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On the origin of (combinatorial) species: a category-theoretical (de)construction of the theory of Joyal species

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ABSTRACT: We give a brief overview of alternative theories of combinatorial species other than that given by Joyal [1]. Then, we construct a general theory of species which extends the concept of species for any two categories with finite categories as objects, and invertible morphisms (functors) in the first category, and this construct is equivalent to that of all the existing species defined on categories as the aforementioned, so that ordinary species, all types of weighted species, q-species [5], Möbius species [4], and $\mathbb{L}$-species are just special cases. We adapt to this new scenario the classic constructs of assemblies of species, derivation and pointing. An analogue for exponentiation is constructed. Adapting the concept of arithmetic product yields a new type of function called "rational Dirichlet series".

Key Words: Discrete Mathematics, Combinatorics, Category theory, Species.

Contents

1. Introduction

In [1] the author gives a description of how considering functors from the category $\mathbb{B}$ of sets and bijections to the category $\mathbb{E}$ of sets and functions and associating to each functor a series (type, generating exponential, cycle index series) the operations between combinatorial species defined such as sum, product, composition and derivation (among other operations) translate directly into these series as sum, product, composition and derivation of series (except in the "type of isomorphism" series). Also, Joyal introduced notions such as that of multisort species, weighted species, virtual species, among others. The latter was introduced in order to give a notion of "inversion" in the contexts of addition, composition, etc.

Although the construction of virtual species is possible in order to give notions of additive inverse using equivalence classes of structures, it is not so readily and naturally extended to other types of species in the context of composition. Moreover, it is not always possible, as the reader will see, as this depends on the symmetries of the *Euler characteristic* of a finite category. If there are said symmetries, that is, objects in a category such that their Euler characteristics have opposing signs, then it is possible to define naturally the "additive inverse" of a species. So this is a very delicate and specific construct which will not be treated in this article, as it depends on the specific structure of the categories with which one works.

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Roughly speaking, [2] an ordinary combinatorial species is a rule which assigns to each labelled finite set a finite set of structures such as that of a tree, cycle, linear order, etc. Bijections between sets are also transported by the species coherently. In other words, it is a covariant functor from $\mathbb{B}$ to $\mathbb{E}$.

[5] gives a notion of species defined from $\mathbb{V}_q$ the category of vector spaces over finite fields ($GF(q)$) to the category $\mathbb{B}$, whilst [4] gives a notion of species constructed from $\mathbb{B}$ into the category **Int** of posets with maximal and minimal element.

[7] notes that the Boolean algebra B_n is isomorphic to the Lattice of subspaces $L_n(q)$ of a vector space over $GF(q)$ quotient the automorphism group of said lattice, and [6] also gives a notion of partition of such vector spaces.

[3] give an interesting construct that is the arithmetic product of species. This product translates well to the product of the *Dirichlet* series. This gives an interesting scenario on which one can analyse arithmetic functions and convolutions of arithmetic functions, such as the divisor function, which are very important in number theory. Moreover, the arithmetic product of two species M and N is isomorphic to M-assemblies of N *cloned* structures, and vice-versa (commutativity is present). Examples of arithmetic product of seemingly simple labelled structures such as the species of cycles and that of the linear orders give rise to important arithmetic functions in their Dirichlet series [3].

Let us not forget that the product of two Dirichlet series $\sum_{n\geq 1}\frac{a(n)}{n^s}$ and $\sum_{n\geq 1}\frac{b(n)}{n^s}$ is given by $\sum_{n\geq 1}\frac{(a*b)(n)}{n^s}$, where $*$ denotes the convolution of the arithmetic functions a and b.

We try to extend this using the more general notion of species given here, adapting the arithmetic product in a sense which allows to study "arithmetic" functions which take (positive and negative) rational values.

With these ideas in mind, one can also then begin to ask whether can these species be joined to produce new species, such as a "q-Möbius species" consisting of posets constructed with vector spaces or similarly, and if these constructions could lead to a new playground for further research of combinatorial properties. That is the aim of this article: to give a notion of general (categorical, abstract) species that could be applied to more areas, allowing the combinatorial and categorical interpretation and analysis of mathematical phenomena.

In fact, a central role in the general theory of what we will call $\Phi;\Psi$ species is played by the Euler Characteristic of a category; furthermore, the Euler characteristic can be defined, for example, for a finite exact sequence.

Thus the general theory of species which is developed here can also be applied to the study of exact sequences.

Note: In the following, in order to build the foundations of the theory, all subsections with a citation means that the results and definitions are taken from that author, unless otherwise stated in the same subsection, and the proofs can be found in the respective articles.

2. Survey of some existing theories

We give a brief introduction to the aspects of the theories relevant to this work

2.1. q-species

[[5]]

Definition 2.1 *A q-species is a functor F from the category $\mathbb{V}_q$ whose objects are vector spaces over a finite field $GF(q)$ and morphisms are automorphism of vector spaces to the category $\mathbb{B}$ whose objects are finite sets and morphisms are bijective functions. That is, $F[n]$ are structures constructed with vectors of an n-dimensional vector space over $GF(q)$.*

Let $|E|$ denote the cardinality of the finite set E.

Let $\gamma_n = \prod_{i=0}^{n-1}(q^n - q^i)$ denote the order of $GL_n(q)$ of square matrices over $GF(q)$, with $\gamma_0 = 1$. This comes from the fact that all bases of a vector space over $GF(q)$ can be written as upper triangular (reduced row echelon form) matrices with entries in $GF(q)$, and there are exactly γ_n such bases.

Definition 2.2 *Let F be a q-species, $f_n = |F[V_n]|$ for an n-dimensional object in $\mathbb{V}_{_{\shortparallel}}$. Then we define:*

- *Generating series:* $\hat{F}(x) = \sum_{n\geq 0} \frac{f_n}{\gamma_n} x^n$
- *Type series:* $\tilde{F}(x) = \sum_{n\geq 0} |\tilde{f}_n| x^n$

  Where $|\tilde{f}_n|$ is the number of isomorphism classes or orbits of $F[V_n]$ due to the action of the automorphisms.
- *Cycle index series:* $Z_F = \sum_{n\geq 0} \frac{1}{\gamma_n} \sum_{\sigma\in Aut(E_n)} \mathit{fix}F[\sigma] \prod_{\phi,i} x_{\phi,i}^{e_{\phi,i}(\sigma)}$

  Where $\mathit{fix}F[\sigma]$ is the number of fixed points of $F[\sigma]$, ϕ ranges over all the irreducible non trivial polynomials in $\mathbf{F}_q[z]$, i is a positive integer such that V decomposes as direct sum of primary cyclic modules of the form $\mathbf{F}_q[z]/\phi^i$, and $e_{\phi,i}$ is the number of copies of $\mathbf{F}_q[z]/\phi^i$ that appears in the decomposition of $\mathbf{F}_q[z]$.

Definition 2.3 *Let F, G, be two q-species. We define their*

- *Sum:*

  $(F+G)[V] = F[V] \sqcup G[V]$ *(disjoint union)*

  $$(F+G)[\phi](s) = \begin{cases} F[\phi](s), & \text{if } s\in F[V] \\ G[\phi](s), & \text{if } s \in G[V]. \end{cases}$$
- *Product:*

  $(F\cdot G)[V] = \bigcup_{V_1\oplus V_2 = V} F[V_1]\times G[V_2]$

Let $E : \mathbb{B} \to \mathbb{B}$ be the ordinary species of sets, which has generating function $E(x) = exp(x)$, and E_n be the species of sets on only the sets of cardinality n and which gives the structure of an empty set to any set of cardinality distinct from n. Clearly $E = \sum_{n\geq 0} E_n$.

We say that two species are isomorphic if they are isomorphic functors.

