (PB) at $x = 0$ is measured by the parameter $\mu_{\tilde{a}}$ defined by

$$\mu_{\tilde{a}} = \sup_{0<x\leq 1} \frac{x|\tilde{a}'(x)|}{\tilde{a}(x)}.$$

They obtained optimal polynomial stability of the solutions. Moreover, the degeneracy does not affect the decay rates of the energy.

Very recently in [19], Zerkouk, Aichi and Benaissa considered the stabilization for the following degenerate wave equation with fractional damping acting on the degenerate boundary $x = 0$, that is,

$$\begin{cases} u_{tt}(x,t) - (x^\gamma u_x(x,t))_x = 0 & \text{on } (0,1) \times (0,+\infty), \\ (x^\gamma u_x)(0,t) = \varrho \partial_t^{\alpha,\eta} u(0,t) & t \in (0,+\infty), \\ u(1,t) = 0 & t \in (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } (0,1), \end{cases} \tag{PC}$$

where $0 < \gamma < 1$. They obtained optimal exponential and polynomial stability of the solutions. Moreover, the degeneracy affect the decay rates of the energy.

To our best knowledge, there is no results on the stability of the problem (PC) when instead of x^γ we take a weakly degenerate function $a(x)$.

We underline that this is the first paper to consider the stabilization for the system (P) that couples a degenerate variable coefficient $a(x) = \sqrt{1-x^2}$ in the principal part with a fractional damping acting at a degenerate boundary.

The outline of this paper as follows. In section 2, we introduce our notations, functional space. In section 3, we show the well-posedness of our problem by semigroup theory. In section 4, we show the lack of exponential stability by spectral analysis. In the last section, we also show a logarithmic type decay rate using a recent theorem of Rozendaal, stahn and Seifertr.

2. Preliminaries results

Now, we introduce, as in [5], the following weighted Sobolev spaces:
For $a(x) = \sqrt{1-x^2}$, we define the Hilbert space $H^1_{1,a}(-1,1)$, as

$$H^1_{1,a}(-1,1) = \{u \in L^2(-1,1): \ \sqrt{a(x)}u_x \in L^2(-1,1)/\ u(-1) = 0\},$$

$$H^1_a(-1,1) = \{u \in L^2(-1,1): \ \sqrt{a(x)}u_x \in L^2(-1,1)\}.$$

We remark that $H^1_a(-1,1)$ is Hilbert space with the scalar product

$$(u,v)_{H^1_a(-1,1)} = \int_{-1}^{1} \left(u\bar{v} + a(x)u'(x)\overline{v'(x)}\right) dx, \quad \forall u, v \in H^1_a(-1,1).$$

Let us also set

$$|u|_{H^1_{1,a}(-1,1)} = \left(\int_{-1}^{1} a(x) \mid u'(x) \mid^2\right)^{\frac{1}{2}}, \quad \forall u \in H^1_a(-1,1).$$

Actually, $|\ .\ |_{H^1_{1,a}(-1,1)}$ is an equivalent norm on the closed subspace $H^1_{1,a}(-1,1)$ to the norm of $H^1_a(-1,1)$ This fact is a simple consequence of the following version of Poincaré inequality.

Proposition 2.1 *There is a positive constant $C^* = C(\ a)$ such that*

$$\|u\|^2_{L^2(\Omega)} \leq C^* \mid u \mid^2_{H^1_{1,a}(-1,1)}, \quad \forall u \in H^1_{1,a}(-1,1).$$

Proof: Let u $\in H^1_{1,a}(-1,1)$. For any $x \in (-1,1)$ we have that

$$|u(x)| = \left|\int_{-1}^{x} |u'(s)|\, ds\right| \leq |u|_{H^1_{1,a}(-1,1)} \left\{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} ds\right\}^{\frac{1}{2}}.$$

Therefore

$$\int_{-1}^{1} |u(x)|^2 dx \leq \pi |u|_{H^1_{1,a}(-1,1)}.$$

□

Next, we define

$$H^2_a(-1,1) = \left\{u \in H^1_a(-1,1) : \sqrt{1-x^2}u'(x) \in H^1(-1,1)\right\}.$$

Remark 2.1 Notice that if $u \in H^2_a(-1,1), 1/a \notin L^1(-1,1)$,we have $(a(x)u_x)(\pm 1) \equiv 0$. Indeed, if $a(x)u_x(x) \to L$ when $x \to \pm 1$, then $a(x)|u_x(x)|^2 \sim L/a(x)$ and therefore $L = 0$ otherwise $u \notin H^1_a(-1,1)$.

2.1. Augmented model

In this section we reformulte (P) into an augmented system. For that, we need the following proposition.

Proposition 2.2 (see [15]) *Let μ be the fonction:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, -\infty < \xi < +\infty, 0 < \alpha < 1, \tag{2.1}$$

then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, -\infty < \xi < +\infty, \eta \geq 0,\ t > 0, \tag{2.2}$$

$$\phi(\xi,0) = 0, \tag{2.3}$$

$$O(t) = \pi^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\eta}DU = D^{\alpha,\eta}U, \tag{2.5}$$

where

$$[I^{\alpha,\eta}](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)d\tau. \tag{2.6}$$

Lemma 2.1. [1] If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ then

$$F(\lambda) = \int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}d\xi = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\eta)^{\alpha-1}.$$

Using now Proposition 2.2 and relation (2.5), system (P) may be recast into the following augmented system

$$(P') \qquad \begin{cases} u_{tt}(x,t) - (\sqrt{1-x^2}u_x(x,t))_x = 0, \\ \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - u_t(1,t)\mu(\xi) = 0, -\infty < \xi < +\infty, t > 0, \\ u(-1,t) = 0, \\ (\sqrt{1-x^2}u_x)(1,t) = -\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases}$$

where $\zeta = \rho(\pi)^{-1}\sin(\alpha\pi)$.
We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\int_{-1}^{1}(|u_t|^2 + \sqrt{1-x^2}|u_x|^2)dx + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi. \tag{2.7}$$

Lemma 2.2. Let(u,ϕ) be a regular solution of the problem (P'). Then, the energy functional defined by (7) satisfies

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi \leq 0. \tag{2.8}$$

Proof of Lemma 2.2. Multiplying the first equation in (P') by $\overline{u}_t$, integrating over $(-1,1)$ and using integration by parts, we get

$$\int_{-1}^{1}u_{tt}(x,t)\overline{u_t}dx - \int_{-1}^{1}(\sqrt{1-x^2}u_x(x,t))_x\overline{u_t}dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-1}^{1}|u_t(x,t)|^2dx\right) + \frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\sqrt{1-x^2}|u_x(x,t)|^2dx - \mathcal{R}\left[(\sqrt{1-x^2}u_x)(x,t)\overline{u_t}\right]_{-1}^{1} = 0.$$

Then from the boundary conditions $(P')_3 - (P')_4$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(|u_t(x,t)|^2+\sqrt{1-x^2}|u_x(x,t)|^2)dx+\zeta\mathcal{R}\overline{u_t}(1,t)\int_{-\infty}^{+\infty}u(\xi)\phi(\xi,t)d\xi=0. \tag{2.9}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}$ and integrating over $(-\infty,+\infty)$, to obtain

$$\zeta\int_{-\infty}^{+\infty}\phi_t(\xi,t)\overline{\phi}d\xi+\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi-\zeta u_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)d\xi=0.$$

Hence

$$\frac{\zeta}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi+\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi-\zeta\mathcal{R}u_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)d\xi=0. \tag{2.10}$$

From (2.7), (2.9) and (2.10) we obtain

$$E'(t)=-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi\le 0.$$

This completes the proof of the lemma.

