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The Continuous Quaternion Wavelet Transform on Function Spaces

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ABSTRACT: In this paper, boundedness results for the continuous quaternion wavelet transform on Besov, BMO and Hardy H^p spaces are established. Furthermore, the continuous quaternion wavelet transform is also studied on the weighted Besov, BMO_k and H^p_k spaces associated with a tempered weighted function.

Key Words: Quaternions, continuous quaternion wavelet transform, Besov space, Hardy space, BMO space.

Contents

1. Introduction

In 1843, Irish mathematician Sir W. R. Hamilton introduced the quaternions as a generalisation of the complex numbers in which the scalar (real) axis is left unchanged whereas the vector (imaginary) axis is supplemented by adding two more vectors axes. The set of quaternions is denoted by $\mathbb{H}$, and each element $q \in \mathbb{H}$ is written as a linear combination of three imaginary units with real coefficients and real scalars [3,4,5,18,22,24]

$$q = a_0 + ia_1 + ja_2 + ka_3\,;\ a_0, a_1, a_2, a_3 \in \mathbb{R}. \tag{1.1}$$

The three imaginary units i, j, k are square roots of -1 and are related through the following multiplication relations:

$$ij = -ji = k;\ jk = -kj = i;\ ki = -ik = j;\ i^2 = j^2 = k^2 = ijk = -1.$$

The following properties hold in the quaternion algebra:

(i) The conjugate $\overline{q}$ of q is defined as

$$\overline{q} = a_0 - ia_1 - ja_2 - ka_3;\ a_0, a_1, a_2, a_3 \in \mathbb{R}.$$

(ii) The modulus of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\,\overline{q}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

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(iii)
$$|\overline{q}| = |q|, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{p\,q} = \overline{q}\,\overline{p} \quad \text{and} \quad |q\,p| = |q||p|, \text{ for all } \ p, q \in \mathbb{H}. \tag{1.2}$$

According to (1.1), a quaternion-valued function $f : \mathbb{R}^2 \to \mathbb{H}$ can be expressed as

$$f(\mathbf{x}) = f_0(\mathbf{x}) + i f_1(\mathbf{x}) + j f_2(\mathbf{x}) + k f_3(\mathbf{x})\,; \ f_0, f_1, f_2, f_3 \in \mathbb{R}. \tag{1.3}$$

Definition 1.1 (The $L^p(\mathbb{R}^2;\mathbb{H})$ space)**.** *Let $L^p(\mathbb{R}^2;\mathbb{H}), 1 \le p < \infty$, denotes the space of all measurable quaternion-valued functions f on $\mathbb{R}^2$ such that*

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^p \, d^2\boldsymbol{x} < \infty, \ \text{where } d^2\boldsymbol{x} = dx_1 dx_2.$$

The space $L^p(\mathbb{R}^2;\mathbb{H})$ is a normed linear space under the norm defined by

$$\|f\|_{L^p(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^p d^2\boldsymbol{x} \right)^{1/p}.$$

Definition 1.2. *[3]. The right sided quaternion Fourier transform (right-sided QFT) of a function $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is denoted by $\mathscr{F}_q\{f\}$ and defined as*

$$\mathscr{F}_q\{f\}(\boldsymbol{\xi}) := \hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-2\pi i \xi_1 x_1} e^{-2\pi j \xi_2 x_2}\, d^2\boldsymbol{x},$$

where $\boldsymbol{x} = x_1 e_1 + x_2 e_2$, $\boldsymbol{\xi} = \xi_1 e_1 + \xi_2 e_2$, $e_1 = (1,0)$ and $e_2 = (0,1)$, and $e^{-2\pi i \xi_1 x_1} e^{-2\pi j \xi_2 x_2}$ is called the quaternion Fourier kernel.

Theorem 1.3 ([3])**.** *Suppose that $f \in L^2(\mathbb{R}^2;\mathbb{H})$ and $\hat{f} \in L^1(\mathbb{R}^2;\mathbb{H})$. Then the inverse right-sided QFT of $\hat{f}$ is defined as*

$$\mathscr{F}_q^{-1}\{\hat{f}\}(\boldsymbol{x}) = f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\xi})\, e^{2\pi j \xi_2 x_2} e^{2\pi i \xi_1 x_1} d^2\boldsymbol{\xi}.$$

For two functions $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, we obtain Plancherel's formula, specific to the right-sided QFT

$$\langle f, g \rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \langle \mathscr{F}_q\{f\}, \mathscr{F}_q\{g\} \rangle_{L^2(\mathbb{R}^2;\mathbb{H})}.$$

In particular, if $f = g$ then we get Parseval's formula,

$$||f||^2_{L^2(\mathbb{R}^2;\mathbb{H})} = ||\mathscr{F}_q\{f\}||^2_{L^2(\mathbb{R}^2;\mathbb{H})}.$$

Definition 1.4. *The convolution of $f \in L^2(\mathbb{R}^2;\mathbb{H})$ and $g \in L^2(\mathbb{R}^2;\mathbb{H})$, denoted by $f * g$, is defined as*

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{x}\text{-}\boldsymbol{y})\, d^2\boldsymbol{y},$$

and also satisfy distributive law:

$$f * (g + h) = f * g + f * h.$$

For the basic properties of the quaternion Fourier transformation, we may refer to [3,4,5,11,13,14].