Definition 2.4 *Let F be a q-species with $F[0] = \emptyset$. Then the species $E\circ F$ is the species of assemblies of F-structures, and we note $F^{\{n\}} = E_n \circ F$ the species of n assemblies of F-structures or "nth symmetric power of F". Moreover: $F^{\{n\}} = F^n S_n$, where S_n is the group of permutations on n elements. Clearly $E\circ F = 1 + \sum_{n\geq 1} E_n \circ F$.*

We have that this is exactly what it means verbally: $F^{\{n\}}$ is a multiset of n F-structures, and assemblies of F-structures are multisets

Theorem 2.1 *Let F and G be two q-species as before. Then there is an isomorphism $(E\circ F)\cdot(E\circ G) = E\circ(F+G)$.*

Theorem 2.2 *There is an isomorphism $(F+G)^{\{n\}} = \sum_{m=0}^{n} F^{\{m\}} \cdot G^{\{n-m\}}$.*

Theorem 2.3 *The series for q-species obey the following rules:*

- $\overline{(F+G)}(x) = \overline{F(x)} + \overline{G(x)}$
- $\overline{(F\cdot G)}(x) = \overline{F}(x)\overline{G}(x)$

  Where $\overline{H}$ means $\hat{H}$ or $\tilde{H}$
- $Z_{F\cdot G} = Z_F Z_G$
- $\widehat{E\circ F} = exp(\hat{F}(x))$
- $\widetilde{E\circ F} = exp(\sum_{n\geq 1} \frac{\tilde{F}(x^n)}{n})$
- $\hat{F}(x) = Z_F((x,0,0...),(0,0....),(0,...),...)$
- $\tilde{F}(x)$ *is obtained from Z_F setting $x_{\phi,i} = x^i$ $\forall i$.*

2.2. Möbius species

[[4]]

We will develop this theory as simply as possible without delving into the categorical aspects of c-monoids (needed to define properly additive, multiplicative inverses for Möbius species and composition and its inverse) since the subject is deep enough in its own right. For further information refer to the work of [4].

We consider the category **Int** whose objects are finite families of partially ordered sets with a finite number of nodes and with a unique maximal and minimal element $\hat{1}$ and $\hat{0}$ respectively. We shall call these objects finite bounded posets, or simply "posets" for short. The morphisms of this category are order isomorphisms, that is, if $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ are posets, then $f : P \to Q$ is an order isomorphism only when $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$.

We now define the Möbius function μ on posets, that is the extension of the usual Möbius function defined in number theory (which acts on the poset of the integers):

Definition 2.5 *Let $P \in$ **Int**. Then for any $x, y \in P$:*

$$\mu(x,y) = \begin{cases} 1 \textit{ if } x = y \\ -\sum_{z: x \leq z < y} \mu(x,z) \textit{ if } x < y \\ 0 \textit{ otherwise} \end{cases}$$

Definition 2.6 *Let $A \in$ **Int**.*

We define the Möbius cardinality $|| \cdot ||$ as:

$$||A|| = \sum_{I \in A} \mu_I(\hat{0}, \hat{1})$$

So for $A \in$ Int one has that its Möbius cardinality is essentially the sum of the *Euler characteristic* of its connected components $I \in A$, that is, the posets in A, and recall that every poset is essentially a category. This moment will be very important to further generalize the theory of species. If C_k is the number of chains of length k between $\hat{0}$ and $\hat{1}$, then $\mu_I(\hat{0}, \hat{1}) = \sum_k (-1)^k C_k$. This result is known as *Philip Hall's Theorem* ([8], proposition 3.8.5). So now we have an intuitive notion of what *really* the Möbius cardinality *is*.

We understand for $[x, y]$ for $x, y \in I$ as the interval $\{z : x \leq z \leq y\}$. The product of two posets I and I' is the poset $I \times I' = \{(x, y) : x \in I, y \in I'\}$, and the order is $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$, being the elements non comparable otherwise. The sum of two posets is given by the disjoint union of both and the order preserved as it is on both, being two nodes of distinct posets non comparable in the sum.

Definition 2.7 *The product $\times :$ **Int** $\times$ **Int** $\to$ **Int** is given by:*

$$A \times B = \{I \times I' : I \in A, I' \in B\}$$

Lemma 2.1 *Let $A, B \in ob$ **Int**, and let Id denote the set whose only object is the singleton poset. The Möbius cardinality obeys the following rules:*

a) $||A \times B|| = ||A|| \cdot ||B||$

b) $||Id \times A|| = ||A|| = ||A \times Id||$

Definition 2.8 *A Möbius species is a covariant functor $M : \mathbb{B} \to$ **Int**.*

If $|E| = |F|$ for a pair of sets then given a Möbius species M, $||M[E]|| = ||M[F]||$ [4].

Definition 2.9 *Let M be a Möbius species. We define the (Möbius) generating series of M as:*

$$M(x) = \sum_{n\geq 0} \frac{||M[n]||}{n!} x^n$$

Definition 2.10 *Let M and N be two Möbius species. Let M' denote the "derivated" species and $M^\bullet$ be the pointed species as in the sense of [1] or [2]. E or n is a set with n elements. We define:*

- *Sum:* $(M+N)[n] = M[n] + N[n]$
- *Product:* $(MN)[n] = \sum_{a+b=n} M[a] + N[b]$
- *For* $N[\emptyset] = \emptyset$, *the Möbius species of* k*-assemblies of* N*-structures* $\gamma_k(N)$*:*

$$\gamma_k(N)[n] = \sum_{\substack{\pi\in\Pi[n] \\ |\pi|=k}} \prod_{B\in\pi} N[B]$$

  where π *is a partition of* n, $\Pi[n]$ *the set of all partitions of* n, *and* $|\pi|$ *its number of parts.*

- *Derivated species:* $M'[n] = M[n \cup \{n\}]$
- *Pointed species:* $M^\bullet[n] = \{0_e : e \in n\} \times M[n]$, *where* 0_e *is the singleton poset.*

Theorem 2.4 *Möbius species behave similarly to q-species and ordinary species with their operations of product and sum:*

- $(M+N)(x) = M(x) + N(x)$
- $(MN)(x) = M(x)N(x)$
- *If* $N[\emptyset] = \emptyset$, *then* $\gamma_k(N)(x) = \frac{(N(x))^k}{k!}$
- $(M')(x) = (M(x))'$
- $M^\bullet(x) = x(M(x))'$

There is a notion of additive and multiplicative inverse for Möbius species, as well as composition and its inverse, but we shall not duel on this here for the subject is too long.

3. The Origin of Species

We will give a general abstract notion of species in the following. In order to do this one must first define the Euler Characteristic for a category. Such work has been done by [9] and we shall expose the key aspects of this.

But first let us formalize some notions of category theory.

Definition 3.1 ([10]) *A* ***category*** *is:*

- *A collection of objects* $Ob(\Phi)$*:* $X, Y, Z...$
- *A collection of morphisms* $mor(\Phi)$, *noted* $f, g, ...$
- *Two operations Dom and Cod assigning to each morphism* f *two objects called domain (source) and codomain (target) of* f
- *An operation* Id *assigning to each object* Y *a morphism* Id_Y *with* $dom(Id_Y) = cod(Id_Y) = Y$.
- *An operation called composition noted by* $\circ$ *assigning to each pair of morphisms* f, g *with* $dom(f) = cod(g)$ *a new morphism* $f \circ g$ *with* $dom(f \circ g) = dom(g)$ *and* $cod(f \circ g) = cod(f)$.

And also the identity and composition must satisfy:

Identity law:
for each pair of morphisms f and g such that $dom(g) = Y = cod(f)$:
$Id_Y \circ f = f$
$g \circ Id_Y = g$
associative law:
for any morphisms f, g, h such that $dom(f) = cod(g)$ and $dom(g) = cod(h)$:
$(f \circ g) \circ h = f \circ (g \circ h)$

We call the *opposite* category to Φ, and note it Φ^{op} the category with the same objects as Φ and whose morphisms are the same as in Φ only with the domain and codomain swapped. That is, if $f(X) = Y$ in Φ, then $f(Y) = X$ in Φ^{op}.