3. Well-posedness of the system

In order to study the system (P') we use a reduction order argument. First, we introduce the Hilbert space $\mathcal{H}=H^1_{1,a}(-1,1)\times\in L^2(-1,1)\times\in L^2(-\infty,+\infty)$ equipped with the scalar product

$$<U,\tilde{U}>_{\mathcal{H}}=\int_{-1}^{1}\sqrt{1-x^2}u_x\overline{\tilde{u}_x}dx+\int_{-1}^{1}v\overline{\tilde{v}}dx+\zeta\int_{-\infty}^{+\infty}\phi\overline{\tilde{\phi}}d\xi,\ \forall U,\tilde{U}\in\mathcal{H}, U=(u,v,\phi),\tilde{U}=(\tilde{u},\tilde{v},\tilde{\phi}).$$

Then we consider the unbounded operator

$$\begin{array}{l}\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}\\ \quad U=(u,v,\phi)\mapsto\mathcal{A}U=\left(v,(\sqrt{1-x^2}u_x)_x,-(\xi^2+\eta)\phi+v(1)\mu(\xi)\right),\end{array} \tag{3.1}$$

where

$$D(\mathcal{A})=\left\{\begin{array}{c}(u,v,\phi)\text{ in }\ \mathcal{H}:u\in H^2_a(-1,1)\cap H^1_{1,a}(-1,1),v\in H^1_{1,a}(-1,1),\\ -(\xi^2+\eta)\phi+v(1)\mu(\xi)\in L^2(-\infty,+\infty),\\ (\sqrt{1-x^2}u_x)(1)+\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi=0,\\ |\xi|\phi\in L^2(-\infty,+\infty).\end{array}\right\}. \tag{3.2}$$

So the system (P') is formally equivalent to

$$\begin{cases}U'=\mathcal{A}U,\\ U(0)=U_0,\end{cases} \tag{3.3}$$

where $U_0=(u_0,u_1,0)^T$.

Theorem 3.1 *The operator $\mathcal{A}$ is an m-dissipative operator on $\mathcal{H}$ and thus it generates a C_0-semigroup.*

Proof: To prove this result we shall use the Lumer-Phillips' theorem. Since for every $U=(u,v,\phi)\in D(\mathcal{A})$ we have

$$\mathcal{R}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2d\xi. \tag{3.4}$$

Hence the operator $\mathcal{A}$ is dissipative.
We would like to show that there exists $\lambda > 0$ such that $(\lambda I - \mathcal{A})$ is surjective. Let $\lambda > 0$ be given. For $(f_1, f_2, f_3) \in \mathcal{H}$, we look for $(u, v, \phi) \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{3.5}$$

Equation (3.5) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - (\sqrt{1-x^2}u_x)_x = f_2, \\ \lambda \phi + (\xi^2 + \eta) - v(1)\mu(\xi) = f_3, \end{cases} \tag{3.6}$$

Suppose u is found with the appropriate regularity. Then $(3.6)_1$ and $(3.6)_3$ yeild

$$v = \lambda u + f_1 \in H^1_{1,a}(-1,1), \tag{3.7}$$

$$\phi = \frac{f_3(\xi) + \mu(\xi)v(1)}{\xi^2 + \eta + \lambda}. \tag{3.8}$$

By using $(3.6)_1$ and $(3.6)_2$ it can easily be shown that u satisfies

$$\lambda^2 u - (\sqrt{1-x^2}u_x)_x = f_2 + \lambda f_1. \tag{3.9}$$

Solving equation (3.9) is equivalent to finding $u \in H^2_a(-1,1) \cap H^1_{1,a}(-1,1)$ such that

$$\int_{-1}^{1} (\lambda^2 u\overline{w} - (\sqrt{1-x^2}u_x)_x \overline{w})dx = \int_{-1}^{1} (f_2 + \lambda f_1)\overline{w}dx \tag{3.10}$$

for all $w \in H^1_{1,a}(-1,1)$ By using (3.10), the boundary condition $(3.2)_3$, (3.8) and (3.7), the function u satisfies the following equation :

$$\begin{aligned} &\textstyle\int_{-1}^{1} (\lambda^2 u\overline{w} + (\sqrt{1-x^2}u_x)\overline{w}_x)dx + \overline{\zeta}v(1)\overline{w}(1) \\ &\qquad = \textstyle\int_{-1}^{1} (f_2 + \lambda f_1)\overline{w}dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)d\xi\overline{w}(1), \end{aligned} \tag{3.11}$$

where $\overline{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} d\xi$. Using again (16), we deduce that

$$v(1) = \lambda u(1) - f(1). \tag{3.12}$$

Inserting (3.12) into (3.11), we get

$$\begin{cases} \int_{-1}^{1} (\lambda^2 u\overline{w} + \sqrt{1-x^2}u_x\overline{w_x})dx + \lambda\overline{\zeta}u(1)\overline{w}(1) \\ = \int_{-1}^{1} (f_2 + \lambda f_1)\overline{w}dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)d\xi\overline{w}(1) + \tilde{\zeta}f_1(1)\overline{w}(1). \end{cases} \tag{3.13}$$

Problem (3.13) is of the form

$$\mathcal{B}(u, w) = \mathcal{L}(w) \tag{3.14}$$

where $\mathcal{B} : H^1_{1,a}(-1,1) \times H^1_{1,a}(-1,1) \to \mathbb{C}$ is the bilinear form defined by

$$\mathcal{B}(u, w) = \int_{-1}^{1} (\lambda^2 u\overline{w} + \sqrt{1-x^2}u_x\overline{w_x})dx + \lambda\overline{\zeta}u(1)\overline{w}(1)$$

and $\mathcal{L} : H^1_{1,a}(-1,1) \to \mathbb{C}$ is the linear functional given by

$$\mathcal{L}(w) = \int_{-1}^{1} (f_2 + \lambda f_1)\overline{w}dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)d\xi\overline{w}(1) + \tilde{\zeta}f_1(1)\overline{w}(1).$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (3.14) has a unique solution $u \in H^1_{1,a}(-1,1)$. By the regularity theory for the linear elliptic equations, it follows that $u \in H^2_a(-1,1)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. □

Now, we are able to state the following existence result of problem (3.3).

