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The Littlewood-Paley g-function in quantum calculus*

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ABSTRACT: The purpose of this paper is to define and study, by the virtue of the q-Hardy-Littlewood maximal function $\mathcal{M}_q^\alpha(f)$ the so called L^p-boundedness of the q-Littlewood-Paley g-function when $p \in (1, 2]$.

Key Words: q-Poisson kernel, q-Littlewood-Paley g-function, q-Hardy-Littlewood maximal function.

Contents

1. Introduction

The theory of Littlewood-Paley was developed by Stein in his book [17], which remains the best reference in the study of this topic and has been an important impact in harmonic analysis. It plays an important role in the study of many functional spaces like the Hardy space, Lipshitz space, and BMO spaces. We point out that many authors have studied the Littlewood-Paley g-function, for instance one can cite [1,2,16,18].
The usual Littlewood-Paley g-function is defined in the n-dimensional Euclidean space $\mathbb{R}^n$ according to [17] by:

$$g(f)(x) := \Big(\int_0^\infty \Big|\nabla P^t f(x)\Big|^2 t dt\Big)^{\frac{1}{2}},$$

where $(P^t)_{t>0}$ is the usual Poisson semigroup defined by:

$$P^t f(x) := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{tf(y)}{(t^2+ \mid x-y \mid^2)^{\frac{n+1}{2}}} dy$$

and $\nabla := \Big(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\Big)$ is the gradient.

The well-known results is that the mapping $f \longmapsto g(f)$ is bounded from the Lebesgue space $L^p(\mathbb{R}^n, dx)$, $p \in (1, \infty)$ into itself.
The aim of this work is to define and study the g-function associated with the bessel operator using many intermediary results in "Quantum calculus" or q-analogs, where the parameter q is supposed to be a number from the interval $(0, 1)$. Our interest in this paper is to prove one of well-known results:

Main Theorem. For $p \in (1, 2]$, there exit two constants $A_{p,q,\alpha} > 0$ and $B_{p,q,\alpha} > 0$ such that for $f \in L^p(\mathbb{R}_{q,+}, d_q\mu_\alpha(x))$

$$B_{p,q,\alpha} \parallel f \parallel_{L^p(\mathbb{R}_{q,+}, d_q\mu_\alpha(x))} \leq \parallel g(f)(x; q^2) \parallel_{L^p(\mathbb{R}_{q,+}, d_q\mu_\alpha(x))} \leq A_{p,q,\alpha} \parallel f \parallel_{L^p(\mathbb{R}_{q,+}, d_q\mu_\alpha(x))} .$$

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This theorem will be proved in the last section of this paper. The techniques used are inspired in major part of the very interesting book of Stein [17].
This work is organized as follows. In the second section, we recall some q-harmonic analysis results associated with the bessel operator related to q-calculus. In the third section we will define and study the Poisson kernel and Poisson integral associated with the bessel operator and we present some technical lemmas that will be useful for the proof of the main result of this manuscript. The last section will be devoted, by the virtue of the q-Hardy-Littlewood maximal function $\mathcal{M}_q^\alpha(f)$ to study the so called L^p-boundedness of the q-Littlewood-Paley g-function when $p \in (1, 2]$.

2. Preliminaries

The aim of this section is to introduce some notions of functions theory in the q-calculus. For $a \in \mathbb{C}$, the q-shifted factorial $(a;q)_k$ is defined as a product of k factors

$$(a;q)_0 = 1, \ (a;q)_k = (1-a)(1-aq)...(1-aq^{k-1}), \quad k = 1, 2, ...$$

This definition remains meaningful for $k = \infty$ as a convergent infinite product

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also write $(a_1, ..., a_r; q)_k$ for the product of r q-shifted factorials

$$(a_1, \cdots, a_r; q)_k = (a_1;q)_k...(a_r;q)_k, \quad k = 1, 2, ..., \infty.$$