2. The Continuous quaternion wavelet transform

The continuous quaternion wavelet transform (CQWT) is the generalisation of continuous wavelet transform. Its application is applied in image denoising. The basic properties and applications of the CQWT may be found in [1,2,4,12,14]. First, we recall the following elementary operations

(1) (Translation) $T_{\mathbf{b}} : f(\mathbf{x}) \to f(\mathbf{x} - \mathbf{b}), \ \mathbf{b} \in \mathbb{R}^2$

(2) (Dilation) $D_a : f(\mathbf{x}) \to a^{-2} f(\mathbf{x}/a),\ a > 0$

(3) (Rotation) $r_\theta : f(\mathbf{x}) \to f(r_\theta \mathbf{x}),\ r_\theta \in SO(2)$,

where $a > 0$ is called the dilation parameter, $\mathbf{b} \in \mathbb{R}^2$ the translation parameter, θ the rotation angle, and the rotation $r_\theta \in SO(2)$ is a 2×2 rotation orthogonal matrix acts on $\mathbf{x} \in \mathbb{R}^2$ as :

$$r_\theta(\mathbf{x}) = (x_1 \cos\theta - x_2 \sin\theta,\ x_1 \sin\theta + x_2 \cos\theta),\ 0 \le \theta < 2\pi.$$

These three operations generate the two-dimensional Euclidean group known as the similitude group SIM(2) denoted by $\mathcal{G}$ on $\mathbb{R}^2$, and defined as [2]

$$\mathcal{G} = \mathbb{R}_+ \times SO(2) \times \mathbb{R}^2 = \left\{ (a, r_\theta, \mathbf{b}) \,|\, a \in \mathbb{R}_+, r_\theta \in SO(2),\ \mathbf{b} \in \mathbb{R}^2 \right\},$$

where SO(2) is the certain orthogonal group of $\mathbb{R}^2$ defined as

$$SO(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in [0, 2\pi) \right\}.$$

Definition 2.1. *Combining the above three operators T_b, D_a and r_θ we define the unitary linear operator $U_{a,\theta,\boldsymbol{b}}$ which acts on a given single function $\psi \in L^2(\mathbb{R}^2; \mathbb{H})$ as*

$$U_{a,\theta,\boldsymbol{b}}(\psi) = \psi_{a,\theta,\boldsymbol{b}}(\boldsymbol{x}) = a^{-2}\, \psi \left(r_{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right).$$

Note that the linear span of the family $\left\{ \psi_{a,\theta,\boldsymbol{b}};\ (a, r_\theta, \boldsymbol{b}) \in \mathcal{G} \right\}$ is dense subspace of $L^2(\mathbb{R}^2; \mathbb{H})$.

Definition 2.2 (Admissible Quaternion wavelet)**.** *A function $\psi \in L^2(\mathbb{R}^2; \mathbb{H})$ satisfies the admissibility condition*

$$0 < C_\psi = \int_{SO(2)} \int_{\mathbb{R}_+} \left| \hat{\psi}\left(a r_{-\theta}(\boldsymbol{\xi}) \right) \right|^2 \frac{da d\theta}{a} < \infty, \tag{2.1}$$

is called admissible quaternion wavelet and AQW denotes the class of such admissible quaternion wavelets.

Using (1.3) we may express the function $\psi \in$ AQW into the following form

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + i\psi_1(\mathbf{x}) + j\psi_2(\mathbf{x}) + k\psi_3(\mathbf{x}),$$

where $\psi_s \in L^2(\mathbb{R}^2; \mathbb{R})$ for $s = 0, 1, 2, 3$. The right-sided QFT of ψ may be defined as

$$\mathscr{F}_q\{\psi\}(\boldsymbol{\xi}) = \mathscr{F}_q\{\psi_0\}(\boldsymbol{\xi}) + i\mathscr{F}_q\{\psi_1\}(\boldsymbol{\xi}) + j\mathscr{F}_q\{\psi_2\}(\boldsymbol{\xi}) + k\mathscr{F}_q\{\psi_3\}(\boldsymbol{\xi}),$$

where $\mathscr{F}_q\{\psi_s\} \in L^2(\mathbb{R}^2; \mathbb{R})$ for $s = 0, 1, 2, 3$. Similar to the classical wavelets [8,10,19], for any $\psi \in$ AQW, we have

$$\int_{\mathbb{R}^2} \psi(\mathbf{x}) d^2\mathbf{x} = 0.$$

Which implies that the integral of each component ψ_s of the quaternion mother wavelet is vanished, i.e.,

$$\int_{\mathbb{R}^2} \psi_s(\mathbf{x}) d^2\mathbf{x} = 0;\ s = 0, 1, 2, 3.$$

Definition 2.3. *[4] The CQWT of a quaternion-valued function $f \in L^2(\mathbb{R}^2; \mathbb{H})$ with respect to $\psi \in AQW$ in 2-dimension is defined as*

$$(W_\psi f)(a, \theta, \boldsymbol{b}) = \langle f, \psi_{a,\theta,\boldsymbol{b}} \rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{a^2} \overline{\psi} \left(r_{-\theta} \left(\frac{\boldsymbol{x} - \boldsymbol{b}}{a} \right) \right) d^2\boldsymbol{x}. \tag{2.2}$$

