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$*$-η-Ricci-Yamabe solitons on LP-Sasakian manifolds

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ABSTRACT: In the present note, we characterize LP-Sasakian manifolds endowed with $*$-η-Ricci-Yamabe solitons. Finally, the existence of $*$-η-Ricci-Yamabe solitons in an LP-Sasakian manifold has been proved by constructing a non-trivial example.

Key Words: LP-Sasakian manifolds, $*$-η-Ricci-Yamabe solitons, M-projective curvature tensor, η-Einstein manifolds, generalized η-Einstein manifolds

Contents

1. Introduction

About three decades ago Hamilton ([13], [14]) introduced the notion of Ricci flow and Yamabe flow. The Yamabe flow on a smooth Riemannian (or semi-Riemannian) manifold M can be defined as the evolution of the Riemannian (or semi-Riemannian) metric g_0 in time t to $g = g(t)$ by the equation

$$\tfrac{\partial}{\partial t} g(t) = -rg, \ g(0) = g_0,$$

where $r(t)$ is the scalar curvature of the metric $g(t)$. In two dimension, the Yamabe flow is equivalent to Ricci flow defined by

$$\tfrac{\partial}{\partial t} g(t) = -2S(g(t)),$$

where S denotes the Ricci tensor. However, for the dimension higher than two, there is not such an equivalence (since the Yamabe flow preserve the conformal class of metric but the Ricci flow does not in general).

The Ricci and Yamabe solitons emerge as the limit of the solutions of the Ricci and Yamabe flows, respectively. On a Riemannian manifold admitting a vector field K the Ricci and Yamabe solitons are defined by

$$\pounds_K g + 2S + 2\Lambda g = 0,$$

and

$$\pounds_K g + 2(r - \Lambda)g = 0$$

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respectively; where $\pounds_K$ is the Lie derivative operator along the vector field K (called the soliton vector field) on M and Λ is a real number.

In 2018, Deshmukh and Chen ([3], [5]) studied Yamabe solitons to find sufficient conditions on the soliton vector field so that the metric of the Yamabe soliton is of constant scalar curvature. Yamabe solitons have also been studied in ([4], [6], [26], [31]) and many others.

Recently in 2019, a new class of geometric flows called Ricci-Yamabe flow of the type (α, β) was introduced by Güler and Crasmareanu [11] and is defined by

$$\frac{\partial}{\partial t}g(t) = -2\alpha S(g(t)) + \beta r(t)g(t), \quad g(0) = g_0.$$

After Güler and Crasmareanu, Dey [7] introduced the notion of Ricci-Yamabe solitons, according to him a Ricci-Yamabe soliton of the type (α, β) is a Riemannian manifold which satisfies the following equation:

$$\pounds_K g + 2\alpha S + (2\Lambda - \beta r)g = 0,$$

where $\alpha, \beta \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers). Also we note that Ricci-Yamabe solitons of type $(\alpha, 0)$ and of type $(0, \beta)$ are known as α-Ricci and β-Yamabe solitons, respectively.

The notion of $*$-Ricci soliton has been studied by Kaimakamis and Panagiotidou [18] within the framework of real hypersurfaces of complex space forms. Here, it is mentioned that the notion of $*$-Ricci tensor was first introduced by Tachibana [27] on almost Hermitian manifolds and further studied by Hamada [12] on real hypersurfaces of non-flat complex space forms.

A Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton if there exists a smooth vector field K and a real number Λ, such that

$$\pounds_K g + 2S^* + 2\Lambda g = 0, \tag{1.1}$$

where

$$S^*(X, Y) = g(Q^*X, Y) = Trace\,\{\phi \circ R(X, \phi Y)\},$$

for all vector fields X, Y on M. Here, ϕ is a tensor field of type $(1, 1)$, S^* is the $*$-Ricci tensor of type $(0, 2)$ and Q^* is the $*$-Ricci operator. In this connection, we recommend the papers ([10], [15], [16], [17], [22], [23], [25], [28]) and the references therein for more details about the study of Ricci solitons, η-Ricci solitons and $*$-Ricci solitons.