Theorem 3.2 *For $(u_0, u_1, 0) \in \mathcal{H}$, the problem (P') admits a unique weak solution*

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

and for $(u_0, u_1, 0) \in D(\mathcal{A})$, the problem (P') admits a unique strong solution

$$(u, u_t, \phi) \in C^0(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{D}(\mathcal{H})).$$

Proof of Theorem 3.2. The regularity of solution of the solution of the problem (P') is a consequence of the semigroup property.

Remark 3.1 • We can easily extend the global existence result for a general function $a(x)$ satisfying (H) instead of $\sqrt{1-x^2}$ (see introduction).

• In the case $\alpha = 1$, we take $\varrho u_t(1,t)$ instead of $\varrho \partial_t^{\alpha,\eta} u(1,t)$. We do not need to introduce an augmented system. In this case the operator $\mathcal{A}$ takes the form

$$\tilde{\mathcal{A}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (\sqrt{1-x^2} u_x)_x \end{pmatrix} \tag{3.15}$$

with domain

$$D(\tilde{\mathcal{A}}) = \left\{ \begin{array}{l} (u,v) \text{ in } \tilde{\mathcal{H}} : u \in H^2_a(-1,1) \cap H^1_{1,a}(-1,1), v \in H^1_{1,0}(-1,1), \\ (\sqrt{1-x^2} u_x)(1) + \rho v(1) = 0, \end{array} \right\}, \tag{3.16}$$

where

$$\tilde{\mathcal{H}} = H^1_{1,a}(-1,1) \times L^2(-1,1)$$

with inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle_{\tilde{\mathcal{H}}} = \int_{-1}^{1} \sqrt{1-x^2} u_x \overline{\tilde{u}}_x dx + \int_{-1}^{1} v \overline{\tilde{v}} dx.$$

The well-posedness result follows exactly as in the case $0 < \alpha < 1$. Moreover, the energy function is defined as

$$\tilde{E}(t) = \frac{1}{2} \int_{-1}^{1} (|u_t|^2 + \sqrt{1-x^2}|u_x|^2) dx \tag{3.17}$$

and decays as follows

$$\tilde{E}'(t) = -\rho |u_t(1,t)|^2 \leq 0.$$

4. Lack of exponential stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (3.3). In order to state and prove our stability results, we need some lemmas.

Theorem 4.1 ([11], [16]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{X}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{4.1}$$

and

$$\overline{\lim_{x\to+\infty}} \left\| (i\beta I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{X})} < \infty. \tag{4.2}$$

Our main result is the following.

Theorem 4.2 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable if $\alpha \neq 1$.*

Proof: We will examine two cases.
• **Case1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $F = (\sin(x+1), 0, 0)^T \in \mathcal{H}$, and assume that there exists $U = (\varphi, u, \phi)^T \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$. Then $\phi(\xi) = -|\xi|^{\frac{2\alpha-5}{2}} \sin 2$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]1, 0[$. So the operator $\mathcal{A}$ is not invertible.
• **Case2** $\eta \neq 0$:
We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, \phi)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - (\sqrt{1-x^2} u_x)_x = 0, \\ \lambda \phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0 \end{cases} \tag{4.3}$$

with boundary conditions

$$\begin{cases} (\sqrt{1-x^2} u_x)(1) + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) d\xi = 0, \\ u(-1) = 0. \end{cases} \tag{4.4}$$

It is well-known that Bessel functions play an important role in this type of problem. From $(4.3)_1$ - $(4.3)_2$ for such λ, we find

$$\lambda^2 u - (\sqrt{1-x^2} u_x)_x = 0. \tag{4.5}$$

Using the boundary conditions and $(4.3)_3$, we deduce that

$$\begin{cases} \lambda^2 u - (\sqrt{1-x^2} u_x)_x = 0, \\ (\sqrt{1-x^2} u_x)(1) + \zeta v(1) \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \lambda + \eta} d\xi = (\sqrt{1-x^2} u_x)(1) + \rho\lambda(\lambda+\eta)^{\alpha-1} u(1) = 0, \\ u(-1) = 0. \end{cases} \tag{4.6}$$

From $(4.6)_1$ for such λ , we find

$$(x^2 - 1)u_{xx} + x u_x + \lambda^2 \sqrt{1-x^2} u = 0. \tag{4.7}$$

We take $u(x) = \Psi(\xi),\ x = \cos\xi,\ \ k\pi < \xi < (k+1)\pi, \quad -1 < x < 1,$ we find

$$\Psi_{\xi\xi} - \lambda^2 \sin(\xi)\Psi = 0, \tag{4.8}$$

(see [**12**], p 454, 2.236), and equation (30) can write by

$$\Psi_{tt} - 4\lambda^2 cos(2t)\Psi = 0, \tag{4.9}$$

where $\xi = (2t + \frac{\pi}{2})$. Then we have

$$\Psi(2t + \frac{\pi}{2}) = C_1 J_0(2\lambda e^{it}) + C_2 Y_0(2\lambda e^{it})$$

(see [**18**], p.665). Then it becomes

$$u(x) = C_1 J_0\Big(2\lambda e^{i(\frac{\arccos x}{2} - \frac{\pi}{4})}\Big) + C_2 Y_0\Big(2\lambda e^{i(\frac{\arccos x}{2} - \frac{\pi}{4})}\Big),$$

where J_n and Y_n are defined by:

$$J_n(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{y}{2}\right)^{2m+n}$$

$$Y_n(y) = \tfrac{2}{\pi}\ln(\tfrac{x}{2})J_n(x) - \tfrac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$
$$-\tfrac{1}{\pi}\sum_{k=0}^{\infty}\left[\frac{\Gamma'(k+1)}{\Gamma(k+1)}+\frac{\Gamma'(k+n+1)}{\Gamma(k+n+1)}\right]\frac{(-1)^k\left(\frac{x}{2}\right)^{n+2k}}{k!(n+k)!}\ \text{if } n\neq 0.$$

If $n=0$, then

$$Y_0(y) = \frac{2}{\pi}\left\{\ln(\frac{x}{2})+\gamma\right\}J_0(x) + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{H_k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}, \quad H_k=\sum_{j=1}^{k}\frac{1}{j},$$

which γ is the Euler-Mascheroni constant,

$$\gamma = \lim_{n\to+\infty}(H_n-\ln(n)) \sim 0,5772...,$$

wehre J_n and Y_n are Bessel functions of the first kind and second kind of order n and are linearly independent and therefore the pair (J_0, Y_0) (classical result) forms a fundamental system of solutions of (4.7). Hence, given C_1 and C_2,

$$u(x) = C_1 J_0\left(2\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right) + C_2 Y_0\left(2\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right) \in H^1_{1,a}(-1,1) \tag{4.10}$$

with the following boundary condition

$$\begin{cases}(\sqrt{1-x^2}u_x)(1)+\rho\lambda(\lambda+\eta)^{\alpha-1}u(1)=0,\\ u(-1)=0.\end{cases} \tag{4.11}$$

Then

$$M(\lambda)\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{4.12}$$

where

$$M(\lambda)=\begin{pmatrix} i\lambda e^{\frac{-i\pi}{4}}J_1(2\lambda e^{\frac{-i\pi}{4}})+\rho\lambda(\lambda+\eta)^{\alpha-1}J_0(2\lambda e^{\frac{-i\pi}{4}}) & i\lambda e^{\frac{-i\pi}{4}}Y_1(2\lambda e^{\frac{-i\pi}{4}})+\rho\lambda(\lambda+\eta)^{\alpha-1}Y_0(2\lambda e^{\frac{-i\pi}{4}}) \\ J_0(2\lambda e^{\frac{i\pi}{4}}) & Y_0(2\lambda e^{\frac{i\pi}{4}})\end{pmatrix}$$

Hence, a non-trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes.
Set $f(\lambda)=detM(\lambda)$ thus the characteristic equation is $f(\lambda)=0$. Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0. Since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leqslant \Re(\lambda)\leqslant 0$ for some $\alpha_0>0$ large enough and for such λ, we remark that J_0, Y_0 remains bounded.