For morphisms there is a notion which formalizes that of "injectivity": $f : X \hookrightarrow Y$ is called a monic morphism, if for any object Z and morphisms $g, h : Z \to X$: $f \circ g = f \circ h$ implies $g = h$. Such a morphism is also called *left-cancellative*. Similarly, an epic morphism (formalization of the notion of "surjectivity") or *right-cancellative* morphism $f : X \twoheadrightarrow Y$ is such that for every object Z and morphisms $g, h : Y \to Z$: $g \circ f = h \circ f$ implies $g = h$. An invertible morphism of "isomorphism" is a right-and-left-cancellative morphism.

Definition 3.2 *A **finite** category (denoted by lowercase greek letters ϕ, ψ,...) is a category with a finite amount of objects and morphisms.*

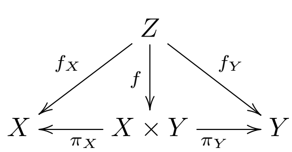
Definition 3.3 ([10]) *Let Φ and Ψ be two categories. A (covariant) **functor** is a pair of operations $\mathfrak{F}_{Ob} : Ob(\Phi) \to Ob(\Psi)$ and $\mathfrak{F}_{mor} : mor(\Phi) \to mor(\Psi)$ which verifies for each $f : X \to Y$ and $g : Y \to Z$ in Φ:*

- $\mathfrak{F}_{mor}[f] : \mathfrak{F}_{Ob}(X) \to \mathfrak{F}_{Ob}(Y)$
- $\mathfrak{F}_{mor}[g \circ f] = \mathfrak{F}_{mor}[g] \circ \mathfrak{F}_{mor}[f]$
- $F_{mor}[Id_X] = Id(F_{Ob}(X))$

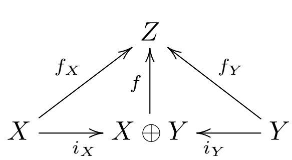
Definition 3.4 ([10]) *Two categories Φ, and Ψ, are said to be equivalent if there are two functors $\mathfrak{F} : \Phi \to \Psi$ and $\mathfrak{L} : \Psi \to \Phi$ such that $\mathfrak{F} \circ \mathfrak{L} = Id_\Psi$ and $\mathfrak{L} \circ \mathfrak{F} = Id_\Phi$.*

If it exists, we define the product and coproduct of two objects in a category in the following way:

Definition 3.5 *If Φ is a category, then for objects X and Y, their product, if it exists, is noted by $X \times Y$, is an object such that there are morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ which satisfy the universal property that for every object Z and pair of morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$, there is a unique morphism $f : Z \to X \times Y$ which makes the diagram commute:*



The coproduct of X and Y exists and is noted $X \oplus Y$, if there exists morphisms $i_X : X \to X \oplus Y$ and $i_Y : Y \to X \oplus Y$ which satisfy the universal property that for every object Z and pair of morphisms $f_X : X \to Z$ and $f_Y : Y \to Z$, there is a unique morphism $f : X \oplus Y \to Z$ which makes the diagram commute:



Definition 3.6 *We say that a category Φ has finite products if the product of any pair of objects in Φ exists, and say that it has finite coproducts if the coproduct of any pair of objects in Φ exists.*

Definition 3.7 *Let ϕ and ψ be two finite categories. We define their product $\times$ (Π for more than a pair) as the Cartesian product of both categories, that is, if $X \in Ob(\phi)$, $Y \in \psi$, $f_i \in mor(\phi)$, $g_i \in mor(\psi)$, then:*

- $Ob(\phi \times \psi) = \{(a, b)\}$
- $mor(\phi \times \psi) = \{(f_i, g_i)\}$
- $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$
- $Id_{\phi\times\psi} = (Id_\phi, Id_\psi)$

Definition 3.8 *Let ϕ and ψ be two finite categories. We define their direct sum $\oplus$, also called weak direct product, ($\sum$ for more than a pair, and it will be understood as direct sum) by $\phi\oplus\psi = (\phi\times\{0\})\sqcup(\psi\times\{1\})$. For more than one it is understood as $\sum_i \phi_i = \bigsqcup_i \phi_i \times \{i\}$*

With these definitions, categories such as $\mathbb{B}$ and **Int**, actually can be seen as "collections" of categories, or rather, since all morphisms considered when talking about species are invertible morphisms of each of the object's labelled elements, "a category of finite categories all of the same type", each of which has a unique "cardinality". This notion of cardinality is actually the Euler characteristic of each (finite) category.

3.1. The Euler characteristic

[[9]]

In this subsection all categories ϕ are finite categories, $\phi(X, Y)$ is the set of morphisms from $X \in \phi$ to $Y \in \phi$.

It is noteworthy that the theory of the Euler characteristic developed by Tom Leinster is compatible with the notions of Euler characteristic of grupoids (as according to the cardinality of a grupoid devised by Baez and Dolan), graphs, topological spaces and orbifolds, to say the least.

Definition 3.9 *Let $R(\phi)$ be the $\mathbb{Q}$-algebra of functions $Ob(\phi) \times Ob(\phi) \to \mathbb{Q}$ with pointwise addition, and scalar multiplication, the latter given by:*

$$(\theta\phi)(X, Z) = \sum_{Y\in\phi} \theta(X, Y)\phi(Y, Z)$$

. Then the Zeta function of a category is defined $\zeta_\phi(X, Y) = |\phi(X, Y)|$, and if it is invertible in $R(\phi)$ then the inverse is called the Möbius function of ϕ and it satisfies:

$$\sum_{Y\in\phi} \mu_\phi(X, Y)\zeta_\phi(Y, Z) = \delta(a, c) = \sum_{Y\in\phi} \zeta_\phi(X, Y)\mu_\phi(Y, Z)$$

where $\delta(X, Z)$ is the Kronecker delta function. A category with a Möbius function defined is said to have Möbius inversion.

[9] gives interesting formulas for the Möbius function for *skeletal* categories, i.e. categories where isomorphic objects are necessarily the same object.

Definition 3.10 *We call a **weighting** on ϕ a function $k^\bullet : Ob(\phi) \to \mathbb{Q}$ such that $\forall X \in \phi$:*

$$\sum_{Y\in\phi} \zeta(X, Y)k^Y = 1$$

*And a **coweighting** $k_\bullet$ is a weighting on ϕ^{op}.*

[9] notes that a weighting may be indeed not unique, and ϕ has Möbius inversion iff it has a unique weighting iff it has a unique coweighting. Moreover, in this case $k^X = \sum_Y \mu(X,Y)$ and $k_Y = \sum_X \mu(X,Y)$.

Theorem 3.1 *Let ϕ and ψ be two equivalent finite categories. Then ϕ admits a weighting iff ψ does.*

Lemma 3.1 *Let $n \geq 1$ and $\phi_1, ..., \phi_n$ be finite categories.*

a) If each ϕ_i has as a weighting $k_i^\bullet$, then $\sum_i \phi_i$ has $l^X = k_i^X$ as a weighting if $X \in \phi_i$. If each ϕ_i has Möbius inversion then so does $\sum_i \phi_i$, and for $X \in \phi_i, Y \in \phi_j$:

$$\mu(X,Y) = \begin{cases} \mu_{\phi_i}(X,Y) \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

b) If each ϕ_i has as a weighting $k_i^\bullet$, then $\Pi_i \phi_i$ has $l^{(X_1,...,X_n)} = k_1^{X_1}...k_n^{X_n}$ as a weighting. If each ϕ_i has Möbius inversion then so does $\Pi_i \phi_i$ and it is given by:

$$\mu((X_1,...,X_n),(Y_1,...,Y_n)) = \mu_{\phi_1}(X_1,Y_1)...\mu_{\phi_n}(X_n,Y_n)$$

Lemma 3.2 *Let ϕ be a finite category with weighting $k^\bullet$ and coweighting $k_\bullet$. Then $\sum_X k^X = \sum_X k_X$.*

Definition 3.11 *A finite category ϕ is said to have **Euler characteristic** $\chi(\phi)$ iff it admits both a weighting and a coweighting, and it is given by $\chi(\phi) = \sum_X k^X = \sum_X k_X$.*

Proposition 3.1 *Let $n \geq 1$ and $\phi_1, ..., \phi_n$ be finite categories that all have Euler characteristic. Then $\sum_i \phi_i$ and $\prod_i \phi_i$ have Euler characteristic, given by:*

$$\chi(\sum_i \phi_i) = \sum_i \chi(\phi_i),$$

$$\chi(\prod_i \phi_i) = \prod_i \chi(\phi_i).$$

An important role in the development of the notion of Cycle index series for general species is the following notion: for a given finite category ϕ, f an endofunctor of ϕ. Fix$[f]$ is the category such that its objects are $X \in \phi$ such that $f(X) = X$ (i.e. objects that form part of the set of orbits of length one for f) and morphisms $G : X \to Y$ such that $f(G) = G$.