A q-hypergeometric series is a power series (for the moment still formal) in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1, ..., a_r$ and s complex lower parameters $b_1, ..., b_s$ as

$${}_r\phi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) := \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k (q;q)_k} [(-1)^k q^{\frac{k(k-1)}{2}}]^{1+s-r} x^k,$$

where $r, s = 1, 2, ...$
The q-derivative of a function f given on a subset of $\mathbb{R}$ or $\mathbb{C}$ is defined [9] by

$$D_{q,x} f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x, q \neq 0,$$

where x and qx should be in the domain of f. By continuity we set $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. For $k = 0, 1, 2, ...$

$$D_{q,x}^k f(x) = \frac{(-1)^k}{x^k(1-q)^k} \sum_{i=0}^{k} (-1)^i \frac{(q;q)_k}{(q;q)_i (q;q)_{k-i}} q^{-(k-i)(k-i-1)/2} f(q^{k-i}x). \tag{2.1}$$

Moreover, for all $n \in \mathbb{N}$,

$$D_{q,x}(f^n(x)) = \frac{f^n(x) - f^n(qx)}{f(x) - f(qx)} D_{q,x} f(x) = \Big[\sum_{k=0}^{n-1} f^k(x) f^{-k}(qx)\Big] f^{n-1}(qx) D_{q,x} f(x). \tag{2.2}$$

$$\begin{aligned} D_{q,x}[x^{2\alpha+1} D_{q,x}(f^n(x))] \ = \ & q \ x^{2\alpha+1} \Big[\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(x) f^{-i}(qx). \Big(D_{q,x} f(qx)\Big)\Big(D_{q,x} f(x)\Big) + q \sum_{k=0}^{n-1} \sum_{i=0}^{n-k-2} f^i(q^2x) f^{-i}(qx) \\ & \times \Big(D_{q,x} f(qx)\Big)^2\Big] f^{n-2}(qx) + \Big[\sum_{k=0}^{n-1} f^k(x) f^{-k}(qx)\Big] f^{n-1}(qx) D_{q,x}(x^{2\alpha+1} D_{q,x} f(x)). \end{aligned} \tag{2.3}$$

Note that when $q \uparrow 1^-$, (2.2) tend to $nf^{n-1}(x)f'(x)$.
We begin by putting

$$\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^k, k \in \mathbb{Z}\}, \quad [q]_n = \frac{1-q^n}{1-q}.$$

The q-Bessel operator is defined as follows

$$\Delta_{q,\alpha,x} f(x) := \frac{1}{x^{2\alpha+1}} D_{q,x}\Big[x^{2\alpha+1} D_{q,x} f\Big](q^{-1}x) = q^{2\alpha+1}\Delta_{q,x} f(x) + D_{q,x} f(q^{-1}x)$$

and $\Delta_{q,x} := \Lambda_{q,x}^{-1} D_{q,x}^2$, where the q-shift operators is $(\Lambda_{q,x}^{-1} f)(x) := f(q^{-1}x)$ and

$$\Delta_{q,\alpha} := \Delta_{q,\alpha,x} + \Delta_{q,t}, (x,t) \in \mathbb{R}_q \times \mathbb{R}_{q,+}. \tag{2.4}$$

For $a > 0$ and a function f given on $(0, a]$ we define the q-integral [9] by

$$\int_0^a f(x) d_q x := (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n.$$

The improper integral is defined in the following way

$$\int_0^{\infty} f(x) d_q x := (1-q) \sum_{k=-\infty}^{+\infty} f\left(q^k\right) q^k. \tag{2.5}$$

Note that for $n \in \mathbb{Z}$, we have

$$\int_0^{\infty} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{\infty} f(x) d_q x, \qquad \int_0^{a} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{aq^n} f(x) d_q x,$$

and if f and g are two suitable functions, the q-integration by parts is given by

$$\int_a^b D_{q,x} f(x) g(x) d_q x = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(qx) D_{qx} g(x) d_q x. \tag{2.6}$$

We denote by μ_α the measure on $\mathbb{R}_{q,+}$ given by

$$\mathrm{d}_q \mu_\alpha(y) = \frac{1}{1-q} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} y^{2\alpha+1} \mathrm{d}_q y = c_{q,\alpha} y^{2\alpha+1} \mathrm{d}_q y, \tag{2.7}$$

where the q-gamma function Γ_{q^2} (see [9,10], Section 1.3.) is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad q \in (0,1), \quad z \neq 0, -1, -2, ...$$

Let us now introduce some q-functional spaces which one will need in this work.