Lemma 4.1 *There exists $N\in\mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*,|k|\geq N}\subset\sigma(\mathcal{A}), \tag{4.13}$$

$$\lambda_k = i\frac{(2k+1)}{4\sqrt{2}}\pi+\frac{\tilde{\alpha}}{k^{1-\alpha}}+\frac{\tilde{\beta}}{k^{1-\alpha}}+o\left(\frac{1}{k^{1-\alpha}}\right),\quad k\geq N,\ \tilde{\alpha}\in i\mathbb{R},\ \beta\in\mathbb{R},\ \beta<0.$$
$$\lambda_k=\overline{\lambda_{-k}}\ \textit{if } k\leq -N,$$

Moreover, for all $|k|\geq N$, the eigenvalues λ_k are simple.

Proof.
For $\alpha\neq 1$
Step 1. We will use the following classical development (see [13] p. 122, (5.11.6)): for all $\delta>0$, the following development holds when $|\arg z|<\pi-\delta$:

$$J_\nu(z)=\left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)-\frac{(\nu-\frac{1}{2})(\nu+\frac{1}{2})}{2}\frac{\sin\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{z}+O\left(\frac{1}{|z|^2}\right)\right], \tag{4.14}$$

$$Y_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\sin\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)+\frac{(\nu-\frac{1}{2})(\nu+\frac{1}{2})}{2}\frac{\cos\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{z}+O\left(\frac{1}{|z|^2}\right)\right]. \tag{4.15}$$

Then

$$\begin{aligned} f(\lambda) &= \tfrac{ie^{-i\frac{\pi}{4}}}{\pi}\left(\cos(2\sqrt{2}\lambda i)+\tfrac{\rho e^{i\frac{\pi}{4}}}{i}(\eta+\lambda)^{\alpha-1}\sin(2\sqrt{2}\lambda i)+O\left(\frac{1}{\lambda}\right)\right)\\ &= \tfrac{ie^{-i\frac{\pi}{4}}}{\pi}\left(\cos(2\sqrt{2}\lambda i)+\tfrac{\rho e^{i\frac{\pi}{4}}\sin(2\sqrt{2}\lambda i)}{i\lambda^{1-\alpha}}+o\left(\frac{1}{\lambda^{1-\alpha}}\right)\right)\\ &= \tfrac{ie^{-i\frac{\pi}{4}}}{2\pi}e^{-2\sqrt{2}\lambda}\left(e^{4\sqrt{2}\lambda}+1-\tfrac{\rho e^{i\frac{\pi}{4}}}{\lambda^{1-\alpha}}(1-e^{4\sqrt{2}\lambda})+o\left(\frac{1}{\lambda^{1-\alpha}}\right)\right). \end{aligned}$$

We set

$$\tilde{f}(\lambda) = f_0(\lambda)+\frac{f_1(\lambda)}{\lambda^{1-\alpha}}+o\Big(\frac{1}{\lambda^{1-\alpha}}\Big), \tag{4.16}$$

where

$$f_0(\lambda) = e^{4\sqrt{2}\lambda}+1, \tag{4.17}$$

$$f_1(\lambda) = -\rho e^{i\frac{\pi}{4}}(1-e^{4\sqrt{2}\lambda}). \tag{4.18}$$

We look at the roots of f_0. From (4.17), f_0 has one family of roots that we denote λ_k^0.

$$f_0(\lambda)=0 \Leftrightarrow e^{4\sqrt{2}\lambda}+1=0$$

Hence

$$\lambda_k^0 = i\frac{(k+\frac{1}{2})}{2\sqrt{2}}\pi, \quad k\in\mathbf{Z}.$$

Note that f remain bounded in the strip $-\alpha_0 \leqslant \Re(\lambda) \leqslant 0$.

Step 2.

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (4.16) the unknown λ by $u=(4\sqrt{2}\lambda)$ then (4.16) becomes

$$\tilde{f}(u) = (e^u+1)+O\left(\frac{1}{u^{1-\alpha}}\right)$$

The roots of f_0 are $u_k=\frac{i(2k+1)}{4\sqrt{2}}$, $k\in\mathbb{Z}$, and setting $u=u_k+re^{it}$, $t\in[0,2\pi]$, we can easily check that there exists a constant $C>0$ independent of k such that $|e^u+1|\geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N\in\mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k=\lambda_k^0+o(1)$ which tends to the roots $\left(\frac{i(2k+1)}{4\sqrt{2}}\right)$ of f_0. Finally for $|k|\geq N$, λ_k is simple since λ_k^0 is.

Step3. From Step 2, we can write

$$\lambda_k = \frac{(2k+1)\pi}{4\sqrt{2}}i+\varepsilon_k. \tag{4.19}$$

Using (4.17) and (4.18), we get

$$f_0(\lambda_k) = -4\sqrt{2}\varepsilon_k+O(\varepsilon_k^2), \tag{4.20}$$

$$f_1(\lambda_k) = -2\rho e^{i\frac{\pi}{4}}+O(\varepsilon_k). \tag{4.21}$$

Substituting (4.20), (4.21) into (4.16), using the fact that $\tilde{f}(\lambda_k)=0$, we get:

$$\tilde{f}(\lambda_k) = \left(-4\sqrt{2}\varepsilon_k+O(\varepsilon_k^2)-\frac{2\rho e^{i\frac{\pi}{4}}+O(\varepsilon_k)}{(\frac{i(2k+1)\pi}{4\sqrt{2}}+\varepsilon_k)^{1-\alpha}}+o\left(\frac{1}{k^{1-\alpha}}\right)\right)$$

$$= \left(-4\sqrt{2}\varepsilon_k - \frac{2\rho e^{i\frac{\pi}{4}}}{\left(\frac{ik\pi}{2\sqrt{2}}\right)^{1-\alpha}} \right) + O(\varepsilon_k^2) + o\left(\frac{1}{k^{1-\alpha}}\right) = 0 \tag{4.22}$$

and hence

$$\varepsilon_k = -\frac{\rho}{(2\sqrt{2})^{\alpha}\pi^{1-\alpha}k^{1-\alpha}} \left(\cos(2\alpha-1)\frac{\pi}{4} + i\sin(2\alpha-1)\frac{\pi}{4} \right) + o\Big(\frac{1}{k^{\alpha-1}}\Big), \quad for\ k \geq 0. \tag{4.23}$$