In 2020, Dey and Roy [8] introduced and studied the notion of $*$-η-Ricci soliton on an n-dimensional Sasakian manifold. A Riemannian manifold (M, g) is called $*$-η-Ricci soliton if

$$\pounds_\xi g + 2S^* + 2\Lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where $\mu \in \mathbb{R}$.

Motivated by the above studies, in this paper we introduce the notion of $*$-η-Ricci-Yamabe soliton of type (α, β) which is a Riemannian manifold satisfying

$$\pounds_K g + 2\alpha S^* + (2\Lambda - \beta r^*)g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where Λ, μ, α and β are constants. A $*$-η-Ricci-Yamabe soliton is said to be shrinking, steady or expanding if it admits a soliton vector for which Λ is negative, zero or positive, respectively. In particular, if $\mu = 0$, then the notion of $*$-η-Ricci-Yamabe soliton $(g, K, \Lambda, \mu, \alpha, \beta)$ reduces to the notion of $*$-Ricci-Yamabe soliton $(g, K, \Lambda, \alpha, \beta)$. Recently, α-cosymplectic manifold admitting $*$-conformal η-Ricci-Yamabe solitons admitting quarter-symmetric metric connection have studied by Zhang et al. [32].

The outline of this paper is as follows: In section 2, we collect the basic results and some basic definitions of LP-Sasakian manifolds. Section 3 deals with the study of $*$-η-Ricci-Yamabe solitons on LP-Sasakian manifolds in which we obtain some important characterizations for the manifold. Section 4 is concerned with the study M-projectively flat and quasi-M-projectively flat LP-Sasakian manifolds admitting $*$-η-Ricci-Yamabe solitons. Finally, we construct a 3-dimensional non-trivial example of an LP-Sasakian manifold to prove the existence of a $*$-η-Ricci-Yamabe soliton on the manifold.

2. Preliminaries

Let M be an n-dimensional smooth manifold equipped with a quartet (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is the unit timelike vector field, η is a 1-form and a Lorentzian metric g on M such that [2,20]

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad rank(\phi) = n-1 \tag{2.2}$$

for all $X \in \chi(M)$; where $\chi(M)$ denotes the collection of all smooth vector fields on M. The manifold M is said to have an almost para-contact metric structure (ϕ, ξ, η, g) when it admits a Lorentzian metric g, such that

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \tag{2.3}$$

for all $X, Y \in \chi(M)$. If moreover,

$$(\nabla_X \phi)Y = \eta(X)\phi^2 X + g(\phi X, \phi Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi X \iff (\nabla_X \eta)Y = g(\phi X, Y) = g(X, \phi Y), \tag{2.5}$$

then (M, ϕ, ξ, η, g) is called a Lorentzian para-Sasakian manifold (briefly, LP-Sasakian manifold), where ∇ denotes the Levi-Civita connection of the manifold [19].
For an n-dimensional LP-Sasakian manifold, the following relations hold [1,24]:

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.6}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$S(X,\xi) = (n-1)\eta(X) \iff Q\xi = (n-1)\xi, \tag{2.8}$$

for any $X, Y \in \chi(M)$.

Definition 2.1 *The M-projective curvature tensor H in an n-dimensional LP-Sasakian manifold is defined by [21]*

$$\begin{aligned} H(Y,Z)W \;=\; & R(Y,Z)W - \frac{1}{2(n-1)}[S(Z,W)Y - S(Y,W)Z \\ & + g(Z,W)QY - g(Y,W)QZ], \end{aligned} \tag{2.9}$$

where R and S are the curvature tensor and the Ricci tensor of M, respectively; and Q is the Ricci operator defined as $S(Z,W) = g(QZ,W)$.

Definition 2.2 *An LP-Sasakian manifold M is said to be a generalized η-Einstein if its non-vanishing Ricci tensor S is of the form [30]*

$$S(X,Y) = A_1 g(X,Y) + A_2 \eta(X)\eta(Y) + A_3 g(\phi X, Y), \tag{2.10}$$

where A_1, A_2 and A_3 are smooth functions on M. If $A_3 = 0$ (resp., $A_2 = A_3 = 0$), then M is called an η-Einstein (resp,. Einstein) manifold.

Definition 2.3 *A vector field K is said to be an affine conformal vector field if it satisfies [9]*

$$(\pounds_K \nabla)(X,Y) = X(\rho)Y + Y(\rho)X - g(X,Y) grad\rho, \tag{2.11}$$

where ρ is a smooth function on M. If ρ is a constant, then K is called an affine vector field.

