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Separable ranges of Henstock-Kurzweil-Pettis integral

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ABSTRACT: In this article, we introduce Pettis integrability type property for Henstock-Kurzweil-Pettis-integrals. We discuss several necessary conditions that X has Henstock-Kurzweil-Pettis-integrability property for weak Baire measure. Necessary and sufficient conditions of the indefinite integral of any Henstock-Kurzweil-Pettis (respectively, Denjoy-Pettis) integrable function with values in a fixed Banach space having separable ranges are discussed.

Key Words: Henstock-Kurzweil-Pettis integrability property, Mazur property, separable ranges.

Contents

1. Introduction

Integration Theory of functions with values in a Banach space has long been a fruitful area of study. Bochner, Gelfand, Pettis, Birkhoff, Phillips, etc have a major contribution in this area of interest. Edgar, Huff, Talagrand discussed Pettis integrability with scalar measurability, Pettis integrability of weak measurable functions with values by several different techniques (see [2,3,12]). Piazza and Musiał' [11] characterized Henstock-Kurzweil-Pettis integrals by cores of the functions and throught the properties of suitable operations defined by integrands. Bongiorno et al. [1] studied about Henstock-Kurzweil-Pettis integrable function and Henstock integrable functions are scalarly approximated in the Alexiewicz norm by a sequence of step functions. In their work authors have done the full approximation of the Henstock-Kurzweil-Pettis integral if and only if the range of the integral is norm relatively compact. notion of Bourgain property of real-valued functions formulated by J. Bourgain. Plebanek [9] has shown that the Pettis integral property of a Banach space X, together with the requirement of separability of X-valued Pettis integral is equivalent to the fact that every weak Baire measure on X is in a certain weak sense, concentrated on a separable subspace. Naralenkov [8] find the necessary and sufficient conditions so that the indefinite integral of any Henstock-Kurzweil-Pettis integrable function with values in a fixed Banach space has a relatively compact range or be continuous, except at most on a countable set. Through out the article, we denote Henstock-Kurzweil-Pettis integrals as HKP-integrals.

Plebanek and Naralenkov's work motivated us to investigate the Henstock-Kurzweil-Pettis integrability property for μ-measure compact and Henstock-Kurzweil-Pettis integrals. We also investigate separable ranges of the Henstock-Kurzweil-Pettis integrable functions. Finally, we discuss separable ranges of indefinite Henstock-Kurzweil-Pettis integrals.

Our article structured as follows: In Section 2, the basic concepts and terminology are intro- duced together with some definitions and results. In Section 3, several properties of HKP-integral has been discussed. We give several conditions of Henstock-Kurzweil-Pettis integrability in X associate with weak Baire measure and compact measure. In Section 4, separable ranges of HKP-integrable function are discussed.

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2. Preliminaries

Assume Ω be set of intervals, Σ are the σ-algebra of Ω. Throughout the article (Ω, Σ, μ) be finite measure space. Let $[0,1]$ be the unit interval of the real line with usual topology and μ be Lebesgue measure. In the whole article X is an arbitrary Banach space with dual X^*. The closed unit ball of X is denoted by $B(X)$. If $Y \subset X$ then $Y^\perp$ are annihilator of Y in X^*. We denote $\sigma(X, X^*)$ are weak topology and the topology generated on X by a Banach space Y is $\sigma(X, Y)$. Recall Banach-valued Henstock-Kurzweil as follows.

Definition 2.1 [1, Definition 1] A function $f : [0,1] \to X$ is said to be Henstock-Kurzweil integrable on $[0,1]$ if there exists $w \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$||\sum_{i=1}^{p} f(t_i)|I_i| - w|| < \epsilon$$

for each δ-fine perron partition $P = \{(I_1, t_i), (I_2, t_2), .., (I_p, t_p)\}$ of $[0,1]$. We set $w = (H)\int_0^1 f d\mu$. We call $HK([0,1], X)$ the set of all Henstock integrable functions $f : [0,1] \to X$.