From (4.23), we have in that case $|k|^{1-\alpha}\,\Re\lambda_k \sim \beta$, with

$$\beta = -\frac{\rho}{(2\sqrt{2})^{\alpha}\pi^{1-\alpha}} \cos(2\alpha-1)\frac{\pi}{4} < 0, \quad 0<\alpha<1, \quad \eta \geq 0.$$

Now, setting $\tilde{U}_k = (\lambda_k^0 - \mathcal{A})U_k$, where U_k is a normalized eigenfunction associated to λ_k. We then have

$$\begin{aligned} \|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \sup_{U\in\mathcal{H}, U\neq 0} \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}U\|_{\mathcal{H}}}{\|U\|_{\mathcal{H}}} &\geq \frac{\|(\lambda_k^0 - \mathcal{A})^{-1}\tilde{U}_k\|_{\mathcal{H}}}{\|\tilde{U}_k\|_{\mathcal{H}}} \\ &\geq \frac{\|U_k\|_{\mathcal{H}}}{\|(\lambda_k^0 - \mathcal{A})U_k\|_{\mathcal{H}}}. \end{aligned}$$

Hence, by Lemma 4.1, we deduce that

$$\|(\lambda_k^0 - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \geq c|k|^{1-\alpha}.$$

Thus, Theorem (4.1) is not satisfied. So that, the semigroup $e^{t\mathcal{A}}$ is not exponentially stable. Thus the proof is complete.

Remark 4.1 If $\alpha = 1$, similarly, we prove that there exists $N \in \mathbb{N}$ such that

$$\{\lambda_k\}_{k\in\mathbb{Z}^*, |k|\geq N} \subset \sigma(\mathcal{A}),$$

$$\lambda_k = i\frac{1}{4\sqrt{2}}(2k\pi + \theta) + \frac{1}{8\sqrt{2}} \ln \frac{\rho^2 + 1 - 2\rho\cos\frac{\pi}{4}}{\rho^2 + 1 + 2\rho\cos\frac{\pi}{4}} + O\left(\frac{1}{k}\right), \quad k \in \mathbf{Z},$$

and θ is such that

$$\begin{cases} \cos\theta &= \dfrac{(\rho^2 - 1)}{\sqrt{1+\rho^2+2\rho\cos\pi/4}\sqrt{1+\rho^2-2\rho\cos\pi/4}}, \\ \sin\theta &= \dfrac{2\rho\sin\pi/4}{\sqrt{1+\rho^2+2\rho\cos\pi/4}\sqrt{1+\rho^2-2\rho\cos\pi/4}}, \end{cases}$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N,$$

Moreover, for all $|k| \geq N$, the eigenvalues λ_k are simple.

□

5. Energy decay when $\eta \neq 0$

By Lemma 4.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues. Unfortunately we were not able to prove this decay rate by frequency domain method based on multiplier method as the problem (P) is degenerate and the fractional damping is acting on the degenerate boundary. To state and prove our stability results, we need some results from semigroup theory.

Theorem 5.1 ([2]- [14]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup$\{S(t)\}$ a Hilbert space $\mathcal{X}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set, then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{X}} \longrightarrow 0$ as $t \longrightarrow \infty$ for any $z \in \mathcal{X}$.

Theorem 5.2 ([17]) *Let X be a Hilbert space and let $\mathcal{A}$ be the generator of a bounded C_0-semigroup $(S(t))_{t\geq 0}$ on X. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq M(|\beta|),$$

where $M : \mathbb{R}_+ \to (0, \infty)$ is a continuous non-decreasing function of positive increase, then there exists a positive constant $C > 0$ such that

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{M^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t \to \infty.$$

In this section, by an explicit representation of the resolvent of the generator on the imaginary axis and the use of the Theorem by Rozendaal, stahn and Seifertr, we prove some decay rate. Our main result is the following.

Theorem 5.3 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (11) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0$$

If $\eta \neq 0$, then the global solution of the problem (P) has the following energy decay property

$$E(t) = \|e^{t\mathcal{A}}U_0\|^2_{\mathcal{H}} \leq \frac{c}{(\ln t)^2}\|U_0\|^2_{D(\mathcal{A})}$$

Proof. For the proof of Theorem 5.3, we need the following two lemmas.

Lemma 5.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof: We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.
Step 1. Solving for $\mathcal{A}U = 0$ leads to the system

$$\begin{cases} v = 0, \\ (\sqrt{1-x^2}u_x)_x = 0, \\ (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0 \end{cases} \tag{5.1}$$

together with the boundary conditions (4.4). Then $v = 0, \phi = 0, (\sqrt{1-x^2}u_x)(1) = 0$ and

$$(\sqrt{1-x^2}u_x)(x) = c,$$

where c is a constant. As $(\sqrt{1-x^2}u_x)(1) = 0$, we have $(\sqrt{1-x^2}u_x)(x) = 0$. Hence

$$u_x(x) = 0 \text{ for } x \in (-1, 1).$$

As $u(-1) = 0$, then $u = 0$. we have $U = 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.

Step 2. Let $\lambda \in \mathbb{R} - \{0\}$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$. Let $U = (u, v, \phi)^T$ with $\|U\|_{\mathcal{H}} = 1$, be such that

$$\mathcal{A}U = i\lambda U. \tag{5.2}$$

Using the definition of $\mathcal{A}$ it follows that $\mathcal{A}U = i\lambda U$ if and only if

$$\begin{cases} i\lambda u - v = 0, \\ i\lambda v - (\sqrt{1-x^2}u_x)_x = 0, \\ i\lambda\phi + (\xi^2 + \eta)\phi - v(1)\mu(\xi) = 0 \end{cases} \tag{5.3}$$

together with the boundary conditions (4.4). Using (3.4) and (5.2), we find

$$\phi \equiv 0, \tag{5.4}$$

then, using the third equation in (5.3), we deduce that

$$v(1)=0. \tag{5.5}$$

Therefore, from $(5.3)_1$ and $(4.4)_2$, we get

$$u(1)=0 \quad \text{and} \quad (\sqrt{1-x^2}u_x)(1)=0. \tag{5.6}$$

Thus, by eliminating v, the system (5.3) implies that

$$\begin{cases} \lambda^2 u + (\sqrt{1-x^2}u_x)_x = 0 \text{ on } (-1,1), \\ u(-1)=u(1)=0, \\ (\sqrt{1-x^2}u_x)(1)=0. \end{cases} \tag{5.7}$$

The solution of the equation (5.7) is given by

$$u(x)=C_1\Phi_+(x)+C_2\Phi_-(x),$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x)=J_0\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right), \quad \Phi_-(x)=Y_0\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right). \tag{5.8}$$