Lemma 2.4 *The $*$-Ricci tensor of an n-dimensional LP-Sasakian manifold is given by [15]*

$$S^*(X,Y) = S(X,Y) + (n-2)g(X,Y) - g(X,\phi Y)\theta + (2n-3)\eta(X)\eta(Y), \tag{2.12}$$

where θ is the trace of ϕ and $X, Y \in \chi(M)$.

From (2.12) it follows that

$$r^* = r + n^2 - 4n + 3 + \theta^2. \tag{2.13}$$

3. $*$-η-Ricci-Yamabe solitons on LP-Sasakian manifolds

Theorem 3.1 *If (M,g) is an n-dimensional LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,\xi,\Lambda,\mu,\alpha,\beta)$, then M is a generalized η-Einstein manifold. Moreover, the scalars Λ and μ are related by $\Lambda-\mu=\frac{\beta r^*}{2}$.*

Proof: Let the metric of an n-dimensional LP-Sasakian manifold be a $*$-η-Ricci-Yamabe soliton $(g,\xi,\Lambda,\mu,\alpha,\beta)$, then (1.3) turns to

$$g(\nabla_Y\xi,Z)+g(Y,\nabla_Z\xi)+2\alpha S^*(Y,Z)+(2\Lambda-\beta r^*)g(Y,Z)+2\mu\eta(Y)\eta(Z)=0 \tag{3.1}$$

for any vector fields Y and Z on M. By making use of (2.5) and (2.12), (3.1) leads to

$$S(Y,Z)=A_1g(Y,Z)+A_2g(Y,\phi Z)+A_3\eta(Y)\eta(Z), \tag{3.2}$$

where $A_1=-[n-2+\frac{1}{\alpha}(\Lambda-\frac{\beta r^*}{2})]$, $A_2=-(\frac{1}{\alpha}-\theta)$ and $A_3=-(2n-3+\frac{\mu}{\alpha})$, $\alpha\neq 0$.
By putting $Z=\xi$ in (3.2), then using (2.1) and (2.2), we get

$$S(Y,\xi)=(A_1-A_3)\eta(Y), \tag{3.3}$$

where $A_1-A_3=n-1+\frac{1}{\alpha}(\mu-\Lambda+\frac{\beta r^*}{2})$ and $\alpha\neq 0$.

In view of (2.8), from (3.3) it follows that

$$\Lambda-\mu=\frac{\beta r^*}{2}. \tag{3.4}$$

Thus, from (3.2) and (3.4), we state the above Theorem 3.1. □

On contracting (3.2), we find

$$r=-(n-1)(n-3+\frac{\mu}{\alpha})+\theta^2-\frac{\theta}{\alpha}, \tag{3.5}$$

where μ and $\alpha(\neq 0)$ are constants. In particular, if we take $\theta=\alpha=1$ and $\mu=0$ in (3.2) and (3.5), then we find $S(Y,Z)=-(n-2)g(Y,Z)-(2n-3)\eta(Y)\eta(Z)$ and $\Lambda=\frac{\beta}{2}$, respectively; being $r=-(n-1)(n-3)$ and $r^*=\theta^2=1$. Thus we have

Corollary 3.2 *Let M be an n-dimensional LP-Sasakian manifold. If the manifold admits a $*$-Ricci-Yamabe soliton $(g,\xi,\Lambda,0,1,\beta)$ with $\theta=1$, then M is an η-Einstein manifold. Moreover, the manifold is shrinking, steady or expanding according to $\beta<0,=0$ or >0.*

Theorem 3.3 *If (M,g) is an n-dimensional LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,K,\Lambda,\mu,\alpha,\beta)$ such that the vector field K is an affine conformal vector field, then (M,g) is a generalized η-Einstein manifold and K is an affine vector field, provided trace ϕ =constant.*

Proof: By using (2.12) in (1.3), we have

$$\begin{aligned}(\pounds_Kg)(Y,Z)= & -2\alpha S(Y,Z)-[(2\Lambda-\beta r^*)+2\alpha(n-2)]g(Y,Z)\\ & +2\theta\alpha g(Y,\phi Z)-2[\alpha(2n-3)+\mu)]\eta(Y)\eta(Z).\end{aligned} \tag{3.6}$$