The $HK[0,1]$ is endowed with the Alexiewicz norm

$$||h||_A = \sup_{0 \le \alpha \le 1} |(HK)\int_0^\alpha h(t) d\mu(t)|.$$

The completion of the $HK[0,1]$ is isomorphic to the space of distribution which are distributional derivatives of continuous function. The conjugate space $HK^*[0,1]$ is linearly isometric to the space $BV[0,1]$.

Definition 2.2 [13] A function $f : [0,1] \to X$ is said to be scalarly measurable (resp scalarly integrable) if for every $x^* \in X^*$ the function $x^* f$ is Lebesgue measurable(resp integrable). A scalarly integrable function $f : [0,1] \to X$ is said to be Pettis integrable if for each $A \in L$ there exists a vector $w_A \in X$ such that

$$\langle x^*, w_A \rangle = \int_A x^* f d\mu \; for \; every \; x^* \in X^*.$$

We call w_A the Pettis integral of f over A and we write $w_A = (P)\int_A f d\mu$.

Definition 2.3 [10] A function $f : [0,1] \to X$ is said to be scalarly Henstock-Kurzweil integrable if for each $x^* \in X^*$ the function $x^* f$ is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function f is said to be Henstock-Kurzweil-Pettis integrable if for each $I \in [0,1]$ there exists $w_I \in X$ such that

$$\langle x^*, w_I \rangle = (HK)\int_I x^* f d\mu \; for \; every \; x^* \in X^*.$$

We call w_I the Henstock-Kurzweil-Pettis integral of f over I and we write $w_I = (HKP)\int_I f d\mu$.

We call the additive interval function $F(I) = (HKP)\int_I f d\mu$ the HKP-primitive of f. Let $HKP([0,1], X)$ the set of all X-valued HKP-integrable function on $[0,1]$ (functions that are scalarly equivalent are identified). Henstock-Kurzweil-primitive (resp HKP-primitives) F of a function f is continuous (resp weakly continuous i.e. $x^* F$ is continuous for every $x^* \in X^*$). One can see [1]. The Alexiewicz norm on $HKP([0,1], X)$ is

$$||f||_A = \sup_{0 \le t \le 1} ||(HKP)\int_0^1 f d\mu||.$$

If $|x^* f| \le K$ a.e. for some constant K and every $x^* \in X^*$ then f is weakly bounded. If $f : [0,1] \to X$ is a scalarly bounded function then the image measure λ given by

$$\lambda(E) = \mu(f^{-1}(E))$$

is a weak Baire measure on X.

Definition 2.4 [9] Given a weak Baire measure μ on a Banach space X, we say μ is scalarly concentrated on a subspace Γ of X if $x^*|_\Gamma = 0$ implies $x^* = 0$ μ-a.e..

Recall a topological space X is measure compact if every non zero-σ-additive Baire measure on X has non empty support. Measure-compact implies real compact but not conversely. It is well known that every compact measure is perfect. For a measure to be perfect it is necessary and sufficient that it's restriction to any σ-subalgebra with a countable number of generators be compact.

Recall a Banach space X is said to have Pettis Integral property (PIP) if for every X-valued weakly bounded function f defined on an arbitrary finite measure space is Pettis integrable. We say that X has Mazur's property if each $weak^*$-continuous functional on X^* is in X.

Definition 2.5 *A scalarly measurable function $f : [0,1] \to X$ is determined by a space $Y \subseteq X$ if for every $x^* \in Y^\perp$ the equality $x^*f = 0$ hold μ-a.e..*

Theorem 2.1 [1, Theorem 2] If $f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable then there exists a sequence (f_n) of X-valued step function such that for all $x^* \in X^*$, $x^*f = \lim_n x^* f_n$ a.e. if and only if f is determined by a separable subspace of X.