Definition 3.12 *The Lefschetz number $L(f)$ for an endofunctor f is $\chi(Fix[f])$, when this exists.*

In words of T. Leinster: "The Lefschetz number is, then, the sum of the (co)weights of the fixed points. This is analogous to the standard Lefschetz fixed point formula, (co)weight playing the role of index."

Proposition 3.2 *Let ϕ be a finite category. Then $L(1_\phi) = \chi_\Phi(\phi)$, one side being defined iff the other is. (1_ϕ being the identity functor).*

Note 1 *We want to note that from [9] we have that $\chi(G) = \frac{1}{|G|}$ for any finite group G, and that if ϕ is a discrete category, then $\chi(\phi) = |ob(\phi)|$.*

We are now ready to proceed with the general theory of Species, out of which all existing ones are just particular cases.

3.2. The Origin of Species

Let Φ and Ψ be two categories, the first with finite coproducts, and the latter with finite products and coproducts and initial object ψ_0. The objects of these categories (ϕ and ψ respectively) are finite categories themselves, and whose morphisms are: invertible functors (thus, since $\phi \in \Phi$ are categories, isofunctors) in the first. The set (in fact, a group in some cases, if a functor acting as the identity is present) of invertible morphisms in Φ with same domain and codomain, $\phi \to \phi \in \Phi$ (itself) will be noted as $Aut(\phi)$. It is very important to note that this is the set of morphisms in $\underline{\Phi}$ which act on ϕ, and, are not part of the structure of ϕ, the object (a finite category), itself. The latter category Ψ may be defined as having any morphisms (for example, taking $\Psi = \mathbb{E}$. This last one is because we will be actually interested in the image of the morphisms of ϕ by a functor from Φ to Ψ..

For example, when talking about sets, the invertible morphisms are permutations of the elements of the set, or when talking about vector spaces, vector space automorphisms, or order isomorphisms in case of totally ordered sets (which are the input of $\mathbb{L}$-species).

In the following all categories Φ and Ψ are to be understood as above.

We may ask that $\forall \phi \in \Phi$ there exists a unique smallest object ϕ^+ such that there is a monic morphism (and certainly not an an isomorphism for this would be absurdly trivial) $I : \phi \hookrightarrow (\phi^+)$ which satisfies $I \simeq Id_\phi$. In other words, the objects in Φ are "nested":

$$\phi \hookrightarrow \phi^+ \hookrightarrow \phi^{++} \hookrightarrow \phi^{+++} \hookrightarrow \phi^{++++} \hookrightarrow ...$$

That is, in terms of category theory, that ϕ^+ is universal in the sense that any monic morphism $\phi \hookrightarrow \phi'$ factors through I.

Due to the finiteness of the objects, this means that the objects in Φ satisfy a descending chain condition. We *could* state this as saying that Φ is *Artinian*. This is necessary in order to properly define the derivative of a species.

Observe that not necessarily the objects of Φ form a totally ordered set. Rather, they form several disjoint infinite chains. A category in which all objects are nested successively will be called a **Linear** category.

In the case of Φ being Linear, what this means is that Φ has an initial object, and if it is removed from the category, the remaining category still has an initial object.

Definition 3.13 *A $\Phi;\Psi$ species is a functor $\mathfrak{F} : \Phi \to \Psi$, and $\mathfrak{F}[\phi]$ is called an $\mathfrak{F}$-structure on ϕ.*

We define the basic operations of $\Phi;\Psi$ species:

Definition 3.14 (Operations) *Let $\mathfrak{F}, \mathfrak{L}$ be two $\Phi;\Psi$ species. Let $\oplus$ and $\times$ denote respectively the coproduct and product of finite categories. We define the:*

- *Sum:* $(\mathfrak{F} + \mathfrak{L})[\phi] = \mathfrak{F}[\phi] \oplus \mathfrak{L}[\phi]$

  If f is a morphism in Φ:

$$(\mathfrak{F} + \mathfrak{L})[f](s) = \begin{cases} \mathfrak{F}[f](s), & \text{if } s \in \mathfrak{F}[\phi] \\ \mathfrak{L}[f](s), & \text{if } s \in \mathfrak{L}[\phi] \end{cases}$$

- *Product:* $(\mathfrak{F}\mathfrak{L})[\phi] = \sum_{\hat{\phi} \oplus \tilde{\phi} = \phi} \mathfrak{F}[\hat{\phi}] \times \mathfrak{L}[\tilde{\phi}]$

  If f is a morphism in Φ, $s \in \hat{\phi}$, $t \in \tilde{\phi}$:

$$(\mathfrak{F}\mathfrak{L})[f](s,t) = \mathfrak{F}[f|_{\hat{\phi}}](s) \times \mathfrak{L}[f|_{\tilde{\phi}}](t)$$

- *The pointed species is defined as: $\mathfrak{F}^\bullet[\phi] = \{\psi_{0,p} : p \in \phi\} \times \mathfrak{F}[\phi]$; where $\psi_{0,p}$ is the initial object ψ_0 indexed by some $p \in \phi$ (that is, given ϕ, the pointed species consists of all structures of the type $\psi_{0,p} \times \mathfrak{F}[\phi]$ with p ranging in ϕ, $\mathfrak{F}[\phi]$ an $\mathfrak{F}$-structure on ϕ).*

- *If Φ is linear, the Derivative is defined as:* $\mathfrak{F}'[\phi] = \mathfrak{F}[\phi^+]$

Clearly, the sum and product are commutative, and the pointing and derivation are distributive with respect to the sum and product.

Definition 3.15 *A Partition $\pi = \{\phi_1, ..., \phi_m\}$ of an object $\phi \in \Phi$ is a decomposition of ϕ as coproduct of other (non empty) objects in Φ, $\phi = \sum_{i=1}^{m} \phi_i$.*

The set of all partitions of ϕ will be noted $\Pi[\phi]$ and $|\pi| = k$ means that $\phi = \phi_1 \oplus ... \oplus \phi_k$, i.e. the partition π consists of exactly k disjoint objects. π itself as an object is taken as a set of cardinality k. Thus a partition can be interpreted as a sort of "sheaf".

Let κ_Φ be a weighting on Φ, κ_Ψ be a weighting on Ψ, and both produce the respective euler characteristics χ_Φ and χ_Ψ.

It is clear from this that a partition $\pi = \{\phi_i\}$ of ϕ satisfies $\chi_\Phi(\phi) = \sum_i \chi_\Phi(\phi_i)$

In the spirit of [4] and [2] we define the species of k-assemblies of $\mathfrak{F}$-structures and $\mathfrak{L}$-assemblies of $\mathfrak{F}$-structures as follows:

Definition 3.16 *Let $\phi \in \Phi$, $\mathfrak{F}$ a $\Phi; \Psi$ species with $\mathfrak{F}[\emptyset] = \emptyset$, and we write $\forall i$: $\pi_i = \{\phi_{i,j}\}_{j=1}^{k}$.*

Then the species of k-assemblies of $\mathfrak{F}$-structures on ϕ is the species γ_k given by:

$$\gamma_k(\mathfrak{F})[\phi] = \sum_{\substack{\pi_i \in \Pi[\phi] \\ |\pi| = k}} \prod_{\phi_{i,j}} \mathfrak{F}[\phi_{i,j}]$$

Let $\mathfrak{L}$ be a $\mathbb{B}, \Psi$ species.