From the above equation, we can easily find

$$\begin{aligned}(\nabla_X\pounds_Kg)(Y,Z) &= 2\alpha\theta[g(Z,X)\eta(Y)+g(Y,X)\eta(Z)+2\eta(Y)\eta(X)\eta(Z)]\\ &\quad -2(\alpha(2n-3)+\mu)[g(Y,\phi X)\eta(Z)+g(Z,\phi X)\eta(Y)]\\ &\quad -2\alpha(\nabla_XS)(Y,Z)+\beta X(r^*)g(Y,Z).\end{aligned} \tag{3.7}$$

Following Yano [29], the following formula:

$$(\pounds_K\nabla_X g-\nabla_X\pounds_K g-\nabla_{[K,X]}g)(Y,Z)=-g((\pounds_K\nabla)(X,Y),Z)-g((\pounds_K\nabla)(X,Z),Y)$$

is well-known for any X,Y,Z on M. As g is parallel with respect to ∇, the above relation turns to

$$(\nabla_X\pounds_K g)(Y,Z)=g((\pounds_K\nabla)(X,Y),Z)+g((\pounds_K\nabla)(X,Z),Y),$$

which in view of (2.11) takes the form

$$(\nabla_X\pounds_K g)(Y,Z)=2X(\rho)g(Y,Z). \tag{3.8}$$

Putting $Y=Z=\xi$ in (3.7), and using (2.1), (2.2), (2.6) and (3.8), we get

$$2X(\rho)=\beta X(r^*). \tag{3.9}$$

From (3.7)-(3.9), we find

$$\begin{aligned}\alpha(\nabla_X S)(Y,Z)&=\alpha\theta[g(Z,X)\eta(Y)+g(Y,X)\eta(Z)+2\eta(Y)\eta(Z)\eta(X)]\\&\quad-(\alpha(2n-3)+\mu)[g(Y,\phi X)\eta(Z)+g(Z,\phi X)\eta(Y)],\end{aligned}$$

which by replacing $Z=\xi$ turns to

$$(\nabla_X S)(Y,\xi)=-\theta(g(Y,X)+\eta(Y)\eta(X))+(2n-3+\frac{\mu}{\alpha})g(Y,\phi X). \tag{3.10}$$

Now by the covariant differentiation of (2.8), we have

$$(\nabla_X S)(Y,\xi)=-S(Y,\phi X)+(n-1)g(Y,\phi X). \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$S(Y,\phi X)=\theta[g(Y,X)+\eta(Y)\eta(X)]-(n-2+\frac{\mu}{\alpha})g(Y,\phi X). \tag{3.12}$$

By replacing X by ϕX in (3.12) and using (2.1), we get

$$S(X,Y)=-(n-2+\frac{\mu}{\alpha})g(X,Y)+\theta g(Y,\phi X)-(2n-3+\frac{\mu}{\alpha})\eta(X)\eta(Y). \tag{3.13}$$

By contracting (3.13) over X and Y, we get $r=-(n-1)(n-3+\frac{\mu}{\alpha})+\theta^2$. If θ=trace ϕ=constant, then r and hence r^* is constant. Therefore, from (3.9) we have $X(\rho)=0$. This implies that ρ is constant. Therefore, K is an affine vector field. This completes the proof. □

We have the following lemma for the further use:

Lemma 3.4 *An n-dimensional LP-Sasakian manifold satisfies the following relations [15]*

$$(\nabla_Z Q)\xi=(n-1)\phi Z-Q\phi Z, \tag{3.14}$$

$$(\nabla_\xi Q)Z=-2Q\phi Z+2\theta Z+2\theta\eta(Z)\xi, \tag{3.15}$$

where Q is the Ricci operator.