3. Henstock-Kurzweil-Pettis integral

The Henstock-Kurzweil-Pettis integral is the natural generalization of the Pettis integral for a function, obtained by replacing the Lebesgue integrability for scalar functions by the Henstock-Kurzweil integrability. If X contains no copy of c_0, it is well known that each X-valued strongly measurable scalarly integrable function is also Pettis integrable. In the case of the Henstock-Kurzweil integrability, a similar property is fulfilled by weakly sequentially complete Banach spaces. Gordon proved that a separable Banach space is weakly sequentially complete if and only if each X-valued Henstock-Kurzweil scalarly integrable function is also HKP integrable.

Lemma 3.1 *Let $f : [0,1] \to X$ be scalarly Henstock-Kurzweil integrable. Define $T_f : X^* \to HK(\mu)$ by*

$$T_f(x^*)(t) = < f(t), x^* >, \;\; x^* \in X^*, \;\; t \in [0,1]$$

then T_f is weak*-pointwise continuous.

Proof: If $t \in [0,1]$ and $X_a^* \to x^*$ $(weak^*)$, then

$$\begin{aligned} T_f(x_a^*)(t) &= < f(t), x_a^* > \\ &\to < f(t), x^* > \\ &= T_f(x^*)(t) \end{aligned}$$

so, $T_f(x_a^*) = T_f(x^*)$ is pointwise. □

Proposition 3.1 *Let $f : [0,1] \to X$ be scalarly Henstock-Kurzweil integrable then f is HKP-integrable if and only if $T_f : X^* \to HK([0,1], \mu)$ is weak*-weakly continuous. In particular if f is Henstock-Kurzweil-Pettis integrable then T_f is necessarily a weakly compact operator.*

Proof: Let T_f is $weak^*$-weakly continuous. Assume $A \in \Sigma$. Then $\chi_A \in BV(\mu)$. In this situation linear functional $\mathfrak{a} \in X^{**}$

$$\begin{aligned} < x^*, \mathfrak{a} > &= (HK) \int_A < f(t), x^* > d\mu(t) \\ &= (HK) \int T_f(x^*)(t).\chi_A(t) d\mu(t) \\ &= < T_f(x^*), \chi_A > \end{aligned}$$

is $weak^*$ continous. So, for $x_A \in X$ with

$$< x_A, x^* >= (HK) \int_A < f(t), x^* > d\mu(t) \ for \ all \ x^* \in X^* \tag{3.1}$$

gives f is Henstock-Kurzweil-Pettis integrable.
Conversely, assume f is Henstock-Kurzweil-Pettis integrable then for $A \in \Sigma$ there exists $x_A \in X$ with (3.1). So, from the representation theorem and convergence theorem we have:

$$< T_f(x^*), \chi_A >= (HK) \int_A < f(t), x^* > d\mu(t) =< x_A, x^* >$$

is $weak^*$-continuous function of x. Hence $< T_f(x^*), g >$ is $weak^*$ continuous function of x^* for any step function g. Again, using the boundedness of f, we get $< T_f(x^*), g >$ is $weak^*$-continuous function of x^* for any $g \in BV(\mu)$. □

Corollary 3.1 *Let $f : [0,1] \to X$ be scalarly Henstock-Kurzweil integrable then f is HKP-integrable if and only if $T_f : B(X^*) \to HK([0,1], \mu)$ is $weak^*$-weakly continuous.*

From Proposition 3.1 and Corollary 3.1 we can deduce the following corollary as follows:

Corollary 3.2 *A scalarly Henstock-Kurzweil integrable function $f : [0,1] \to X$ determined by a subspace Y of X if and only if $T : X \to HK(\mu)$ is defined by $x^* \to x^* f$ is a $\sigma(X^*, Y)$-weakly continuous.*

The action of T_f on $weak^*$ neighborhoods in X^* as follows:

Definition 3.1 [5] Let Y be a finite set in X, and $\epsilon > 0$, let

$$\mathcal{O}(Y, \epsilon) = \{x^* \in X^* : \ ||x^*|| \leq 1 \ and \ x^*(x) \leq \epsilon \ for \ every \ x \ in \ Y\}.$$