3. The Continuous quaternion wavelet transform on Besov spaces

Besov spaces have been intensively studied and they naturally appear in many branches of analysis including harmonic analysis, PDEs and approximation theory. Let us recall definitions of Besov space. For an arbitrary $f \in L^p(\mathbb{R}^2, \mathbb{H}), 1 \leq p \leq \infty$, the modulus of continuity is defined by $w_p(f, \mathbf{h}) = ||f(\cdot + \mathbf{h}) - f(\cdot)||_{L^p}$, where $0 < \mathbf{h} \in \mathbb{R}^2$.

Definition 3.1. *For $1 \leq p, q \leq \infty$, Besov space $B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H})$, $0 < \alpha < 1$, is defined as*

$$B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in L^p(\mathbb{R}^2, \mathbb{H}) : \int_{\mathbb{R}^2} [w_p(f, \boldsymbol{h})]^q \frac{d^2\boldsymbol{h}}{|\boldsymbol{h}|^{2+\alpha q}} < \infty \right\}, \text{ for } 1 \leq q < \infty,$$

and

$$B_p^{\alpha,\infty}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in L^p(\mathbb{R}^2, \mathbb{H}) : |\boldsymbol{h}|^{-\alpha} w_p(f, \boldsymbol{h}) \in L^\infty \left(\mathbb{R}^2, \mathbb{H}\right) \right\}, \text{ for } q = \infty,$$

where $|\boldsymbol{h}|$ is an Euclidean norm of $\boldsymbol{h} \in \mathbb{R}^2$. It is easy to see that $B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H})$ is a Banach space associated with the following norms

$$||f||_{B_p^{\alpha,q}} = ||f||_{L^p} + \left(\int_{\mathbb{R}^2} [w_p(f, \boldsymbol{h})]^q \frac{d^2\boldsymbol{h}}{|\boldsymbol{h}|^{2+\alpha q}} \right)^{\frac{1}{q}} \text{ for } q < \infty,$$

and

$$||f||_{B_p^{\alpha,\infty}} = ||f||_{L^p} + \left\| |\boldsymbol{h}|^{-\alpha} w_p(f, \boldsymbol{h}) \right\|_\infty \text{ for } q = \infty.$$

Theorem 3.2. *The operator $W_\psi : B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H}) \to B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H}), f \mapsto (W_\psi f)(a, \theta, \cdot)$, is bounded. Moreover, the following estimate holds*

$$\|(W_\psi f)(a, \theta, \cdot)\|_{B_p^{\alpha,q}} \leq \|\psi\|_{L^1} \|f\|_{B_p^{\alpha,q}}. \tag{3.1}$$

Proof. We claim that $(W_\psi f)(a, \theta, \cdot) \in B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H})$. By change of variable as $\mathbf{x} = a\mathbf{y} + \mathbf{b}$, the equation (2.2) can be rewritten as

$$(W_\psi f)(a, \theta, \mathbf{b}) = \int_{\mathbb{R}^2} f(a\mathbf{y} + \mathbf{b}) \overline{\psi}(r_{-\theta}\mathbf{y}) \, d^2\mathbf{y},$$

and we get

$$\begin{aligned} \|(W_\psi f)(a, \theta, \cdot)\|_{L^p} &= \left\| \int_{\mathbb{R}^2} f(a\mathbf{y} + \mathbf{b}) \overline{\psi}(r_{-\theta}\mathbf{y}) \, d^2\mathbf{y} \right\|_{L^p} \\ &= \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f(a\mathbf{y} + \mathbf{b}) \overline{\psi}(r_{-\theta}\mathbf{y}) \, d^2\mathbf{y} \right|^p d^2\mathbf{b} \right)^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} |f(a\mathbf{y} + \mathbf{b})|^p \, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\ &\leq ||\psi||_{L^1} ||f||_{L^p}. \end{aligned} \tag{3.2}$$

For $0 < \alpha < 1, 1 \leq p < \infty$, by the Minkowski's inequality, we obtain

$$\begin{aligned} w_p\left((W_\psi f)(a, \theta, \cdot), \mathbf{h} \right) &= \|(W_\psi f)(a, \theta, \cdot + \mathbf{h}) - (W_\psi f)(a, \theta, \cdot)\|_{L^p} \\ &= \left\| \int_{\mathbb{R}^2} [f(a\mathbf{y} + \mathbf{b} + \mathbf{h}) - f(a\mathbf{y} + \mathbf{b})] \overline{\psi}(r_{-\theta}\mathbf{y}) \, d^2\mathbf{y} \right\|_{L^p} \\ &= \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} [f(a\mathbf{y} + \mathbf{b} + \mathbf{h}) - f(a\mathbf{y} + \mathbf{b})] \overline{\psi}(r_{-\theta}\mathbf{y}) \, d^2\mathbf{y} \right|^p d^2\mathbf{b} \right)^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left| (f(a\mathbf{y} + \mathbf{b} + \mathbf{h}) - f(a\mathbf{y} + \mathbf{b})) \overline{\psi}(r_{-\theta}\mathbf{y}) \right|^p d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\ &= \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} |f(a\mathbf{y} + \mathbf{b} + \mathbf{h}) - f(a\mathbf{y} + \mathbf{b})|^p \, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\ &\leq \|\psi\|_{L^1} \, w_p(f, \mathbf{h}). \end{aligned}$$