The species of $\mathfrak{L}$-assemblies of $\mathfrak{F}$-structures, noted $\mathfrak{L} \circ \mathfrak{F}$ or $\mathfrak{L}(\mathfrak{F})$, and also called the composition of species, on ϕ is given by the pairs (f, l), where $f = (f_1, ..., f_k)$ is a $k - ary$ $\mathfrak{F}$-structure such that f_j a $\mathfrak{F}$-structure on $\phi_{i,j}$, and l is a $\mathfrak{L}$-structure on the associated partition π:

$$\mathfrak{L} \circ \mathfrak{F}[\phi] = \sum_{\pi_i \in \Pi[\phi]} \mathfrak{L}[\pi_i] \times \prod_{\phi_{i,j} \in \pi_i} \mathfrak{F}[\phi_{i,j}]$$

Essentially, a k-assembly of $\mathfrak{F}$-structures is a multiset of cardinality k whose components are $\mathfrak{F}$-structures, and a $\mathfrak{L}$-assembly is the same except that on the multiset, taken as a set, an $\mathfrak{L}$-structure is imposed.

Theorem 3.2 *Let $\mathfrak{F}$ and $\mathfrak{L}$ be two $\Phi; \Psi$-species. Then:*

$$\gamma_n(\mathfrak{F} + \mathfrak{L}) = \sum_{0 \leq k \leq n} \gamma_k(\mathfrak{F}) \gamma_{n-k}(\mathfrak{L})$$

Proof: An n-assembly of $\mathfrak{F} + \mathfrak{L}$ on ϕ is constructed by partitioning ϕ into n parts, and onto each part imposing either an $\mathfrak{F}$-structure or a $\mathfrak{L}$-structure. Since the order does not matter, we may group together all those parts onto which an $\mathfrak{F}$-structure has been imposed, and group together those onto which a $\mathfrak{L}$-structure has been imposed. Then $\phi = \hat{\phi} \oplus \tilde{\phi}$ and on the first only the $\mathfrak{F}$-structure is present and on the latter only the $\mathfrak{L}$-structure. Then in the partition, if $\hat{\phi}$ is coproduct of k parts, then $\tilde{\phi}$ is coproduct of $n - k$ parts. And this is so for all $0 \leq k \leq n$. Therefore, $\gamma_n(\mathfrak{F} + \mathfrak{L}) = \sum_{0 \leq k \leq n} \gamma_k(\mathfrak{F}) \gamma_{n-k}(\mathfrak{L})$. □

Definition 3.17 (Series) *Let $\mathfrak{F}$ be a $\Phi; \Psi$ species. We define its:*

- *Generating series:*

$$\mathfrak{F}(z) = \sum_{\phi \in \Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$$

- *Type series:*

$$\widetilde{\mathfrak{F}}(z) = \sum_{\phi\in\Phi} \chi_{\hat{\Psi}}(\widetilde{\mathfrak{F}[\phi]}) z^{\chi_\Phi(\phi)}$$

Where $\widetilde{\mathfrak{F}[\phi]}$ *is the set of orbits or equivalence classes* $\overline{\lambda}$ *of* $\mathfrak{F}[\phi]$ *in the quotient category* $\widetilde{\mathfrak{F}[\phi]} = \mathfrak{F}[\phi]/Mor(\mathfrak{F}[\phi])$*, because* $Mor(\mathfrak{F}[\phi])$ *consists entirely of invertible morphisms, and* $\chi_{\hat{\Psi}}$ *is the euler characteristic of the quotient category. Essentially one removes isomorphic copies and counts only those that are not, and the same happens with the Euler characteristic: If there were* m *isomorphic copies of* $\lambda \in \mathfrak{F}[\phi]$*, then* $\chi_{\hat{\Psi}}(\overline{\lambda}) = \frac{\chi_\Psi(\lambda)}{m}$*. This happens due to the additive and multiplicative nature of* χ*. (Two* $\mathfrak{F}$*-structures* α, β *are isomorphic if* $\exists f \in mor(\Phi) : \mathfrak{F}[f](\alpha) = \beta$*)*

- *Cycle index series: Let* $f \in Aut(\phi)$*. We can decompose* f *as a "direct sum of automorphisms" in the following way:* $f = \sum_l f_l$ *such that* $\forall l$*,* f_l *has only one orbit, and successive composition of an* f_l *with itself* l *times produces a cycle, i.e.* $f_l^l = Id$ *(and* l *is the smallest such integer of course).*

  This means that we may write $\phi = \sum_l \phi_{f,l}$*, where* $\phi_{f,l}$=coproduct of all $\tilde{\phi}$ such that $f^l(\tilde{\phi}) = \tilde{\phi}$ and l is the smallest such exponent; *that is,* ϕ *is the coproduct of all the* $\phi_{f,l}$*; then* $f_l = f|_{\phi_{f,l}}$ *and* f_l *has* $\phi_{f,l}$ *as its orbit. Therefore,* f *induces a partition* π_f *of* ϕ*.*

  Observe that there may be more than one copy of some $\tilde{\phi}$*, part of* $\phi_{f,l}$*, that is, it has a multiplicity greater than one. For example, for some* f *and* ϕ*, we may have* $\phi = \phi_{f,1} \oplus \phi_{f,2} \oplus \phi_{f,7}$*, and* $\pi_f = \{\phi_{f,1}, \phi_{f,2,1}, \phi_{f,2,2}, \phi_{f,7,1}, \phi_{f,7,2}\}$*, means that* $\phi_{f,i}$ *is the coproduct of all orbits of length* i*,* $\phi_{f,i} = \coprod_j \phi_{f,i,j}$*, so this multiplicity is already implicit in the notation we have devised.*

  Also, for some $k \in \mathbb{N}$ *it could be that* $\phi_{f,k} = \emptyset$*. In that case we invoke the convention* $\chi(\emptyset) = 0$*.*

  We denote by $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ *the vector whose entries are indexed by the natural numbers.*

  We define the cycle index series as:

$$\mathfrak{Z}_{\mathfrak{F}}(\mathbf{x}) = \sum_{\phi\in\Phi} \frac{1}{|Aut(\phi)|}\Big(\sum_{f\in Aut(\phi)} L(\mathfrak{F}[f]) \prod_{l\geq 0} x_l^{\chi_\Phi(\phi_{f,l})}\Big)$$

Observe that this series are actually of the type called "Puiseux series": series for which the the exponents of z are not necessarily integers, but can be rational numbers! This series appear as a local expansion of y in terms of x for a bivariate Polynomial equation $P(x, y) = 0$. This result is known as the *Newton-Puiseux Theorem* [11].

The generating series is a function of the type called in literature "exponential generating function" (EGF) and the type series is a function of the type called "ordinary generating function" (OGF).

Note that if Φ is Linear, then it is possible to label the objects in order $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, ...$ according to the order in the chain. **Otherwise**, one can label them according to the chains with a double index, as if labelling rational numbers.

Theorem 3.3 *The type and generating series verify the following:*

- $(\mathfrak{F} + \mathfrak{L})(z) = \mathfrak{F}(z) + \mathfrak{L}(z)$
- $(\mathfrak{F}\mathfrak{L})(z) = \mathfrak{F}(z)\mathfrak{L}(z)$
- $(\widetilde{\mathfrak{F} + \mathfrak{L}})(z) = \widetilde{\mathfrak{F}}(z) + \widetilde{\mathfrak{L}}(z)$
- $(\widetilde{\mathfrak{F}\mathfrak{L}})(z) = \widetilde{\mathfrak{F}}(z)\widetilde{\mathfrak{L}}(z)$
- $\mathfrak{Z}_{\mathfrak{F}+\mathfrak{L}} = \mathfrak{Z}_{\mathfrak{F}} + \mathfrak{Z}_{\mathfrak{L}}$

Proof: It follows from the definition of sum and product of species, and the additive and multiplicative properties of the Euler characteristic. We give a brief proof of the first two.