Proposition 3.2 *If f is scalarly Henstock-Kurzweil integrable, then for all $Y \subseteq X$, $\epsilon > 0$ the set $T_f(\mathcal{O}(Y, \epsilon))$ is closed and convex in $HK(\mu)$.*

Proof: The proof is of similar arguement of the [5, Lemma 2]. □

Proposition 3.3 *If f is scalarly Henstock-Kurzweil integrable. Then the following are equivalent:*

1. *$f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable.*
2. *T_f is a weakly compact operator*

$$\mathbb{Z} = \bigcap \left\{ T((\mathcal{O}(Y, \epsilon)) : \ Y \subset X, \ Y \ is \ finite, \ \epsilon > 0 \right\} = \{0\}.$$

Proof: The proof is of similar arguement of the [5, Proposition 3]. □

Henstock-Kurzweil-Pettis Integral property

Next, we give several conditions of Henstock-Kurzweil-Pettis integrability in X in the context of weak Baire measure and compact measure. We called a Banach space X has the μ-HKP integral property if each X-valued, scalarly μ-measurable function is μ-HKP integrable. If such a property holds true for all complete measure, then X is said to have the HKP-integral property. It is not hard to see a Banach space X has Henstock-Kurzweil-Pettis Integral property (HKPIP) if for every X-valued weakly bounded function f defined on an arbitrary finite measure space is Henstock-Kurzweil-Pettis integrable.

Theorem 3.1 *If X is μ-measure compact then X has the μ-HKPIP. In addition, if X is measure compact then X has HKPIP.*

Proof: Suppose X is μ-measure compact. Let $f:[0,1]\to X$ be bounded and scalarly measurable then f is weakly equivalent to Henstock-Kurzweil measurable function $g:[0,1]\to X$. Since g is bounded and measurable, it is Henstock-Kurzweil integrable. Let x_A be Henstock-Kurzweil integral.
In this case if $x_A=(HK)\int_A g d\mu$ then for every $x^*\in X^*$, $x^*f=x^*g$ a.e.. So,

$$\begin{aligned} x_A x^* &= (HK)\int_A g x^* d\mu \\ &= (HK)\int_A f x^* d\mu. \end{aligned}$$

Hence x_A is Henstock-Kurzweil-Pettis integral $(HKP)\int_A f d\mu$. □

Conditions of Mazur in Banach space X are not helpful for HKPIP.

Remark 3.1 The condition of Mazur in X implies Pettis Integral Property in X (see [3]). Ye Guoju et al. in [4] presented that Pettis integrals are subset of Henstock-Kurzweil-Pettis integral. In [11, Proposition 4], it is very clearly mention that Mazur property in X (particularly separable or weakly compactly generated) implies HKPIP if and only if $f:[0,1]\to X$ is scalarly Henstock-Kurzweil integrable and each infinite subset of $\{x^*f:\ ||x^*||\le 1\}$ contains an equi-Henstock-Kurzweil integrable sequences.

Theorem 3.2 *Let X be a Banach space. If the condition of Mazur in X, then HKPIP does not hold on X.*

4. Henstock-Kurzweil-Pettis integrals with separable range

In this Section, we discuss separable ranges of HKP-integrable function. We start this section with the known result.

Theorem 4.1 [1, Theorem 2] $f:[0,1]\to X$ be a function. If there exists a sequence of Henstock-Kurzweil-Pettis integrable functions $f_n:[0,1]\to X$ such that

1. The set $\{x^*f_n:\ ||x^*||\le 1,\ n\in\mathbb{N}\}$ is uniformly μ-integrable.
2. $\lim_n x^*f_n = x^*f$ in μ-measurable for each $x^*\in X^*$.

Then f is Henstock-Kurzweil-Pettis μ-integrable and

$$\lim_n (HK)\int_E f_n d\mu = (HK)\int_E f d\mu \ E\subseteq[0,1]$$

weakly in X for each $E\in\Sigma$.