Therefore, for $q < \infty$,

$$\left(\int_{\mathbb{R}^2} \left[w_p\left(\left(W_\psi f\right)(a,\theta,\cdot),\mathbf{h}\right)\right]^q \frac{d^2\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}} \leq \|\psi\|_{L^1} \left(\int_{\mathbb{R}^2} \left[w_p(f,\mathbf{h})\right]^q \frac{d^2\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}}\right)^{\frac{1}{q}}. \tag{3.3}$$

From (3.2) and (3.3), we get

$$\left\|\left(W_\psi f\right)(a,\theta,\cdot)\right\|_{B_p^{\alpha,q}} \leq \|\psi\|_{L^1} \|f\|_{B_p^{\alpha,q}}.$$

□

Corollary 3.3. *If ψ, ϕ are two basic wavelets and $f, g \in B_p^{\alpha,q}(\mathbb{R}^2, \mathbb{H})$ then*

$$\left\|\left(W_\psi f\right)(a,\theta,\cdot) - \left(W_\phi g\right)(a,\theta,\cdot)\right\|_{B_p^{\alpha,q}} \leq \|\psi - \phi\|_{L^1} \|f\|_{B_p^{\alpha,q}} + \|\phi\|_{L^1} \|f-g\|_{B_p^{\alpha,q}}.$$

Now, we shall define the weighted Besov space by means of the weighted Lebesgue space associated with the tempered weight function.

Definition 3.4. *A positive function k defined on $\mathbb{R}^2$ is called a tempered weight function if there exists positive constants C and N such that [15]*

$$k(\boldsymbol{x}+\boldsymbol{y}) \leq (1+C|\boldsymbol{x}|)^N k(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2. \tag{3.4}$$

Definition 3.5. *For $1 \leq p < \infty$, the weighted Lebesgue space $L_k^p(\mathbb{R}^2, \mathbb{H})$ is defined as the space of all measurable quaternion-valued functions f on $\mathbb{R}^2$ such that*

$$||f||_{L_k^p} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^p k(\boldsymbol{x})\, d^2\boldsymbol{x}\right)^{\frac{1}{p}} < \infty,$$

where $k(\boldsymbol{x})$ is a tempered weight function.

For $f \in L_k^p(\mathbb{R}^2, \mathbb{H})$, $k > 0$ and $\mathbf{h} > 0$, we define the modulus of smoothness as $w_{p,k}(f,\mathbf{h}) = \|f(\cdot+\mathbf{h}) - f(\cdot)\|_{L_k^p}$.

Definition 3.6. *For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the weighted Besov space $B_{p,k}^{\alpha,q}(\mathbb{R}^2, \mathbb{H}), 0 < \alpha < 1$, is defined as*

$$B_{p,k}^{\alpha,q}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in L_k^p(\mathbb{R}^2, \mathbb{H}) : \int_{\mathbb{R}^2} (w_{p,k}(f,\boldsymbol{h}))^q \frac{d^2\boldsymbol{h}}{|\boldsymbol{h}|^{2+\alpha q}} < \infty \right\} \text{ for all } 1 \leq q < \infty,$$

and

$$B_{p,k}^{\alpha,\infty} = \left\{ f \in L_k^p(\mathbb{R}^2, \mathbb{H}) : |\boldsymbol{h}|^{-\alpha} w_{p,k} \in L^\infty\left(\mathbb{R}^2, \mathbb{H}\right) \right\} \text{ for } q = \infty.$$

It is easy to see that the space $B_{p,k}^{\alpha,q}, 1 \leq q < \infty$, is a Banach space associated with the norm defined by

$$||f||_{B_{p,k}^{\alpha,q}} = ||f||_{L_k^p} + \left(\int_{\mathbb{R}^2} (w_{p,k}(f,\boldsymbol{h}))^q \frac{d^2\boldsymbol{h}}{|\boldsymbol{h}|^{2+\alpha q}}\right)^{\frac{1}{q}}$$

and if $q = \infty$,

$$||f||_{B_{p,k}^{\alpha,\infty}} = ||f||_{L_k^p} + \left\||\boldsymbol{h}|^{-\alpha} w_{p,k}(f,\boldsymbol{h})\right\|_\infty.$$

Theorem 3.7. *Let ψ be a compactly supported basic wavelet whose support is contained in a disk centred at the origin and of radius r. Then, the operator $W_\psi : B_{p,k}^{\alpha,q} \to B_{p,k}^{\alpha,q}, f \mapsto \left(W_\psi f\right)(a,\theta,\cdot)$, for any $a > 0$, is bounded and*

$$\left\|\left(W_\psi f\right)(a,\theta,\cdot)\right\|_{B_{p,k}^{\alpha,q}} \leq (1+Cra)^{\frac{N}{p}} \|\psi\|_{L^1} \|f\|_{B_{p,k}^{\alpha,q}}.$$