Now we prove the following theorem:

Theorem 3.5 *If (M,g) is an LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,K,\Lambda,\mu,\alpha,\beta)$ such that the vector field K is the gradient Dr of r defined by (1.3), then either $K=-\xi(r)\xi$ or $\beta=-2$, provided $\theta=trace\phi=constant$*

Proof: We assume that (M,g) is an LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,K,\Lambda,\mu,\alpha,\beta)$ such that the vector field K is the gradient Dr of r, i.e., $K=Dr$. Then from (1.3) we find

$$\nabla_Y Dr = -\alpha QY - [\alpha(n-2)+\Lambda-\frac{\beta r^*}{2}]Y + \theta\alpha\phi Y - [\mu+\alpha(2n-3)]\eta(Y)\xi \tag{3.16}$$

for any Y on M.
Taking the covariant derivative of (3.16) with respect to Z, and using (2.5) and (2.7), we get

$$\begin{aligned}\nabla_Z\nabla_Y Dr &= -\alpha[(\nabla_Z Q)Y + Q(\nabla_Z Y)] - [\alpha(n-2)+\Lambda-\frac{\beta r^*}{2}]\nabla_Z Y + \frac{\beta}{2}Z(r^*)Y\\ &\quad +\theta\alpha[g(Y,Z)\xi+\eta(Y)Z+2\eta(Y)\eta(Z)\xi] + \theta\alpha\phi\nabla_Z Y\\ &\quad -(\mu+\alpha(2n-3))(g(\phi Y,Z)\xi+\eta(\nabla_Z Y)\xi+\eta(Y)\phi Z).\end{aligned} \tag{3.17}$$

By interchanging Y and Z in (3.17), we have

$$\begin{aligned}\nabla_Y\nabla_Z Dr &= -\alpha[(\nabla_Y QZ + Q(\nabla_Y Z)] - [\alpha(n-2)+\Lambda-\frac{\beta r^*}{2}]\nabla_Y Z + \frac{\beta}{2}Y(r^*)Z\\ &\quad +\theta\alpha[g(Z,Y)\xi+\eta(Z)Y+2\eta(Z)\eta(Y)\xi] + \theta\alpha\phi\nabla_Y Z\\ &\quad -(\mu+\alpha(2n-3))(g(\phi Z,Y)\xi+\eta(\nabla_Y Z)\xi+\eta(Z)\phi Y).\end{aligned} \tag{3.18}$$

In view of (3.16), we also have

$$\begin{aligned}\nabla_{[Y,Z]}Dr &= -\alpha Q(\nabla_Y Z) + \alpha Q(\nabla_Z Y) - (\lambda-\frac{\beta r^*}{2}+\alpha(n-2))\nabla_Y Z\\ &\quad +(\Lambda-\frac{\beta r^*}{2}+\alpha(n-2))\nabla_Z Y + \theta\alpha\phi\nabla_Y Z - \theta\alpha\phi\nabla_Z Y\\ &\quad -(\mu+\alpha(2n-3))\eta(\nabla_Y Z)\xi + (\mu+\alpha(2n-3))\eta(\nabla_Z Y)\xi.\end{aligned} \tag{3.19}$$

From (3.17)-(3.19), we get

$$\begin{aligned}R(Y,Z)Dr &= \theta\alpha(\eta(Z)Y-\eta(Y)Z) + (\mu+\alpha(2n-3))(\eta(Y)\phi Z-\eta(Z)\phi Y).\\ &\quad +\alpha((\nabla_Z Q)Y-(\nabla_Y Q)Z) + \frac{\beta}{2}(Y(r^*)Z-Z(r^*)Y).\end{aligned} \tag{3.20}$$

By replacing Y by ξ in (3.20), and using (2.1), (2.6), (3.14) and (3.15) we get

$$\begin{aligned}Z(r)\xi-\xi(r)Z &= \frac{\beta}{2}(\xi(r^*)Z-Z(r^*)\xi) + \theta\alpha(\eta(Z)\xi+Z) - (\mu+\alpha(2n-3))\phi Z\\ &\quad +\alpha[(n-1)\phi Z - Q\phi Z + 2Q\phi Z - 2\theta Z - 2\theta\eta(Z)\xi].\end{aligned} \tag{3.21}$$

Taking the inner product of (3.21) with ξ, and using (2.1), (2.2) and (2.13) we lead to

$$(1+\frac{\beta}{2})(Z(r)+\xi(r)\eta(Z)) = 0. \tag{3.22}$$

Therefore, we have either $\beta=-2$, or $K=Dr=-\xi(r)\xi$. This completes the proof. □