Theorem 4.2 *If $f\in HKP([0,1],X)$ then the followings are equivalent:*

1. $\left\{x^*f:\ x^*\in B(X^*)\right\}$ *is separable subset of $HK([0,1])$.*
2. *There exists a sequence (f_n) of $X-$valued step function, such that for each $x^*\in X^*$ the sequence $(x^*f_n)_{n\in\mathbb{N}}$ is converges weakly to x^*f in $HK([0,1])$.*
3. *$\mu_f(\Sigma)$ is a separable subset of X.*

Proof: For (1) $\implies$ (2) : Let $\{x^*f:\ x^*\in B(X^*)\}$ be separable. In this situation there exists a sequence (x_n^*) in $B(X^*)$ such that $\{x_n^*f:\ n\in\mathbb{N}\}$ is dense in $\{x^*f:\ x^*\in B(X^*)\}$. Since $f\in HKP([0,1],X)$ then by [1, Theorem 2], the topology of convergence in measure and the weak topology of $HK[0,1]$ coincide on the set $\{x^*f:\ x^*\in B(X^*)\}$ and there exists a step function $f_n:[0,1]\to X$ such that $x^*f_n\to x^*f$ weakly in $HK[0,1]$ for every $x^*\in X^*$.

For (2) $\implies$ (3) : By condition (2), for each $E \in \Sigma$ the sequence $\mu_{f_n}(E)$ is weakly convergent to $\mu_f(E)$. Since ranges of μ_{f_n} are finite dimensional, $\mu_f(\Sigma)$ contained in the weak closure of the set $\bigcup_{n=1}^{\infty} \mu_{f_n}(\Sigma)$ as separable.

For (3) $\implies$ (1) : Let us use contradiction metheod here. If possible assume $\{x^*f \ : x^* \in B(X^*)\}$ is non-separable and $x_1^* \in U(X^*)$ where

$$U(X^*) = \left\{(x,s) : \ | < x^*, f(x) > | \geq s||x^*|| \right\} \ for \ x^* \in X^*$$

and $h \in BV(\mu)$ such that $< h, x_1^* f >= 1$. Let for an ordinal $b < x_1$, the family $\left\{(x_a^*, h_a) : \ \ a < b\right\}$ satisfies the followings:

a) $x_a^* \in U(X^*)$,

b) $h_a \in BV(\mu)$,

c) $x_c^* f \in \overline{\lim} \{x_a^* f : \ a < c\}$ for each $c < b$,

d)

$$< h_c, x_a^* f >= \begin{cases} 1 \ \ if \ a = c < b \\ 0 \ if \ a < c < b \end{cases}$$

As $\left\{x^*f \ : \ \ x^* \in B(X^*)\right\}$ is non-separable, so $x_b^* \in U(X^*)$ such that $x_b^* f \notin \overline{\lim} \{x_a^* f : \ a < b\}$. Now from Hahn-Banach Theorem for all $a < b$ such that $< h_b, x_b^* f >= 1$ and $< h_b, x_a^* f >= 1$ implies $h_b \in BV(\mu)$. From the net $\left\{(x_a^*, h_a) : \ \ a < x_1\right\}$ satisfying $(a) - (d)$ for all a, b, c less than x_1 implies $||T_f^*(h_b) - T_f^*(h_a)|| \geq 1$ whenever $a < b$. So, $T_f^*\left(BV(\mu)\right)$ is non-separable on X^{**}. Since $\{\chi_E : \ E \in \Sigma\}$ is norm dense in $BV(\mu)$ and so $\mu(\Sigma)$ is norm dense in $T_f^*\left(BV(\mu)\right)$. This is a contradiction. Hence $\mu_f(\Sigma)$ is separable. □

Amalgamate the Theorem 4.1 and the Theorem 4.2 we can find out the following conclusion.