Proof. Let us give a proof of Theorem 3.7 in the case $1 \le p \le q < \infty$. By the Minkowski's inequality, it can be easily seen that

$$\begin{aligned}
\| (W_\psi f)(a,\theta,\cdot) \|_{L^p_k} &= \left\| \int_{\mathbb{R}^2} f(\mathbf{b}+\mathbf{y})\, \overline{\psi}_{a,\theta}(\mathbf{y})\, d^2\mathbf{y} \right\|_{L^p_k} \\
&= \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f(\mathbf{b}+\mathbf{y})\, \overline{\psi}_{a,\theta}(\mathbf{y})\, d^2\mathbf{y} \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left| f(\mathbf{b}+\mathbf{y})\, \overline{\psi}_{a,\theta}(\mathbf{y}) \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&= \int_{\mathbb{R}^2} |\psi_{a,\theta}(\mathbf{y})| \left(\int_{\mathbb{R}^2} |f(\mathbf{b}+\mathbf{y})|^p \; k(-\, \mathbf{y} + \mathbf{b} + \mathbf{y})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le \int_{\mathbb{R}^2} (1 + C|\mathbf{y}|)^{\frac{N}{p}} |\psi_{a,\theta}(\mathbf{y})| \left(\int_{\mathbb{R}^2} |f(\mathbf{b}+\mathbf{y})|^p \; k(\mathbf{b}+\mathbf{y})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le \int_{|\mathbf{y}|\le r} (1 + C|\mathbf{y}|)^{\frac{N}{p}} |\psi_{a,\theta}(\mathbf{y})| \left(\int_{\mathbb{R}^2} |f(\mathbf{z})|^p \; k(\mathbf{z})\, d^2\mathbf{z} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le (1 + Car)^{\frac{N}{p}} \, \|\psi\|_{L^1} \, \|f\|_{L^p_k} .
\end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned}
&w_{p,k}\left((W_\psi f)(a,\cdot), \mathbf{h} \right) \\
&= \| (W_\psi f)(a,\theta,\cdot + \mathbf{h}) - (W_\psi f)(a,\theta,\cdot) \|_{L^p_k} \\
&= \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} [\, f(a\mathbf{y}+\mathbf{b}+\mathbf{h}) - f(a\mathbf{y}+\mathbf{b}) \,]\, \overline{\psi}(r_{-\theta}\mathbf{y})\, d^2\mathbf{y} \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{y})\,| \left(\int_{\mathbb{R}^2} |f(a\mathbf{y}+\mathbf{b}+\mathbf{h}) - f(a\mathbf{y}+\mathbf{b})|^p \, k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le \int_{\mathbb{R}^2} (1 + C|a\mathbf{y}|)^{\frac{N}{p}} \, |\psi(r_{-\theta}\mathbf{y})\,| \left(\int_{\mathbb{R}^2} |f(a\mathbf{y}+\mathbf{b}+\mathbf{h}) - f(a\mathbf{y}+\mathbf{b})|^p \, k(a\mathbf{y}+\mathbf{b}) d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le \int_{|\mathbf{y}|\le r} (1 + Ca|\mathbf{y}|)^{\frac{N}{p}} \, |\psi(r_{-\theta}\mathbf{y})\,| \left(\int_{\mathbb{R}^2} |f(a\mathbf{y}+\mathbf{b}+\mathbf{h}) - f(a\mathbf{y}+\mathbf{b})|^p \, k(a\mathbf{y}+\mathbf{b}) d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le (1 + Car)^{\frac{N}{p}} \, \|\psi\|_{L^1} \, w_{p,k}(f, \mathbf{h}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2} \left(w_{p,k}\left((W_\psi f)(a,\theta,\cdot), \mathbf{h} \right) \right)^q \frac{d^2\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} \right)^{\frac{1}{q}} \\
&\le (1 + Car)^{\frac{N}{p}} \, \|\psi\|_{L^1} \left(\int_{\mathbb{R}^2} \left(w_{p,k}(f, \mathbf{h}) \right)^q \frac{d^2\mathbf{h}}{|\mathbf{h}|^{2+\alpha q}} \right)^{\frac{1}{q}} .
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we obtain the final result. □

Corollary 3.8. *Let ψ and ϕ be two compactly supported basic wavelets whose supports are contained in a disk centred at the origin and of radius r. If $f, g \in B^{\alpha,q}_{p,k}(\mathbb{R}^2, \mathbb{H})$, then*

$$\begin{aligned}
&\| (W_\psi f)(a,\theta,\cdot) - (W_\phi g)(a,\theta,\cdot) \|_{B^{\alpha,q}_{p,k}} \\
&\le (1 + Cra)^{\frac{N}{p}} \left(\|\psi - \phi\|_{L^1} \, \|f\|_{B^{\alpha,q}_{p,k}} + \|\phi\|_{L^1} \, \|f - g\|_{B^{\alpha,q}_{p,k}} \right) .
\end{aligned}$$

4. The Continuous quaternion wavelet transform on Hardy H^p spaces

Hardy spaces play an important role in various areas of pure and applied mathematics including harmonic analysis and PDEs. In this section, the continuous quaternion wavelet transform is studied on Hardy spaces.