To prove the sum we operate as follows:

$$(\mathfrak{F}+\mathfrak{L})(z) = \sum_{\phi\in\Phi} \frac{\chi_\Psi((\mathfrak{F}+\mathfrak{L})\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)} = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi]\oplus\mathfrak{L}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$$

$$= \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi])+\chi_\Psi(\mathfrak{L}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)} = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)} + \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{L}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$$

$$= \mathfrak{F}(z) + \mathfrak{L}(z)$$

Now we prove the property for the product:

$$(\mathfrak{F}\mathfrak{L})(z) = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}\mathfrak{L}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)} = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\sum_{\hat{\phi}\oplus\tilde{\phi}=\phi}\mathfrak{F}[\hat{\phi}]\times\mathfrak{L}[\tilde{\phi}])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$$

$$= \sum_{\phi\in\Phi} \frac{\sum_{\hat{\phi}\oplus\tilde{\phi}=\phi}\chi_\Psi(\mathfrak{F}[\hat{\phi}]\times\mathfrak{L}[\tilde{\phi}])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)} = \sum_{\phi\in\Phi} \frac{\sum_{\hat{\phi}\oplus\tilde{\phi}=\phi}\chi_\Psi(\mathfrak{F}[\hat{\phi}])\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$$

Since there are $|Aut(\phi)|$ ways to perform the decomposition in the numerator without taking into account the automorphisms of the parts (i.e. ways to "scramble" ϕ, because such is the quantity of isofunctors acting on ϕ), but there are $|Aut(\hat{\phi})|$, and $|Aut(\tilde{\phi})|$ such (ways to scramble $\hat{\phi}$ and $\tilde{\phi}$), there are actually $\binom{\phi}{\hat{\phi}} := \frac{|Aut(\phi|)}{|Aut(\hat{\phi})||Aut(\tilde{\phi})|}$ ways to perform the decomposition in the numerator, $\tilde{\phi}\oplus\hat{\phi}=\phi$, so

$$= \sum_{\hat{\phi},\tilde{\phi}\in\Phi} \frac{\binom{\phi}{\hat{\phi}}\chi_\Psi(\mathfrak{F}[\hat{\phi}])\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\phi)|} z^{\chi_\Phi(\hat{\phi}\oplus\tilde{\phi})} = \sum_{\hat{\phi},\tilde{\phi}\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\hat{\phi}])\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\hat{\phi})||Aut(\tilde{\phi})|} z^{\chi_\Phi(\hat{\phi})+\chi_\Phi(\tilde{\phi})}$$

$$= \sum_{\hat{\phi}\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\hat{\phi}])}{|Aut(\hat{\phi})|} z^{\chi_\Phi(\hat{\phi})} \sum_{\tilde{\phi}\in\Phi} \frac{\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\tilde{\phi})|} z^{\chi_\Phi(\tilde{\phi})} = \mathfrak{F}(z)\mathfrak{L}(z)$$

The fifth property comes from the definition of $\mathfrak{F}+\mathfrak{L}$-structure, $(\mathfrak{F}+\mathfrak{L})[\phi] = \mathfrak{F}[\phi]\oplus\mathfrak{L}[\phi]$. So, the Lefschetz number of such will be, due to the additivity of the Euler characteristic, the sum of the individual characteristics. The result then follows trivially.

□

The following theorems extends a well known results in the theory of species (albeit with a slightly different notation).

Theorem 3.4

$$\mathfrak{Z}_{\mathfrak{F}}(x,0,0,0,0,...) = \mathfrak{F}(x)$$

Proof: In this case, just inputting this in the definition of cycle series implies that, if it were so that $x_1^{\chi_\Phi(\phi_{f,1})}0^{\chi_\Phi(\phi_{f,2})}0^{\chi_\Phi(\phi_{f,3})}0^{\chi_\Phi(\phi_{f,4})}... \neq 0$ then necessarily f is the identity, because it does not have orbits of length greater than 1, and it fixes $\mathfrak{F}[\phi]$ itself in its entirety for every $\phi\in\Phi$. Therefore $\mathfrak{Z}_{\mathfrak{F}}(x,0,0,...) = \sum_{\phi\in\Phi}\frac{1}{|Aut(\phi)|}\Big(L(\mathfrak{F}[Id])x^{\chi_\Phi(\phi)}\Big)$. And $L(\mathfrak{F}[Id]) = \chi(\text{Fix}[\mathfrak{F}[Id]])$, let us remember that Fix$[f]$ has as objects those $X\in\phi$ such that $f(X)=X$ and morphisms those untouched by the action of $\mathfrak{F}[f]$. Since $f = Id$, the single object of Fix$[\mathfrak{F}[Id]]$ is $\mathfrak{F}[\phi]$ itself and its morphisms are all those of it. So $L(\mathfrak{F}[Id]) = \chi_\Psi(\mathfrak{F}[\phi])$. Substituting in the formula obtained before yields the result. □

Theorem 3.5 *If $\mathfrak{F}$ is a $\Phi;\mathbb{B}$ species, then:*

$$\mathfrak{Z}_{\mathfrak{F}}(x,x,x,x,...) = \widetilde{\mathfrak{F}}(x)$$

Proof: We begin by noting that $\Pi_l x^{\chi_\Phi(\phi_{f,l})} = x^{\sum_l \chi_\Phi(\phi_{f,l})} = x^{\chi_\Phi(\phi)}$ because the parts of the product have the same base and due to the fact that f induces a partition of ϕ, and since the euler characteristic is additive we have the final equality. So we have that:

$$\mathfrak{Z}_{\mathfrak{F}}(x,x,x,x,...) = \sum_{\phi\in\Phi} \frac{1}{|Aut(\phi)|}\Big(\sum_{f\in Aut(\phi)} L(\mathfrak{F}[f]) \prod_{l\geq 0} x^{\chi_\Phi(\phi_{f,l})}\Big)$$
$$= \sum_{\phi\in\Phi} \frac{1}{|Aut(\phi)|}\Big(x^{\chi_\Phi(\phi)} \sum_{f\in Aut(\phi)} L(\mathfrak{F}[f])\Big)$$

Let us remember that $L(\mathfrak{F}[f]) = \chi_\Psi(\text{Fix}[\mathfrak{F}[f]])$ where $\text{Fix}[\mathfrak{F}f]$ is the category whose objects are the fixed points of $\mathfrak{F}[f]$ and morphisms those that are also fixed, $\mathfrak{F}[f][g] = g$. So, due to the additivity of χ again we have $\sum_{f\in Aut(\phi)} L(\mathfrak{F}[f]) = \chi_\Psi\Big(\coprod_{f\in Aut(\phi)} \text{Fix}[\mathfrak{F}[f]]\Big)$.

Since $\Psi = \mathbb{B}$, $\chi_\Psi = |\cdot|$ the usual notion of cardinality.

The cardinality of the coproduct, i.e. disjoint union, that results is, by Burnside's Lemma, that of all the equivalence classes of $\mathfrak{F}$-structures on ϕ and there are as many isomorphic copies of them (i.e with the same characteristic) as automorphisms of ϕ, then the normalizing factor $\frac{1}{|Aut(\phi)|}$ removes this copies from the sum. So all in all, $\frac{\chi_\Psi(\coprod_{f\in Aut(\phi)} \text{Fix}[\mathfrak{F}[f]])}{|Aut(\phi)|} = \chi_\Psi(\widetilde{\mathfrak{F}}[\phi])$. □

It is left as an exercise to the enthusiastic reader to prove *(if possible)* the previous result for Ψ in general.

3.3. Arithmetic Product and Rational Dirichlet series

We give a notion of arithmetic product for species, and this gives rise to a new type of series: **Rational** "Dirichlet" series.

Definition 3.18 *Let $\phi \in \Phi$, ϕ_0 be the minimal object in the chain of objects in Φ that contains ϕ.*

A partition π of ϕ is simply a decomposition of ϕ as coproduct of objects $\phi = \sum_{i,\pi} \phi_{i,\pi}$.

A regular partition is a partition of ϕ into a direct sum of $k \in \mathbb{N}$ isomorphic copies of a same object $\hat{\phi}$.