From boundary conditions $(5.7)_2$ and $(5.7)_3$, we deduce that

$$\begin{array}{l} C_1J_0\left(2\lambda e^{i\frac{\pi}{4}}\right)+C_2Y_0\left(2\lambda e^{i\frac{\pi}{4}}\right)=0, \\ C_1J_1\left(2\lambda e^{i\frac{\pi}{4}}\right)+C_2Y_1\left(2\lambda e^{i\frac{\pi}{4}}\right)=0, \\ C_1J_0\left(2i\lambda e^{i\frac{\pi}{4}}\right)+C_2Y_0\left(2i\lambda e^{i\frac{\pi}{4}}\right)=0. \end{array}$$

Using the fact that (the Wronskian $w(J_0(z),Y_0(z))$

$$J_0\left(2\lambda e^{i\frac{\pi}{4}}\right)Y_1\left(2\lambda e^{i\frac{\pi}{4}}\right)-J_1\left(2\lambda e^{i\frac{\pi}{4}}\right)Y_0\left(2\lambda e^{i\frac{\pi}{4}}\right)=-\frac{1}{\pi\lambda e^{i\frac{\pi}{4}}},$$

we deduce that $C_1=C_2=0$. Hence

$$u\equiv 0.$$

Therefore $U=0$, which contradicts $\|U\|_{\mathcal{H}}=1$. This completes the proof of Lemma 5.1.

□

Lemma 5.2
If $\lambda\neq 0$, the operator $i\lambda I-\mathcal{A}$ is surjective.
If $\lambda=0$ and $\eta\neq 0$, the operator $i\lambda I-\mathcal{A}$ is surjective.

Proof: To prove this, we need the following generalization of the Lax-Milgram Lemma.

Lemma 5.3 (Lax-Milgram-Fredholm, see [9]) *Let V and H be Hilbert spaces such that the embedding $V\subset H$ is compact and dense. Suppose that $a_V: V\times V\to\mathbb{C}$ and $a_H: H\times H\to\mathbb{C}$ are two bounded sesquilinear forms such that a_V is V-coercive and $G: V\to\mathbb{C}$ is a continuous conjugate linear form. The equation*

$$a_H(u,v)+a_V(u,v)=G(v), \qquad \forall v\in V$$

has either a unique solution $u\in V$ for all $G\in V'$ or has a nontrivial solution for $G=0$.

Case 1: $\lambda\neq 0$. Let $F=(f_1,f_2,f_3)^T\in\mathcal{H}$ be given, and let $U=(u,v,\phi)^T\in D(\mathcal{A})$ be such that

$$(i\lambda I-\mathcal{A})U=F. \tag{5.9}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (\sqrt{1-x^2}u_x)_x = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - \mu(\xi)v(1) = f_3 \end{cases} \tag{5.10}$$

together with the conditions (4.4).

We divide the proof into three steps, as follows:

Step 1. Inserting $(5.10)_1$ into $(5.10)_2$, we get

$$-\lambda^2 u - (\sqrt{1-x^2}u_x)_x = (f_2 + i\lambda f_1). \tag{5.11}$$

Solving system (5.11) is equivalent to finding $u \in H^2_a \cap H^1_{1,a}(-1,1)$ such that

$$\int_{-1}^{1} (-\lambda^2 u\overline{w} - (\sqrt{1-x^2}u_x)_x\overline{w}\, dx = \int_{-1}^{1} (f_2 + i\lambda f_1)\overline{w}\, dx. \tag{5.12}$$

for all $w \in H^1_{1,a}(-1,1)$. By using $(5.10)_3$ and $(5.10)_1$ the function u satisfies the following system

$$\begin{cases} \displaystyle\int_{-1}^{1} (-\lambda^2 u\overline{w} + \sqrt{1-x^2}u_x\overline{w}_x)\, dx + i\varrho\lambda(i\lambda+\eta)^{\alpha-1}u(1)\overline{w}(1) \\ \displaystyle= \int_{-1}^{1} (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_3(\xi)\, d\xi\ \overline{w}(1) + \varrho(i\lambda+\eta)^{\alpha-1}f_1(1)\overline{w}(1). \end{cases} \tag{5.13}$$

We can rewrite (5.13) as

$$L_\lambda(u,w) + a_{H^1_{1,a}(-1,1)}(u,w) = l(w), \tag{5.14}$$

where the sesquilinear forms $L_\lambda : L^2(-1,1) \times L^2(-1,1) \to \mathbb{C}$, $a_{H^1_{1,a}(-1,1)} : H^1_{1,a}(-1,1) \times H^1_{1,a}(-1,1) \to \mathbb{C}$ and the antilinear form $l : H^1_{1,a}(-1,1) \to \mathbb{C}$ are defined by

$$L_\lambda(u,w) = -\int_{-1}^{1} \lambda^2 u\overline{w}\, dx,$$

$$a_{H^1_{1,a}(-1,1)}(u,w) = \int_{-1}^{1} \sqrt{1-x^2}u_x\overline{w}_x\, dx + i\rho\lambda(i\lambda+\eta)^{\alpha-1}u(1)\overline{w}(1)$$

and

$$\begin{aligned} l(w) = \int_{-1}^{1} (f_2 + i\lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+i\lambda} f_3(\xi)\, d\xi\ \overline{w}(1) \\ +\rho(i\lambda+\eta)^{\alpha-1}f_1(1)\overline{w}(1). \end{aligned}$$

One can easily see that $L_\lambda, a_{H^1_{1,a}(-1,1)}$ and l are bounded. Furthermore

$$\begin{aligned} \Re a_{H^1_{1,a}(-1,1)}(u,u) &= \|(1-x^2)^{1/4}u_x\|_2^2 + \varrho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)|u(1)|^2 \\ &\geq \|(1-x^2)^{1/4}u_x\|_2^2, \end{aligned}$$

where we have used the fact that

$$\rho\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right) = \zeta\lambda^2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\, d\xi > 0.$$

Thus $a_{H^1_{1,a}(-1,1)}$ is coercive. Consequently, by Lemma 5.3, proving the existence of u solution of (5.14) reduces to proving that (5.14) with $l \equiv 0$ has a notrivial solution. Indeed if there exists $u \neq 0$, such that

$$L_\lambda(u,w) + a_{H^1_{1,a}(-1,1)}(u,w) = 0 \quad \forall w \in H^1_{1,a}(-1,1). \tag{5.15}$$

In particular for $w = u$, it follows that

$$\lambda^2\|u\|^2_{L^2(-1,1)} - i\rho\lambda(i\lambda+\eta)^{\alpha-1}|u(1)|^2 = \|(1-x^2)^{1/4}u_x\|^2_{L^2(-1,1)}.$$

Hence, we have

$$u(1) = 0. \tag{5.16}$$

From (5.15), we obtain

$$(\sqrt{1-x^2}u_x)(1) = 0 \tag{5.17}$$

and then

$$\begin{cases} -\lambda^2 u - (\sqrt{1-x^2}u_x)_x = 0, \\ u(1) = (\sqrt{1-x^2}u_x)(1) = 0, \\ u(-1) = 0. \end{cases} \tag{5.18}$$

We deduce that $U = 0$. Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Case 2: $\lambda = 0$ and $\eta \neq 0$. Using Lax-Milgram Lemma, we obtain the result.