Definition 4.1 (Schwartz space). *The Schwartz space $S(\mathbb{R}^2,\mathbb{H})$ be the space of all infinitely differentiable functions ϕ on $\mathbb{R}^2$ such that $\sup_{\boldsymbol{x}\in\mathbb{R}^2} |\boldsymbol{x}^\beta D^\gamma \phi(\boldsymbol{x})| < \infty$ for all multi-indices β and γ.*

Definition 4.2. *Hardy space $H^p(\mathbb{R}^2,\mathbb{H})$ is defined as the space of all functions $f \in L^p(\mathbb{R}^2,\mathbb{H})$ such that*

$$\|f\|_{H^p(\mathbb{R}^2,\mathbb{H})} = \left(\int_{\mathbb{R}^2} \sup_{t>0} |(\overline{f} * \varphi_t)(\boldsymbol{x})|^p d^2\boldsymbol{x}\right)^{\frac{1}{p}},$$

where $\varphi_t = t^{-2}\varphi(\frac{\boldsymbol{x}}{t})$, $t > 0$, $\boldsymbol{x} \in \mathbb{R}^2$, and φ be a function in the Schwartz space such that $\int_{\mathbb{R}^2} \varphi(\boldsymbol{x})d\boldsymbol{x} \neq 0$.

Theorem 4.3. *The operator $W_\psi : H^p(\mathbb{R}^2,\mathbb{H}) \to H^p(\mathbb{R}^2,\mathbb{H}), f \to (W_\psi f)(a,\theta,\cdot)$ is bounded. Moreover, the following estimate holds*

$$\|(W_\psi f)(a,\theta,\cdot)\|_{H^p} \le \|\psi\|_{L^1} \|f\|_{H^p}.$$

Proof. By the change of variable as $\mathbf{x} = a\mathbf{y} + \mathbf{b}$, we have

$$\begin{aligned}\overline{(W_\psi f)(a,\theta,\mathbf{b})} &= \int_{\mathbb{R}^2} \overline{f(a\mathbf{y}+\mathbf{b})\,\overline{\psi}(r_{-\theta}\mathbf{y})}\, d^2\mathbf{y} \\ &= \int_{\mathbb{R}^2} \psi(r_{-\theta}\mathbf{y})\,\overline{f}(a\mathbf{y}+\mathbf{b})\, d^2\mathbf{y}\end{aligned}$$

and hence,

$$\begin{aligned}\left(\overline{(W_\psi f)(a,\theta,\cdot)} * \varphi_t\right)(\mathbf{b}) &= \int_{\mathbb{R}^2} \psi(r_{-\theta}\mathbf{y}) \left(\int_{\mathbb{R}^2} \overline{f}(a\mathbf{y}+\mathbf{b}-\mathbf{x})\,\varphi_t(\mathbf{x})\, d^2\mathbf{x}\right) d^2\mathbf{y} \\ &= \int_{\mathbb{R}^2} \psi(r_{-\theta}\mathbf{y}) \left(\overline{f} * \varphi_t\right)(a\mathbf{y}+\mathbf{b})\, d^2\mathbf{y}.\end{aligned} \tag{4.1}$$

Using the Minkowski's inequality, we get

$$\begin{aligned}\|(W_\psi f)(a,\theta,\cdot)\|_{H^p} &= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\left(\overline{(W_\psi f)(a,\theta,\cdot)} * \varphi_t\right)(\mathbf{b})\right|^p d^2\mathbf{b}\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\int_{\mathbb{R}^2} \psi(r_{-\theta}\mathbf{y}) \left(\overline{f} * \varphi_t\right)(a\mathbf{y}+\mathbf{b})\, d^2\mathbf{y}\right|^p d^2\mathbf{b}\right)^{\frac{1}{p}} \\ &\le \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\psi(r_{-\theta}\mathbf{y}) \left(\overline{f} * \varphi_t\right)(a\mathbf{y}+\mathbf{b})\right|^p d^2\mathbf{b}\right)^{\frac{1}{p}} d^2\mathbf{y} \\ &= \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} \sup_{t>0} \left|\left(\overline{f} * \varphi_t\right)(a\mathbf{y}+\mathbf{b})\right|^p d^2\mathbf{b}\right)^{\frac{1}{p}} d^2\mathbf{y} \\ &\le \|\psi\|_{L^1} \|f\|_{H^p}.\end{aligned}$$

∎

Corollary 4.4. *If ψ, ϕ are two basic wavelets and $f, g \in H^p(\mathbb{R}^2,\mathbb{H})$, then*

$$\|(W_\psi f)(a,\theta,\cdot) - (W_\phi g)(a,\theta,\cdot)\|_{H^p} \le \|\psi - \phi\|_{L^1} \|f\|_{H^p} + \|\phi\|_{L^1} \|f - g\|_{H^p}.$$

Definition 4.5. *Weighted Hardy space $H_k^p(\mathbb{R}^2, \mathbb{H})$ is defined as the space of all functions $f \in L_k^p(\mathbb{R}^2, \mathbb{H})$ such that*

$$||f||_{H_k^p} = \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| (\overline{f} * \varphi_t)(\boldsymbol{x}) \right|^p k(\boldsymbol{x})\, d^2\boldsymbol{x} \right)^{\frac{1}{p}},$$

where $k(\boldsymbol{x})$ is a tempered weight function and $\varphi_t = t^{-2}\varphi(\frac{\boldsymbol{x}}{t})$, $t > 0$, $\boldsymbol{x} \in \mathbb{R}^2$, and φ be a function in the Schwartz space such that $\int_{\mathbb{R}^2} \varphi(\boldsymbol{x}) d\boldsymbol{x} \neq 0$.