If $k\hat{\phi} = \phi$, we will say that $\hat{\phi}$ divides *ϕ.*

A partial rectangle *on ϕ is a pair of partitions π and τ of ϕ such that the least common refinement of both is a partition such that every part of it contains only an object isomorphic to ϕ_0.*

A full rectangle *or simply* rectangle *of ϕ is a pair of regular partitions of ϕ that satisfies the previous condition, as well as there being a non-empty intersection between each pair of parts of π and τ.*

A rectangle on ϕ is noted (π, τ), and k the number of copies of $\hat{\phi}$ in π will be called the height of the rectangle.

The collection of all possible rectangles is noted by $\mathcal{R}[\phi]$, and the collection of rectangles on ϕ of height exactly k by $\mathcal{R}[\phi]|_k$.

Observe that not always it is possible to define a rectangle on an object, or to define a non-trivial one.

Moreover, the partition itself π is not necessarily an object of the category, on which the function can be defined.

This is why the *Arithmetic Product* will be defined only for a very specific type of species: $\mathbb{B};\Psi$ species.

For $\phi \in \mathbb{B}$, if $\chi_{\mathbb{B}}(\phi) = |\phi| = n$, then we write $\phi = n$. In such case clearly $|Aut(\phi)| = n!$, and as [3] give, $|\mathcal{R}[\phi]| = \sum_{d|n} \frac{n!}{d!(\frac{n}{d})!}$, and $|\mathcal{R}[n]|_d = \frac{n!}{d!(\frac{n}{d})!}$. This is because in a rectangle there are $n!$ ways to scramble the labels of the elements it is comprised of, but $d!$ and $\frac{n}{d}!$ ways to scramble the rows and columns repectively, and scrambling these give rise to the same (labelled) object.

Definition 3.19 *The arithmetic product of two $\mathbb{B};\Psi$ species $\mathfrak{F},\mathfrak{L}$ is defined as:*

$$(\mathfrak{F}\boxdot\mathfrak{L})[n] = \sum_{(\pi,\tau)\in\mathcal{R}[n]} \mathfrak{F}[\pi]\times\mathfrak{L}[\tau]$$

Theorem 3.6 *The generating series of the arithmetic product of two species is given by:*

$$(\mathfrak{F}\boxdot\mathfrak{L})(z) = \sum_{\substack{n\in\mathbb{B}\\ \hat{\phi}\oplus\tilde{\phi}=n}} \sum_{d} \frac{|\mathcal{R}[n]||_k\chi_\Psi(\mathfrak{F}[d])\chi_\Psi(\mathfrak{L}(\frac{n}{d}))}{n!}z^n$$

Proof: The proof is identical to that given by [3]:

$$\begin{aligned}\chi_\Psi((\mathfrak{F}\boxdot\mathfrak{L})[n]) &= \chi_\Psi \sum_{(\pi,\tau)\in\mathcal{R}[n]} \mathfrak{F}[\pi]\times\mathfrak{L}[\tau]\\ &= \sum_{d|n}\sum_{\substack{(\pi,\tau)\in\mathcal{R}[n]\\ \text{height}(\pi,\tau)=d}} \chi_\Psi(\mathfrak{F}[d]\times\mathfrak{L}[\frac{n}{d}])\\ &= \sum_{d|n}|\mathcal{R}[n]_d|\chi_\Psi(\mathfrak{F}[d])\chi_\Psi(\mathfrak{L}[\frac{n}{d}])\\ &= \sum_{d|n}\frac{n!}{d!(\frac{n}{d})!}\chi_\Psi(\mathfrak{F}[d])\chi_\Psi(\mathfrak{L}[\frac{n}{d}])\end{aligned}$$

□

Since, as [3] note, there is a bijection between series $s^q \to \frac{1}{q^s}$ we can write any series as a formal series similar to Dirichlet series. Since χ_Ψ takes values in $\mathbb{Q}$, the numerators will not only be natural numbers but can be rational numbers too.

Theorem 3.7 *The arithmetic product of species satisfies the property:*

$$\mathfrak{Q}_{\mathfrak{F}\boxdot\mathfrak{L}}(s) = \mathfrak{Q}_{\mathfrak{F}}(s)\mathfrak{Q}_{\mathfrak{L}}(s)$$

Proof: It is immediate, from performing the substitution $s^q \to \frac{1}{q^s}$ in the definition of the generating series of $\mathfrak{F}\boxdot\mathfrak{L}$ (that is, the expresion for the Dirichlet series) and writing explicitly the number $|\mathcal{R}[n]| = \sum_{d|n}\frac{n!}{d!(\frac{n}{d})!}$. □

One can observe, that there is a similarity (or, rather, an equivalence) between the arithmetic product and a sort of operation on the "cartesian product of sets" (categorical product) in the case of ordinary species. Unfortunately, the property that one may want translated from the arithmetic product series to the Dirichlet series does not generalize so rapidly. In fact, to perform such translation it is necessary for there to be a notion of product.

Definition 3.20 *Let Φ be category with finite products. Then we define the pseudo-arithmetic product of species as:*

$$(\mathfrak{F}\odot\mathfrak{L})[\phi] := \sum_{\substack{\hat{\phi},\tilde{\phi}:\\ \hat{\phi}\times\tilde{\phi}=\phi}} \mathfrak{F}[\hat{\phi}]\times\mathfrak{L}[\tilde{\phi}]$$

.

We will call these series "$\mathbb{Q}$-Dirichlet series". If one thinks about it, they are not so different from usual Dirichlet series: after all, both have rational denominators and both have, although infinitely many, a countable number of them. Thus one could obtain an associated Dirichlet series from a $\mathbb{Q}$-Dirichlet series using a bijection $\mathbb{Q} \leftrightarrow \mathbb{N}$, such as Cantor's diagonal method, but its purpose which will now be made clear is lost in such.

Definition 3.21 *Let $\mathfrak{F}$ be a $\Phi;\Psi$ species, $\Phi : \forall\phi \in \Phi,\ \chi_\Phi(\phi) \neq 0$. Let $\mathfrak{F}(z) = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi])}{|Aut(\phi)|} z^{\chi_\Phi(\phi)}$ be its generating series. The $\mathbb{Q}$-Dirichlet series of $\mathfrak{F}$ is then defined by:*

$$\mathfrak{Q}_{\mathfrak{F}}(s) = \sum_{\phi\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\phi])}{|Aut(\phi)|} \frac{1}{(\chi_\Phi(\phi))^s}$$

Theorem 3.8 *Let Φ be a category with finite products, $\mathfrak{F}$ and $\mathfrak{L}$ be two $\Phi;\Psi$ species, $\mathfrak{Q}_{\mathfrak{F}}(s)$ and $\mathfrak{Q}_{\mathfrak{L}}(s)$ be their respective $\mathbb{Q}$-Dirichlet series. Then the pseudo-arithmetic product of these satisfies:*

$$\mathfrak{Q}_{\mathfrak{F}\odot\mathfrak{L}}(s) = \mathfrak{Q}_{\mathfrak{F}}(s)\mathfrak{Q}_{\mathfrak{L}}(s)$$

Proof:

$$\mathfrak{Q}_{\mathfrak{F}\odot\mathfrak{L}}(s) = \sum_{\phi\in\Phi} \frac{\chi_\Psi((\mathfrak{F}\odot\mathfrak{L})[\phi])}{|Aut(\phi)|} \frac{1}{(\chi_\Phi(\phi))^s}$$

$$= \sum_{\phi\in\Phi} \sum_{\substack{\hat{\phi},\tilde{\phi}: \\ \hat{\phi}\times\tilde{\phi}=\phi}} \frac{\chi_\Psi(\mathfrak{F}[\hat{\phi}] \times \mathfrak{L}[\tilde{\phi}])}{|Aut(\phi)|} \frac{1}{(\chi_\Phi(\phi))^s}$$