4. M-projectively flat and quasi-M-projectively flat LP-Sasakian manifolds admitting $*$-η-Ricci-Yamabe solitons

Theorem 4.1 *If (M,g) is an n-dimensional M-projectively flat LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,\xi,\Lambda,\mu,\alpha,\beta)$, then M is an η-Einstein manifold. Moreover, the scalars Λ and μ are related by $\Lambda-\mu=\frac{\beta r^*}{2}$.*

Proof: First we consider an n-dimensional M-projectively flat LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton, i.e., $H(Y,Z)W = 0$. Then from (2.9), we have

$$R(Y,Z)W = \frac{1}{2(n-1)}[S(Z,W)Y - S(Y,W)Z + g(Z,W)QY - g(Y,W)QZ],$$

which by putting $W = \xi$, and using (2.7) and (3.3) reduces to

$$[n-1+\frac{1}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})](\eta(Z)Y - \eta(Y)Z) = \eta(Z)QY - \eta(Y)QZ. \tag{4.1}$$

By putting $Z = \xi$, (4.1) takes the form

$$[n-1+\frac{1}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})]Y + \frac{2}{\alpha}(\Lambda-\mu-\frac{\beta r}{2})\eta(Y)\xi = QY. \tag{4.2}$$

By taking the inner product of (4.2) with X, we find

$$S(Y,X) = [n-1+\frac{1}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})]g(Y,X) + \frac{2}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})\eta(Y)\eta(X). \tag{4.3}$$

Now taking $X = \xi$ in (4.3), and using (2.1), (2.3) and (2.8), we obtain

$$\Lambda - \mu = \frac{\beta r^*}{2}, \quad \alpha \neq 0. \tag{4.4}$$

□

Next, on contracting (4.3) and making use of (4.4), we get $r = n(n-1)$. In particular, for $\mu = 0$, (4.4) reduces to $\Lambda = \frac{\beta r^*}{2}$. Thus we have

Corollary 4.2 *Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. If the manifold admits a $*$-Ricci-Yamabe soliton $(g,\xi,\Lambda,\alpha,\beta)$, then the manifold is shrinking, steady or expanding according to $\beta < 0, = 0$ or > 0.*

Theorem 4.3 *If (M,g) is an n-dimensional quasi-M-projectively flat LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton $(g,\xi,\Lambda,\mu,\alpha,\beta)$, then M is an η-Einstein manifold. Moreover, the scalars Λ and μ are related by $\Lambda - \mu = \frac{\beta r^*}{2}$.*

Proof: Now we consider an n-dimensional quasi-M-projectively flat LP-Sasakian manifold admitting a $*$-η-Ricci-Yamabe soliton, i. e., $g(H(Y,Z)W,\phi X) = 0$. Then from (2.9) it follows that

$$g(R(Y,Z)W,\phi X) = \frac{1}{2(n-1)}[S(Z,W)g(Y,\phi X) - S(Y,W)g(Z,\phi X) \\ + g(Z,W)S(Y,\phi X) - g(Y,W)S(Z,\phi X)],$$

which by putting $Z = W = \xi$ and using (2.1), (2.2), (2.7) and (3.3) reduces to

$$[n-1+\frac{1}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})]g(Y,\phi X) = S(Y,\phi X). \tag{4.5}$$

By replacing X by ϕX in (4.5) and using (2.1), we obtain

$$S(Y,X) = [n-1+\frac{1}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})]g(Y,X) + \frac{2}{\alpha}(\Lambda-\mu-\frac{\beta r^*}{2})\eta(Y)\eta(X), \quad \alpha \neq 0. \tag{4.6}$$

Now, by taking $X = \xi$ in (4.6), and using (2.1), (2.3) and (2.8), we obtain

$$\Lambda - \mu = \frac{\beta r^*}{2}, \quad \alpha \neq 0. \tag{4.7}$$

□

Next, on contracting (4.6) and making use of (4.7), we get $r = n(n-1)$. In particular, for $\mu = 0$, (4.7) reduces to $\Lambda = \frac{\beta n(n-1)}{2}$. Thus we have

Corollary 4.4 *Let M be an n-dimensional quasi-M-projectively flat LP-Sasakian manifold. If the manifold admits a $*$-Ricci-Yamabe soliton $(g,\xi,\Lambda,\alpha,\beta)$, then the manifold is shrinking, steady or expanding according to $\beta < 0, = 0$ or > 0.*