Theorem 4.6. *Let ψ be a compactly supported basic wavelet whose support is contained in a disk centred at the origin and of radius r. Then, the operator $W_\psi : H_k^p(\mathbb{R}^2, \mathbb{H}) \to H_k^p(\mathbb{R}^2, \mathbb{H})$ is bounded. Moreover*

$$\|(W_\psi f)(a, \theta, \cdot)\|_{H_k^p} \le (1 + Car)^{\frac{N}{p}} \|\psi\|_{L^1} \|f\|_{H_k^p}.$$

Proof. Using (4.1) and the Minkowski's inequality, we get

$$\begin{aligned}
&\|(W_\psi f)(a, \theta, \cdot)\|_{H_k^p} \\
&= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| \left(\overline{(W_\psi f)(a, \theta, \cdot)} * \varphi_t \right)(\mathbf{b}) \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| \int_{\mathbb{R}^2} \psi(r_{-\theta}\mathbf{y}) \left(\overline{f} * \varphi_t \right)(a\mathbf{y} + \mathbf{b})\, d^2\mathbf{y} \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| \left(\overline{f} * \varphi_t \right)(a\mathbf{y} + \mathbf{b}) \right|^p k(\mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y}. \\
&\le \int_{\mathbb{R}^2} (1 + C|a\mathbf{y}|)^{\frac{N}{p}} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| (\overline{f} * \varphi_t)(a\mathbf{y} + \mathbf{b}) \right|^p k(a\mathbf{y} + \mathbf{b})\, d^2\mathbf{b} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le \int_{|\mathbf{y}| \le r} (1 + Ca|\mathbf{y}|)^{\frac{N}{p}} |\psi(r_{-\theta}\mathbf{y})| \left(\int_{\mathbb{R}^2} \sup_{t>0} \left| (\overline{f} * \varphi_t)(\mathbf{x}) \right|^p k(\mathbf{x})\, d^2\mathbf{x} \right)^{\frac{1}{p}} d^2\mathbf{y} \\
&\le (1 + Car)^{\frac{N}{p}} \|\psi\|_{L^1} \|f\|_{H_k^p}.
\end{aligned}$$

□

Corollary 4.7. *Let ψ and ϕ be two compactly supported basic wavelets whose supports are contained in a disk centred at the origin and of radius r. If $f, g \in H_k^p(\mathbb{R}^2, \mathbb{H})$, then*

$$\begin{aligned}
&\| (W_\psi f)(a, \theta, \cdot) - (W_\phi g)(a, \theta, \cdot) \|_{H_k^p} \\
&\le (1 + Cra)^{\frac{N}{p}} \left(\|\psi - \phi\|_{L^1} \|f\|_{H_k^p} + \|\phi\|_{L^1} \|f - g\|_{H_k^p} \right).
\end{aligned}$$

5. The Continuous quaternion wavelet transform on BMO spaces

In 1961, F. John and L. Nirenberg [17] introduced the bounded mean oscillation (BMO) space as the dual space of the real Hardy space H^1. Nowadays, the BMO space is very useful in harmonic analysis, PDEs and theory of functions.

Definition 5.1. *The space $BMO(\mathbb{R}^2, \mathbb{H})$ is defined as the space of all functions $f \in L^1_{loc}(\mathbb{R}^2, \mathbb{H})$ such that*

$$\|f\|_{BMO(\mathbb{R}^2, \mathbb{H})} = \sup_{B \subset \mathbb{R}^2} \frac{1}{|B|} \int_B |f - f_B|\, d^2\boldsymbol{x} < \infty,$$

where the supremum is taken over all the disks B in $\mathbb{R}^2$, and f_B is the mean value of the function f on B defined by $f_B = \frac{1}{|B|}\int_B f(\boldsymbol{y}) d\boldsymbol{y}$ for each disk $B \subset \mathbb{R}^2$.