- $\mathcal{D}_{*,q}(\mathbb{R}_q)$ the space of even functions infinitely q-differentiable on $\mathbb{R}_q$ with compact support in $\mathbb{R}_q$. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.

- $\mathcal{C}_{*,q,0}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ continuous at 0, and satisfying

$$\lim_{x \to \infty} f(x) = 0 \text{ and } \parallel f \parallel_{\mathcal{C}_{*,q,0}} := \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty.$$

▷ $L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)$, $p \in [1, +\infty]$, the space of functions f such that $\| f \|_{L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)} < +\infty$, where

$$\| f \|_{L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)} := \left[\int_0^\infty |f(x)|^p \mathrm{d}_q\mu_\alpha(x) \right]^{\frac{1}{p}} < +\infty, \text{ for } p < \infty,$$

where $\mathrm{d}_q\mu(x)$ is given by (2.7) and

$$\| f \|_{L^\infty(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)} = \sup_{x \in \mathbb{R}_{q,+}} | f(x) | < +\infty.$$

Noting that

$$\| f \|_{L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)} = \sup_{\{h \in \mathcal{D}_{*,q}(\mathbb{R}_q); \|h\|_{L^m(\mathbb{R}_{q,+})} = 1\}} \left| \int_0^\infty f(x)h(x)\mathrm{d}_q\mu_\alpha \right|, \quad 1/p + 1/m = 1. \tag{2.8}$$

2.1. One-parameter family of q-exponential functions

The one-parameter family of q-exponential functions with $\beta \in \mathbb{R}$ has been considered in [8]

$$E_q^{(\beta)}(x) := \sum_{n=0}^\infty q^{\beta n^2/4} \frac{(1-q)^n}{(q;q)_n} x^n, \quad x \in \mathbb{R}. \tag{2.9}$$

Two particular cases of this family with $\beta = 0$ and $\beta = 1$ are well known: they are the q-exponential

$$e_q(x) = E_q^{(0)}(x) := \frac{1}{((1-q)x;q)_\infty} = \sum_{n=0}^\infty \frac{(1-q)^n}{(q;q)_n} x^n,$$

and its reciprocal

$$E_q(x) = e_q^{-1}(x) = E_q^{(1)}(-q^{-1/2}x) := (-(1-q)x;q)_\infty = \sum_{n=0}^\infty q^{n(n-1)/2} \frac{(1-q)^n}{(q;q)_n} x^n,$$

respectively. Another particular example of (2.9) corresponds to the value $\beta = \frac{1}{2}$ and is

$$\mathcal{E}_q(x) = E_q^{(1/2)}(x) := \sum_{n=0}^\infty q^{n^2/2} \frac{(1-q)^n}{(q;q)_n} x^n. \tag{2.10}$$

Note that

$$D_{q,x} E_q^{(\beta)}(x) = q^{\beta/2} E^\alpha(q^\beta x), \tag{2.11}$$

and

$$\lim_{n\to\infty} \mathcal{E}_q(q^{-n}) = \infty, \quad \lim_{n\to\infty} \mathcal{E}_q(-q^{-n}) = \lim_{n\to\infty} \mathcal{E}_q(q^n) = 0. \tag{2.12}$$

2.2. q-even translation and q-bessel Fourier transform

Let f be a function in $L^1(\mathbb{R}_{q,+}), \mathrm{d}_q\mu_\alpha(x)$, the q-even translation operators $T_{q,\alpha,x}$ [6] are defined by

$$T_{q,\alpha,x} f(y) := \int_0^\infty f(z) D_{q,\alpha}(x,y,z) \mathrm{d}_q\mu_\alpha(z),$$

where $D_{q,\alpha}(x,y,z)$ is defined for x and y in $\mathbb{R}_{q,+}$ by

$$D_{q,\alpha}(x,y,z) := \int_0^\infty j_\alpha(xt;q^2) j_\alpha(yt;q^2) j_\alpha(zt;q^2) \mathrm{d}_q\mu_\alpha(t),$$

and the q-Bessel function given in [11] as a series of functions

$$j_\alpha(x;q^2) := \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2};q^2)_n (q^2;q^2)_n} x^{2n} = \sum_{n=0}^\infty (-1)^n b_{n,\alpha}(x;q^2),$$