Taking account of Lemmas 5.1, 5.2 and from Theorem 5.1 The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$.

□

Let us consider the resolvant equation

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - (\sqrt{1-x^2}u_x)_x = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(1)\mu(\xi) = f_3, \end{cases} \tag{5.19}$$

where $F = (f_1, f_2, f_3)^T \in \mathcal{H}$. From $(5.19)_1$ and $(5.19)_2$, we have

$$\lambda^2 u + (\sqrt{1-x^2}u_x)_x = -(f_2 + i\lambda f_1) \tag{5.20}$$

with

$$\begin{cases} (\sqrt{1-x^2}u_x)(1) = -\xi\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi, \\ \qquad u(-1) = 0. \end{cases} \tag{5.21}$$

The substitution of ϕ given by $(5.19)_3$ into $(4.4)_1$ gives us

$$(\sqrt{1-x^2}u_x)(1) = -\rho(i\lambda+\eta)^{\alpha-1}v(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}d\xi. \tag{5.22}$$

Moreover, from $(5.19)_1$, we have

$$v(1) = i\lambda u(1) - f_1(1).$$

Then, the condition (5.22) becomes

$$(\sqrt{1-x^2}u_x)(1) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}u(1) = \rho(i\lambda+\eta)^{\alpha-1}f_1(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}d\xi. \tag{5.23}$$

According to the theory of ordinary differential equations, the general solution of (5.20) can be written as

$$u(x) = C_1\Phi_+(x) + C_2\Phi_-(x) - i\pi\int_{-1}^{x}\Big(f_2(s)+i\lambda f_1(s)\Big)\Big(\Phi_+(s)\Phi_-(x) - \Phi_+(x)\Phi_-(s)\Big)ds, \tag{5.24}$$

where Φ_+ and Φ_- are defined by

$$\Phi_+(x) = J_0\Big(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\Big), \quad \Phi_-(x) = Y_0\Big(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\Big) \tag{5.25}$$

and

$$\Phi_+'(x) = -\frac{\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}}{\sqrt{1-x^2}}J_1\Big(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\Big), \Phi_-'(x) = -\frac{\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}}{\sqrt{1-x^2}}Y_1\Big(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\Big), \tag{5.26}$$

where we have used the following relation

$$J_n'(x) = \frac{1}{x}\Big(nJ_n(x) - xJ_{n+1}(x)\Big), \qquad Y_n'(x) = \frac{1}{x}\Big(nY_n(x) - xY_{n+1}(x)\Big), \quad n \geq 0, \tag{5.27}$$

$$J_0'(x) = -J_1(x), \quad Y_0'(x) = -Y_1(x), \quad n = 0, \tag{5.28}$$

$$u_x(x) = C_1\Phi_+'(x) + C_2\Phi_-'(x) - i\pi\int_{-1}^{x}\Big(f_2(s) + i\lambda f_1(s)\Big)\Big(\Phi_+(s)\Phi_-'(x) - \Phi_+'(x)\Phi_-(s)\Big)ds. \tag{5.29}$$

From where is follows

$$\begin{aligned}(\sqrt{1-x^2})u_x(x) &= -i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\Bigg(C_1 J_1(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}) + C_2 Y_1(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})})\Bigg)\\ &+ i\pi\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\int_{-1}^{x}\Big(f_2(s)+i\lambda f_1(s)\Big)\Bigg(\Phi_+(s)Y_1(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}) - J_1(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})})\Phi_-(s)\Bigg)ds\end{aligned} \tag{5.30}$$

From (5.23), (5.24) and (5.30), we conclude that

$$\begin{aligned}&C_1\Bigg(-i\lambda e^{\frac{-i\pi}{4}}J_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}J_0(2i\lambda e^{\frac{-i\pi}{4}})\Bigg)\\ &+C_2\Big(-i\lambda e^{\frac{-i\pi}{4}}Y_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}Y_0(2i\lambda e^{\frac{-i\pi}{4}})\Big) =\\ &-i\pi\lambda e^{\frac{-i\pi}{4}}\int_{-1}^{1}\Big(f_2(s)+i\lambda f_1(s)\Big)\Big(\Phi_+(s)Y_1(2i\lambda e^{\frac{-i\pi}{4}}) - J_1(2i\lambda e^{\frac{-i\pi}{4}})\Phi_-(s)\Big)ds\\ &+\rho(i\lambda+\eta)^{\alpha-1}f_1(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}d\xi\\ &-\rho\lambda(i\lambda+\eta)^{\alpha-1}\int_{-1}^{1}\Big(f_2(s)+i\lambda f_1(s)\Big)\Big(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s)\Big)ds\end{aligned} \tag{5.31}$$

and

$$u(-1) = C_1\Phi_+(-1) + C_2\Phi_-(-1) = 0, \tag{5.32}$$

where

$$\Phi_+(-1) = J_0(2i\lambda e^{\frac{i\pi}{4}}), \qquad \Phi_-(-1) = Y_0(2i\lambda e^{\frac{i\pi}{4}}).$$

Using (5.31) and (5.32), a linear system in A and B is obtained

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} C_1 \\ C_2\end{pmatrix} = \begin{pmatrix} K \\ \tilde{K}\end{pmatrix}, \tag{5.33}$$

where

$$\begin{cases} a_{11} &= -i\lambda e^{\frac{-i\pi}{4}}J_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}J_0(2i\lambda e^{\frac{-i\pi}{4}}),\\ a_{12} &= -i\lambda e^{\frac{-i\pi}{4}}Y_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}Y_0(2i\lambda e^{\frac{-i\pi}{4}}),\\ a_{21} &= J_0(2i\lambda e^{\frac{i\pi}{4}}),\\ a_{22} &= Y_0(2i\lambda e^{\frac{i\pi}{4}}),\end{cases}$$

$$\begin{cases} K = -i\pi\lambda e^{\frac{-i\pi}{4}}\int_{-1}^{1}\Big(f_2(s)+i\lambda f_1(s)\Big)\Big(\Phi_+(s)Y_1(2i\lambda e^{\frac{-i\pi}{4}}) - J_1(2i\lambda e^{\frac{-i\pi}{4}})\Phi_-(s)\Big)ds\\ -\rho\lambda(i\lambda+\eta)^{\alpha-1}\int_{-1}^{1}\Big(f_2(s)+i\lambda f_1(s)\Big)\Big(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s)\Big)ds\\ \qquad\qquad +\rho(i\lambda+\eta)^{\alpha-1}f_1(1) - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}d\xi,\\ \tilde{K} = 0.\end{cases}$$