Theorem 4.3 *Let $f : [0,1] \to X$ be a given function, μ is a perfect measure and $\mu_f(\Sigma)$ is a separable set. Then $f \in HKP([0,1], X)$ if and only if there exists a sequence (f_n) of X-valued step function such that*

1. *The family $\left\{x^* f_n : \ n \in \mathbb{N}, \ x^* \in B(X^*)\right\}$ is uniformly integrable.*
2. *For each $x^* \in X^*$, $\lim_n x^* f_n = x^* f$ a.e..*

With the analogous way, we can construct the following Lemma from Talagrand [12, 5-1-2], which was used in [9] by G. Plebanek in his work.

Lemma 4.1 *Let $([0,1], \Sigma, \mu)$ be a measure space and C be an absolutely convex set of measurable function which is τ_m-compact and bounded in $BV[0,1]$. Then the followings are equivalent:*

1. *$\mu : C \to \mathbb{R}, \ \mu(g) = (HK) \int_0^1 g d\mu$ is τ_m-continuous.*
2. *There exists a countable subsets $D \subset [0,1]$ such that $(HK) \int_0^1 g d\mu = 0$ whenever $g \in C$ and $g_{|_D} = 0$.*

Theorem 4.4 *If $f : [0,1] \to X$ is scalarly Henstock-Kurzweil integrable function then $f \in HKP([0,1], X)$ and $\mu_f(\Sigma)$ is separable if and only if the measure $f(\mu)$ is scalarly concentrated on a separable subspace of X.*

Proof: Let $f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable function and $\mu_f(\Sigma)$ is separable. Let Γ be separable subspace of $\mu_f(\Sigma)$. From the [11, Theorem 3] we get $x^* \in \Gamma^\perp$ and $x^*f = 0$ a.e. so, $x^* = 0$ $f(\mu)$-a.e..
Conversely, let $\nu = f(\mu)$ and Γ the closure of a countable set of X such that ν is scalarly concentrated on Γ. Next, $x^*_{|\Gamma} = 0$ implies $x^* = 0$ ν a.e.. From Lemma 4.1, $x^* \to (HK)\int_B x^* d\nu$ is τ_m-continuous for every $B \in Ba(X)$. Hence, $X^* \to HK(\nu)$ is τ_m-$(weak)$ continuous where τ_m is the topology of pointwise convergence and $i_X : X \to X$ is Henstock-Kurzweil-Pettis integrable with respect to ν. If $x_0 = (HK)\int_B i_X d\nu$, in this situation $x^*_{|\Gamma} = 0$ gives $x^*x_0 = 0$. Hence $x_0 \in \Gamma$, gives f is Henstock-Kurzweil-Pettis integrable and the range $\mu_f(\Sigma)$ is separable on Γ. □

We begin with a simple characterization of indefinite Henstock-Kurzweil-Pettis integrals that have separable ranges.

Lemma 4.2 *Suppose X is separable Banach spaces. Then $f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable on $[0,1]$ with indefinite integral F of f if the followings are equivalents:*

1. *F is $weak^*$-continuous on $[0,1]$.*
2. *The set $\left\{ \int_I f : \ I \subset [0,1] \right\}$ is separable.*

Proof: For (1) $\implies$ (2) : Let F be $weak^*$-continuous on $[0,1]$. Then $F([0,1])$ is $weak^*$ -compact. Therefore, $\left\{\int_I f : \ I \subset [0,1]\right\}$ is $weak^*$-compact and $weak^*$-topology of X^* restricted to $\left\{\int_I f : \ I \subset [0,1]\right\}$ is metrizable. Hence, $\left\{\int_I f : \ I \subset [0,1]\right\}$ is separable.