Because there are, similarly as for usual dirichlet series, $\frac{|Aut(\phi)|}{|Aut(\hat{\phi})||Aut(\tilde{\phi})|}$ ways to perform the decomposition $\phi = \hat{\phi} \times \tilde{\phi}$ for fixed $\hat{\phi}$ and $\tilde{\phi}$:

$$= \sum_{\hat{\phi},\tilde{\phi}\in\Phi} \frac{|Aut(\phi)|}{|Aut(\phi)|} \frac{\chi_\Psi(\mathfrak{F}[\hat{\phi}])\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\hat{\phi})||Aut(\tilde{\phi})|} \frac{1}{(\chi_\Phi(\hat{\phi})\chi_\Phi(\tilde{\phi}))^s}$$

$$= \sum_{\hat{\phi}\in\Phi} \frac{\chi_\Psi(\mathfrak{F}[\hat{\phi}])}{|Aut(\hat{\phi})|} \frac{1}{\chi_\Phi(\hat{\phi})^s} \sum_{\tilde{\phi}\in\Phi} \frac{\chi_\Psi(\mathfrak{L}[\tilde{\phi}])}{|Aut(\tilde{\phi})|} \frac{1}{\chi_\Phi(\tilde{\phi})^s}$$

$$= \mathfrak{Q}_{\mathfrak{F}}(s)\mathfrak{Q}_{\mathfrak{L}}(s)$$

□

Note 2 *Previously in the definition we asked for $\chi_\Phi(\phi) \neq 0\ \forall\phi \in \Phi$. This is due to the fact that if it were not so, problems could arise when operating between two species with the Dirichlet series.*

3.4. Exponentiation and Assemblies

In ordinary species, the species of "sets" is the functor $E : \mathbb{B} \to \mathbb{E}$ given by $E[n] = \{n\}$.

It exactly has as generating series the exponential map $\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$ and as type series the geometric series.

Now, given Ψ, one may want to find a $\Psi;\Psi$ species such that its generating series *resembles* the exponential map, because such map is essential for finding solutions in the theory of differential equations: Given a differential equation with known solutions, since most ordinary differential equations present solutions in terms of the exponential map, one may find a new combinatorial meaning for these, and thus find new combinatorial properties.

In the example given before, we have that the result of this functor is to send a set to the set of that set. In fact, this can be seen otherwise as the functor which sends a set to the singleton set, which contains one element, and this is sound as a definition, since we have that the characteristic of the object is not that of the source, but of the target, and there is only one such object "$\{n\}$". Similarly, in the case of $\mathbb{V}_q;\mathbb{B}$ species, the exponential is the one given by $V \to \{V\}$.

Definition 3.22 *We call the $\Phi;\Psi$-exponential the functor $I_{\Phi;\Psi} : \Phi \to \Psi$ given by $I[\phi] = \psi_0\ \forall \phi \in \Phi$.*

Proposition 3.3 *The $\Phi;\Psi$-exponential has as generating series:*

$$e^z_{\Phi;\Psi} = \chi_\Psi(\psi_0) \sum_{\phi \in \Phi} \frac{z^{\chi_\Phi(\phi)}}{|Aut(\phi)|}$$

and as type series:

$$\widetilde{e^z_{\Phi;\Psi}} = \chi_\Psi(\psi_0) \sum_{\phi \in \Phi} z^{\chi_\Phi(\phi)}$$

The generating may also be written $\exp_\Psi(z)$ if $\Phi = \mathbb{B}$.

Example 3.1 *The $\mathbb{V}_q;\mathbb{B}$-exponential has generating series:*

$$e^z_{\mathbb{V}_q} = \sum_{n \geq 0} \frac{z^n}{\gamma_n}$$

and type series:

$$\widetilde{e^z_{\mathbb{V}_q}} = \sum_{n \geq 0} z^n$$

We have before, trying to keep close to the usual intuitive definition of an assembly of structures, defined such for general species, using the decomposition by partitions and products. This is compatible with the notion of assembly of structures given in [5], [2], [4].

We give a special emphasis on the notion of exponentiation due to the properties described before.

Traditionally, the species of assemblies of $\mathfrak{F}$-structures and "assemblies of n $\mathfrak{F}$-structures" are defined in terms of the species of Sets as $E \circ \mathfrak{F}$.

But, this is only viable if $\mathfrak{F}$ has as target $\mathbb{B}$.

This traditional approach yields as result, for example for q-species and ordinary species, what can be thought of as an object in the target category which is *enriched* with a $\mathfrak{F}$-structure. such a structure will be called a "Ψ assembly of $\mathfrak{F}$-structures"

Definition 3.23 *Let $\mathfrak{F}$ be a $\Phi;\Psi$ species.*

A Ψ assembly of $\mathfrak{F}$-structures is the species given by the composition of species: $Id_{\mathbb{B};\Psi} \circ \mathfrak{F}$.

From the definitions one immediately has the following:

Proposition 3.4 *The generating series of a Ψ assembly of $\mathfrak{F}$-structures is given by*

$$(I_{\mathbb{B};\Psi} \circ \mathfrak{F})(z) = \chi_\Psi(\psi_0) \sum_{k \geq 0} \gamma_k(\mathfrak{F})(z)$$

Proof:

$$(I_{\mathbb{B};\Psi} \circ \mathfrak{F})(z) = \sum_{\phi \in \Phi} \chi_\Psi\Big(\sum_{\pi_i \in \Pi[\phi]} \psi_0 \times \prod_{\phi_{i,j} \in \pi_i} \mathfrak{F}[\phi_{i,j}]\Big) \frac{z^{\chi_\Phi(\phi)}}{|Aut(\phi)|}$$

$$= \sum_{\phi \in \Phi} \chi_\Psi(\psi_0)\Big(\sum_{\pi_i \in \Pi[\phi]} \prod_{\phi_{i,j} \in \pi_i} \chi_\Psi(\mathfrak{F}[\phi_{i,j}])\Big) \frac{z^{\chi_\Phi(\phi)}}{|Aut(\phi)|}$$

$$= \chi_{\Psi}(\psi_0) \sum_{k \geq 0} \gamma_k(\mathfrak{F})(z)$$

□

If it were so that $\chi_{\Psi}(\psi_0) = 1$ then the definition would indeed be exactly the same as that defined for "exponential" or "assembly of structures" (which are given simultaneously since the former is defined in terms of the latter), as has already been done for the ordinary, q, and Möbius species.

From this result one freely obtains the following:

Corollary 3.1 *Given two* $\Phi; \Psi$ *species* $\mathfrak{F}$ *and* $\mathfrak{L}$*, then:*

$$(I_{\mathbb{B};\Psi} \circ \mathfrak{F})(z)(I_{\mathbb{B};\Psi} \circ \mathfrak{L})(z) = \chi_{\Psi}(\psi_0)(I_{\mathbb{B};\Psi} \circ (\mathfrak{F} + \mathfrak{L}))(z)$$

Proof: Since we have proved that $\gamma_n(\mathfrak{F} + \mathfrak{L}) = \sum_{0 \leq k \leq n} \gamma_k(\mathfrak{F})\gamma_{n-k}(\mathfrak{L})$, then it follows for assemblies of structures that $\sum_{n \geq 0} \gamma_n(\mathfrak{F} + \mathfrak{L})(z) = \sum_{n \geq 0} \gamma_n(\mathfrak{F})(z) \sum_{n \geq 0} \gamma_n(\mathfrak{L})(z)$ by the usual Cauchy product formula for series. □

4. Conclusion

So we have finally settled the basis and extended the notion of species to a more general setting. What is there more to say about the subject? Well, there are a few open questions regarding this which may prove to be interesting.

If one considers the category of all $\Phi; \Psi$ species, with finite categories as objects, functors $\mathfrak{F}$ between them as morphisms, and natural transformations as 2-morphisms, then one obtains [12] a so called (strict) 2-category, which is a subcategory of the primordial 2-category Cat.

This construction may be also done for any pair Φ and Ψ such that there is a notion in each of them of Euler characteristic for their respective objects (categories).

One may then ask, is it possible to extend the notion of species to higher categories, to an n-categorical setting?

Species prove to be a very useful tool for systematically tracking data relating to categories and inferring new qualities about combinatorial constructs with species by manipulating their power series (for example, bounds for the number of constructs, or the existence of a certain construct), so this could be a very fruitful endeavour.

May the Yoneda Lemma have anything to do with it being possible to define a general theory of $\Phi; \Psi$ species for finite categories?

And if so, could a higher categorical analogue of it be the key to creating it?

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