Le the determinant of the linear system given in (5.33) be denoted by D. then

$$\begin{aligned} D &= Y_0(2i\lambda e^{\frac{i\pi}{4}})\left(-i\lambda e^{\frac{-i\pi}{4}}J_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}J_0(2i\lambda e^{\frac{-i\pi}{4}})\right) - \\ & \quad J_0(2i\lambda e^{\frac{i\pi}{4}})\left(-i\lambda e^{\frac{-i\pi}{4}}Y_1(2i\lambda e^{\frac{-i\pi}{4}}) + \rho i\lambda(i\lambda+\eta)^{\alpha-1}Y_0(2i\lambda e^{\frac{-i\pi}{4}})\right) \\ &= -i\lambda e^{\frac{-i\pi}{4}}\left(Y_0(2i\lambda e^{\frac{i\pi}{4}})J_1(2i\lambda e^{\frac{-i\pi}{4}}) - Y_1(2i\lambda e^{-\frac{i\pi}{4}})J_0(2i\lambda e^{\frac{i\pi}{4}})\right) \\ & \quad +\rho i\lambda(i\lambda+\eta)^{\alpha-1}\left(Y_0(2i\lambda e^{\frac{i\pi}{4}})J_0(2i\lambda e^{\frac{-i\pi}{4}}) - Y_0(2i\lambda e^{-\frac{i\pi}{4}})J_0(2i\lambda e^{\frac{i\pi}{4}})\right) \\ &= -\frac{e^{-\frac{i\pi}{4}}}{\pi}\cos 2\sqrt{2}\lambda - \frac{\rho}{\pi}(i\lambda+\eta)^{\alpha-1}\sin 2\sqrt{2}\lambda + O\left(\frac{1}{\lambda}\right) \end{aligned}$$

As $D \neq 0$ for all $\lambda \neq 0$, then C_1 and C_2 are uniqueley determined by (5.33). Hence the operator $(i\lambda I - \mathcal{A})$ is surjective for all $\lambda \neq 0$.
Now, it is easy to prove that

$$|D| \geq c|\lambda|^{\alpha-1} \text{ for large } \lambda. \tag{5.34}$$

In the following Lemma we will prove some technical inequalities which will be useful for showing the optimal polynomial decay of the solution.

Lemma 5.4 **(I)** *for all $\lambda \in \mathbb{R} - \{0\}$ large, we have*

$$\|\Phi_+\|_{L^2(-1,1)}, \|\Phi_-\|_{L^2(-1,1)} \leq \frac{c}{\sqrt{|\lambda|}}e^{|\lambda|}, \tag{5.35}$$

(II)

$$\begin{aligned} &\left\|(1-x^2)^{-1/4}J_0\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right)\right\|_{L^2(-1,1)}, \\ &\left\|(1-x^2)^{-1/4}Y_0\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right)\right\|_{L^2(-1,1)} \leq \frac{c}{\sqrt{|\lambda|}}e^{|\lambda|}, \end{aligned} \tag{5.36}$$

(III) *There exists a constant $C > 0$ such that, for all $f_1 \in H^1_{1,a}(-1,1), f_2 \in L^2(-1,1)$ and $\lambda \in \mathbb{R} - \{0\}$,*

$$\left|\int_{-1}^1 (f_2(s) + i\lambda f_1(s))(\Phi_+(s)\Phi_-(1) - \Phi_+(1)\Phi_-(s))\,ds\right| \leq ce^{|\lambda|}\left(\|f_1\|_{H^1_{1,a}(-1,1)} + \|f_2\|_{L^2(-1,1)}\right). \tag{5.37}$$

The proof of Lemma 5.4 will be given in Appendix A.

Now

$$C_1 = \frac{1}{D}(Ka_{22} - \tilde{K}a_{12})$$

$$C_2 = \frac{1}{D}(\tilde{K}a_{11} - Ka_{21})$$

Considering only the dominant terms of λ, the following is obtained:

$$\begin{aligned} |D||C_1| &\leq \leq c\sqrt{|\lambda|}\ e^{|\lambda|}, \\ |D||C_2| &\leq \leq c\sqrt{|\lambda|}\ e^{|\lambda|}. \end{aligned}$$

Then

$$\|u\|_{L^2(-1,1)} \leq |\lambda|^{1-\alpha}\, e^{2|\lambda|}(\|f_1\|_{H^1_{1,a}(-1,1)} + \|f_2\|_{L^2(-1,1)}). \tag{5.38}$$

Using $(5.19)_1$ and (5.38), we get

$$\|v\|_{L^2(-1,1)} \leq c\,|\lambda|^{2-\alpha}\, e^{2|\lambda|}(\|f_1\|_{H^1_{1,a}(-1,1)} + \|f_2\|_{L^2(-1,1)}).$$

From (5.29), we get

$$\|(1-x^2)^{\frac{1}{4}}u_x\|_{L^2(-1,1)} \leq c\,|\lambda|^{2-\alpha}\, e^{2|\lambda|}(\|f_1\|_{H^1_{1,a}(-1,1)} + \|f_2\|_{L^2(-1,1)}).$$

From (3.4), we have

$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \frac{1}{\eta}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2 d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}} \leq c\,|\lambda|^{2-\alpha}\, e^{2|\lambda|}, \qquad as\ \ |\lambda|\to\infty.$$

The conclusion then follows by applying Theorem 5.1.

□

5.1. Appendix A. Proof of Lemma 5.4

Proof: Suppose that $\lambda \neq 0$. It is enough to consider $\lambda > 0$.

$$\|\Phi_+\|^2_{L^2(-1,1)} = \int_{-1}^{1}\left|J_0\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right)\right|^2 dx. \tag{5.39}$$

Let $s=\frac{\arccos x}{2}+\frac{\pi}{4}$ in equation (5.39), we get

$$\begin{aligned}\|\Phi_+\|^2_{L^2(-1,1)} &= 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}|\cos 2s|\left|J_0\left(2\lambda e^{is}\right)\right|^2 ds \leq 2|I_0(2|\lambda|)|^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}|\cos 2s|\,ds\\ &\leq \frac{c}{|\lambda|}e^{2|\lambda|},\end{aligned}$$

where I_0 is the modified Bessel function of the first kind of order 0. It is defined as

$$I_n(z)=\sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}.$$

(we have used $I_0(z)\equiv e^z/\sqrt{2\pi z}$ as $|z|\to\infty$). Similarly, we prove that

$$\|\Phi_-\|_{L^2(-1,1)} \leq \frac{c}{\sqrt{|\lambda|}}e^{|\lambda|}.$$

(II)

$$\begin{aligned}\left\|(1-x^2)^{-1/4}J_1\left(2i\lambda e^{i(\frac{\arccos x}{2}-\frac{\pi}{4})}\right)\right\|^2_{L^2(-1,1)} &= 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left|J_1\left(2\lambda e^{is}\right)\right|^2 ds\\ &\leq \frac{c}{|\lambda|}e^{2|\lambda|}\end{aligned}$$

(III) Let $f_1\in H^1_{1,a}(-1,1)$ and $f_2\in L^2(-1,1)$. We have

$$\begin{aligned}\left|\int_{-1}^{1}f_i(s)\Phi_+(s)\,ds\right| &\leq \|f_i\|_{L^2(-1,1)}\|\Phi_+\|_{L^2(-1,1)}\\ &\leq c\|f_i\|_{L^2(-1,1)}\frac{e^{|\lambda|}}{\sqrt{|\lambda|}}\end{aligned}$$

□

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