For (2) $\implies$ (1) : Let (f_n) be a sequence of step function on $[0,1]$ and F_n be the indefinite integral of f_n. By [1, Theorem 1], $\lim_n ||x^*F - x^*F_n||_n = 0$. since F_n is $weak^*$-continuous on $[0,1]$, then F is $weak^*$-continuous on $[0,1]$. □

Theorem 4.5 *Let $f : [0,1] \to X$ be scalarly Henstock-Kurzweil integrable function and X is a Mazur space then followings are hold:*

1. *If $f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable on $[0,1]$ then the indefinite integral of f is $weak^*$-continuous on $[0,1]$.*
2. *If $f : [0,1] \to X$ is Henstock-Kurzweil-Pettis integrable on $[0,1]$ then the set $\left\{ \int_I f : \ I \subset [0,1] \right\}$ is separable.*

Proof: For (1) : Let F be the indefinite integral of f. From [8, Theorem 1] x^*F is $weak^*$-continuous on $[0,1]$. Due to the Mazur property of X, we can find F is $weak^*$-continuous on $[0,1]$.

For (2) : It is direct implication of the Lemma4.2. □

Lemma 4.3 *Let $f : [0,1] \to X$ be Henstock-Kurzweil-Pettis integrable with X is separable and $T : X^* \to HK[0,1]$ be a compact. In this situation range $R_\mu\{F(I)\}$ is separable if and only if the set $\left\{x^*f : \ ||x^*|| \leq 1\right\}$ is separable subset of $HK[0,1]$.*

Proof: Let $T : X^* \to HK[0,1]$ defined by $T(x^*) = x^*f$ be compact. Then $T^* : BV[0,1] \to X^{**}$ shall be

$$
\begin{aligned}
< x^*, T^*_{\chi_I} > &=< T(x^*), \chi_I > \\
&= (HK)\int_I x^* f d\mu \\
&=< x^*, (HKP)\int_I f d\mu > \\
&=< x^*, F(I) > .
\end{aligned}
$$

So, $F(I) = T^*_{\chi_I}$. By Schauder Theorem $F(I)$ is compact. Let $X = \bigcup_n B(0,n)T(B(0,n))$ be separable. It is known that union of separable spaces is separable. Hence, $\bigcup_n T\left(B(0,n)\right) = R_\mu(F)$ is separable space. □

Theorem 4.6 *If X is separable Banach space and let $f : [0,1] \to X$ be Henstock-Kurzweil-Pettis integrable function. Assume $F : I \to X$ be its primitive then the followings are equivalents:*

1. *The family $\{x^* f_n : \ n \in \mathbb{N}, \ x^* \in B(X^*)\}$ is uniformly integrable.*
2. *For each $x^* \in X^*$ we have $\lim_n x^* f_n = x^* f$ a.e.*
3. *$R_\mu(F)$ is separable subset of X where $R_\mu(F)$ is the range spaces.*

Proof: For (1) $\implies$ (2) is (1) $\to$ (2) of the Theorem 4.2.

For (2) $\implies$ (3) : The condition (2) gives for each $E \in \Sigma$ the sequence $\left\{(HK)\int_E f_n d\mu : \ n \in \mathbb{N}\right\}$ is weakly convergent to $(HK)\int_E f d\mu$. Hence $R_\mu(F)$ is contained in the weak closure of the set $\bigcup_n F_n(\Sigma)$ where F_n is the indefinite Henstock-Kurzweil-Pettis integral of f_n. Each set $F_n(\Sigma)$ is of finite dimensional so the union is weakly separable. Since weak and norm separability coincide in a Banach space so by Mazur Theorem $R_\mu(F)$ is separable.

For (3) $\implies$ (1) is similar as (3) $\implies$ (1) of the Theorem 4.2. □

Lemma 4.4 *Let Y be a subspace of X and assume Y contains no isomorphic copy of c_0. If Y is WCG (has Mazur's property) then $f : [0,1] \to X$ a bounded scalarly Henstock-Kurzweil integral determined by Y is Henstock-Kurzweil-Pettis integrable.*