Theorem 5.2. *Let ψ be a compactly supported basic wavelet. For any $a > 0$, the operator W_ψ : $BMO(\mathbb{R}^2, \mathbb{H}) \to BMO(\mathbb{R}^2, \mathbb{H}), f \mapsto (W_\psi f)(a, \theta, \cdot)$, is bounded. Furthermore, the following estimate holds*

$$\|(W_\psi f)(a,\theta,\cdot)\|_{BMO(\mathbb{R}^2,\mathbb{H})} \leq \|\psi\|_{L^1} \|f\|_{BMO(\mathbb{R}^2,\mathbb{H})}.$$

Proof. Let B be an arbitrary disk $\subset \mathbb{R}^2$. Then, we have

$$\begin{aligned}\int_B |(W_\psi f)(a,\theta,\mathbf{b})| d^2\mathbf{b} &\leq \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{x})| \left(\int_B |f(a\mathbf{x}+\mathbf{b})| d^2\mathbf{b} \right) d^2\mathbf{x} \\ &= \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{x})| \left(\int_Q |f(\mathbf{y})| d^2\mathbf{y} \right) d^2\mathbf{x},\end{aligned}$$

where $Q = a\mathbf{x} + B$. Since $Q \subset a \operatorname{supp} \psi + B$ is a compact set in $\mathbb{R}^2$ and $f \in L^1_{loc}(\mathbb{R}^2, \mathbb{H})$, it follows that

$$\int_B |(W_\psi f)(a,\theta,\mathbf{b})| d^2\mathbf{b} \leq K \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{x})| d^2\mathbf{x} = K \|\psi\|_{L^1} < \infty,$$

and hence $(W_\psi f)(a, \theta, \cdot) \in L^1_{loc}(\mathbb{R}^2, \mathbb{H})$. By the Fubini's theorem, we have

$$(W_\psi f)_B(a,\theta,\mathbf{b}) = \int_{\mathbb{R}^2} \left(\frac{1}{|B|} \int_B f(a\mathbf{x}+\mathbf{b}) \overline{\psi}(r_{-\theta}\mathbf{x}) \, d^2\mathbf{b} \right) d^2\mathbf{x} = \int_{\mathbb{R}^2} f_Q \overline{\psi}(r_{-\theta}\mathbf{x}) \, d^2\mathbf{x}.$$

Using the Minkowski's inequality, we get

$$\begin{aligned}&\|(W_\psi f)(a,\theta,\cdot)\|_{BMO(\mathbb{R}^2,\mathbb{H})} \\ &= \sup_{B\subset\mathbb{R}^2} \frac{1}{|B|} \int_B \left|(W_\psi f)(a,\theta,\mathbf{b}) - (W_\psi f)_B(a,\theta,\mathbf{b})\right| d^2\mathbf{b}. \\ &\leq \sup_{B\subset\mathbb{R}^2} \frac{1}{|B|} \int_B \left(\int_{\mathbb{R}^2} |f(a\mathbf{x}+\mathbf{b}) - f_Q| \, |\psi(r_{-\theta}\mathbf{x})| \, d^2\mathbf{x} \right) d^2\mathbf{b} \\ &= \int_{\mathbb{R}^2} |\psi(r_{-\theta}\mathbf{x})| \left(\sup_{Q\subset\mathbb{R}^2} \frac{1}{|Q|} \int_Q |f(\mathbf{y}) - f_Q| \, d^2\mathbf{y} \right) d^2\mathbf{x} \\ &= \|\psi\|_{L^1} \|f\|_{BMO(\mathbb{R}^2,\mathbb{H})}.\end{aligned}$$

□

Corollary 5.3. *Let ψ and ϕ be two compactly supported basic wavelets and $f, g \in BMO(\mathbb{R}^2, \mathbb{H})$. Then*

$$\begin{aligned}&\|(W_\psi f)(a,\theta,\cdot) - (W_\phi g)(a,\theta,\cdot)\|_{BMO(\mathbb{R}^2,\mathbb{H})} \\ &\leq \|\psi - \phi\|_{L^1} \|f\|_{BMO(\mathbb{R}^2,\mathbb{H})} + \|\phi\|_{L^1} \|f - g\|_{BMO(\mathbb{R}^2,\mathbb{H})}.\end{aligned}$$

Definition 5.4. *The weighted bounded mean oscillation space $BMO_k(\mathbb{R}^2, \mathbb{H})$ is defined as the space of all weighted Lebesgue integrable (locally) functions defined on $\mathbb{R}^2$ such that*

$$\| f \|_{BMO_k} = \sup_{B\subset\mathbb{R}^2} \frac{1}{|B|_k} \int_B |f(\boldsymbol{x}) - f_B| k(\boldsymbol{x}) d^2\boldsymbol{x} < \infty,$$

where the supremum is taken over all the disks B in $\mathbb{R}^2$ and $|B|_k = \int_B k(\boldsymbol{x}) d^2\boldsymbol{x}$, and $k(\boldsymbol{x})$ is a tempered weight function.

Theorem 5.5. *Let ψ be a compactly supported basic wavelet. For any $a > 0$, the operator W_ψ : $BMO_k(\mathbb{R}^2, \mathbb{H}) \to BMO_k(\mathbb{R}^2, \mathbb{H}), f \mapsto (W_\psi f)(a, \theta, \cdot)$, is bounded. Furthermore, the following estimate holds*

$$\|(W_\psi f)(a,\theta,\cdot)\|_{BMO_k(\mathbb{R}^2,\mathbb{H})} \leq (1 + Car)^{2N} \|\psi\|_{L^1} \|f\|_{BMO_k(\mathbb{R}^2,\mathbb{H})}.$$

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The authors have no conflict of interest to declare.

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