5. Example

We consider a 3-dimensional manifold $M = \{(t_1, t_2, t_3) \in R^3\}$, where (t_1, t_2, t_3) are the standard coordinates in R^3. Let ϱ_1, ϱ_2 and ϱ_3 be the vector fields on M given by

$$\varrho_1 = cosht_3\frac{\partial}{\partial t_1} + sinht_3\frac{\partial}{\partial t_2}, \quad \varrho_2 = sinht_3\frac{\partial}{\partial t_1} + cosht_3\frac{\partial}{\partial t_2}, \quad \varrho_3 = \frac{\partial}{\partial t_3} = \xi,$$

which are linearly independent at each point of M. Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(\varrho_1, \varrho_1) = \quad g(\varrho_2, \varrho_2) = 1, \quad g(\varrho_3, \varrho_3) = -1, \quad g(\varrho_1, \varrho_2) = g(\varrho_1, \varrho_3) = g(\varrho_2, \varrho_3) = 0.$$

Let η be the 1-form on M defined by $\eta(X) = g(X, \varrho_3)$ for all $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi\varrho_1 = -\varrho_2, \quad \phi\varrho_2 = -\varrho_1, \quad \phi\varrho_3 = 0.$$

The linearity property of ϕ and g yields

$$\eta(\varrho_3) = -1, \quad \phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$.
Now, by direct computations, we obtain

$$[\varrho_1, \varrho_2] = 0, \qquad [\varrho_2, \varrho_3] = -\varrho_1, \qquad [\varrho_1, \varrho_3] = -\varrho_2.$$

Using Koszul's formula, we can easily calculate

$$\begin{array}{lll} \nabla_{\varrho_1}\varrho_1 = 0, & \nabla_{\varrho_2}\varrho_1 = -\varrho_3, & \nabla_{\varrho_3}\varrho_1 = 0, \\ \nabla_{\varrho_1}\varrho_2 = -\varrho_3, & \nabla_{\varrho_2}\varrho_2 = 0, & \nabla_{\varrho_3}\varrho_2 = 0, \\ \nabla_{\varrho_1}\varrho_3 = -\varrho_2, & \nabla_{\varrho_2}\varrho_3 = -\varrho_1, & \nabla_{\varrho_3}\varrho_3 = 0. \end{array}$$

Also, one can easily verify that

$$\nabla_X \xi = \phi X \quad (\nabla_X \phi)Y = \eta(Y)\phi^2 X + g(\phi X, \phi Y)\xi.$$

Thus, the manifold M is an LP-Sasakian manifold. It is known that

$$R(Y, Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]}X.$$

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$\begin{array}{lll} R(\varrho_1, \varrho_2)\varrho_1 = \varrho_2, & R(\varrho_1, \varrho_2)\varrho_2 = -\varrho_1, & R(\varrho_1, \varrho_2)\varrho_3 = 0, \\ R(\varrho_2, \varrho_3)\varrho_1 = 0, & R(\varrho_2, \varrho_3)\varrho_2 = -\varrho_3, & R(\varrho_2, \varrho_3)\varrho_3 = -\varrho_2, \\ R(\varrho_1, \varrho_3)\varrho_1 = -\varrho_3, & R(\varrho_1, \varrho_3)\varrho_2 = 0, & R(\varrho_1, \varrho_3)\varrho_3 = -\varrho_1. \end{array}$$

From these curvature tensors, we can easily calculate

$$S(\varrho_1, \varrho_1) = S(\varrho_2, \varrho_2) = 0, \quad S(\varrho_3, \varrho_3) = -2. \tag{5.1}$$

Putting $Y = Z = \xi$ in (3.2) and using (5.1), we lead to $A_1 - A_3 = 2$, which by using the values of A_1 and A_3 for three dimensional LP-Sasakian manifold gives

$$\Lambda - \mu = \frac{\beta r^*}{2}, \qquad \alpha \neq 0.$$

Hence Λ and μ satisfies the equation (3.4), and so g defines a $*$-η-Ricci-Yamabe soliton on the given 3-dimensional LP-Sasakian manifold.

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