Proof: Let Y is WCG and f is scalarly Henstock-Kurzweil integrable, there exists a countable partition P_i of $[0,1]$ into measurable sets such that for each E in P_i, the function $f.\chi_E$ is weakly bounded. From [6, Lemma2], f is Pettis integrable and hence Henstock-Kurzweil-Pettis integral for all E in P_i. This gives for each $F \in \Sigma$, $(HK)\int_{F\cap E} f d\mu = (HKP)\int_{F\cap E} f d\mu \in X$. Now, for $x^* \in X^*$ and $F \in \Sigma$,

$$\begin{aligned}\sum_{E\in P_i} |x^*\left((HKP)\int_{F\cap E} f d\mu\right)| &= \sum_{E\in P_i} |(HK)\int_{F\cap E} x^* f d\mu| \\ &\leq \sum_{E\in P_i} (HK)\int_{F\cap E} |x^* f| d\mu \\ &= (HK)\int_F |x^* f| d\mu < \infty\end{aligned}$$

Now by Bessaga-Pelezynski characterization theorem, the series $\sum_{E\in P_i}(HKP)\int_{F\cap E} f d\mu$ is an unconditionally norm convergent series for all F in Σ. That is $\sum_{E\in P_i}(HKP)\int_{F\cap E} f d\mu = (HKP)\int_F f d\mu$. □

Proposition 4.1 *Let $f : [0,1] \to X$ be scalarly Henstock-Kurzweil integrable and $T_f : X^* \to HK([0,1])$ defined by $x^* \to x^* f$ is weakly compact. If f is determined by a subspace having Mazur's property, then f is Henstock-Kurzweil-Pettis integrable.*

Proof: Let f is determined by $Y \subset X$ having Mazur's property. This gives each *weak**-continuous functional on Y^* is in Y. Let (x_n) be dense in Y. If possible For each n, choose x^*_n in $\mathcal{O}\left(\{x_j\}_{j=1}^n, \epsilon\right)$ so that $x^*_n f = g$ a.e. for $g \in \bigcap_{(Y,\epsilon)} T\left(\mathcal{O}(Y,\epsilon)\right)$ if possible. We need to claim $g = 0$ a.e.. Let x^* be *weak** cluster of $(x^*_n)_n$ then $g = x^* f$ a.e.. From the Theorem 4.1, $\lim_n x^*_n f = x^* f = g$ a.e. The Mazur's property in Y gives $x^* f = g = 0$. □

References

1. B. Bongiorno, L.Di. Piazza, K. Musiał', *Approximation of Banach space valued Non-absolutely integrable functions by step functions*, Glasgow Math. J. 50, 583-593, (2008).
2. G. A. Edgar, Measurability in a Banach space, Indiana Univ. Math. J. 26, 663-677, (1977).
3. G.A. Edgar, *Measurability in a Banach space*, II, ibid. 28 (1979), 559-579.
4. Ye Guoju and AN. Tianqing, *On Henstock-Dunford and Henstock-Pettis integrals*, IJMMS 25:7, 467-478, (2001).
5. R. Huff, *Remarks on Pettis integrability, proceedings of the American mathematical society*, Volume 96, Number 3, March 1986.
6. H. Kalita, B. Hazarika, *Countable additivity of Henstock–Dunford integrable functions and Orlicz Space*, Analysis and Mathematical Physics, 11(2), 1-13, (2021).
7. K. Musiał', *Martingales of Pettis integrable functions, in Measure Theory*, Lecture Notes in Math. 794, Springer, 324-339, (1980).
8. K. M. Naralenkov, *On continuity and Compactness of some vector-valued integrals*, Rocky Mountain Journal of Mathematics, 43(3), 1-17, (2013).
9. G. Plebanek, *On Pettis integrals with separable range*, Colloq. Math. 64(1), 71-78, (1993).
10. L. D. Piazza, Kazimierz Musiał', *Henstock-Kurzweil-Pettis integrability of compact valued multifunctions with values in an arbitrary Banach space*, J. Math. Anal. Appl. 408, 452-464, (2013).
11. L. D. Piazza, Kazimierz Musiał', *Characterizations of Kurzweil-Henstock-Pettis integrable functions*, Studia Mathematics 176(2), 1-19, (2006).
12. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 307 (1984).
13. G. F. Stefansson, *Pettis Integrability*, Tran. AMS, 330(1), 1-19, (1992).

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