On $\mathbb{R}_q$, these functions are bounded and there they satisfy $\left|j_\alpha(x;q^2)\right| \leq \frac{(-q^2;q^2)^2_\infty(-q^{2\alpha+2};q^2)^2_\infty}{(q^{2\alpha+2};q^2)^2_\infty}$.

In [3], for f be an even function defined on $\mathbb{R}_q$ such that $D^k_{q,x}f(x)$ is continuous at 0 for all $k=1,2,...$, the authors proved that the q-even translation $T_{q,\alpha,x}$ can be written in the following form

$$T_{q,\alpha,y}f(x)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}y^{2n}\Delta^n_{q,\alpha,x}f(x). \tag{2.13}$$

where

$$\Delta^n_{q,\alpha,x}f(x)=\frac{q^{-n(n+1)}}{x^{2n}}\sum_{k=-n}^{n}(-1)^{k+n}\frac{q^{(k+n)(k+n+1)/2-2kn}(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}f(q^kx). \tag{2.14}$$

The q-bessel Fourier transform $\mathcal{F}_{q,\alpha}$ and the q-convolution product are defined for suitable functions f,g as follows

$$\mathcal{F}_{q,\alpha}(f)(\lambda)=\int_0^\infty f(t)j_\alpha(\lambda t;q^2)\mathrm{d}_q\mu_\alpha(t), \tag{2.15}$$

$$f*_{q,\alpha}g(x)=\int_0^\infty T_{q,\alpha,x}f(y)g(y)\mathrm{d}_q\mu_\alpha(y).$$

Note that, the q-translation operators and q-bessel Fourier transform satisfies the following properties

(P_1) *L^1-L^∞-boundedness* ([4], Proposition 5.1). For all $f\in L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$, $\mathcal{F}_{q,\alpha}(f)\in L^\infty(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$ and

$$\|\mathcal{F}_{q,\alpha}(f)\|_{L^\infty(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}\leq\frac{1}{1-q}\frac{(-q^{\cdot}q^2)_\infty(q^{2\alpha+1};q^2)_\infty}{(q;q^2)^2_\infty}\|f\|_{L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}.$$

(P_2) *Inversion theorem* ([11]). Let $f\in L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$, such that $\mathcal{F}_{q,\alpha}(f)\in L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$. Then

$$f(x)=\mathcal{F}_{q,\alpha}(\mathcal{F}_{q,\alpha}(f))(x),\qquad x\in\mathbb{R}_{q,+}.$$

(P_3) *Plancherel theorem* ([7], Theorem 7.7.). The q-bessel Fourier transform $\mathcal{F}_q$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$ onto itself. In particular,

$$\|\mathcal{F}_q(f)\|_{L^2(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha(x))}=\|f\|_{L^2(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha(x))}.$$

(P_4) *q-Gaussian function* ([5], Proposition 6.2). The function $G_\alpha(x,t;q^2)$ given by

$$G_\alpha(x,t;q^2):=\frac{1}{A_\alpha(t,q^2)}e_{q^2}(-\frac{x^2}{q(1+q)^2t})$$

where $A_\alpha(t,q^2)=q^{-\alpha-1}(1+q)^{2\alpha+2}\frac{(-1-q^2q^{2\alpha+2},-(1-q^2)q^{2\alpha},q^2;q^2)_\infty}{(-(1-q^2),-q^{2\alpha+2},-q^2(1-q^2)^{-1};q^2)_\infty t^{\alpha+1}}$, $t>0$, satisfies

$$\mathcal{F}_{q,\alpha}(G_\alpha(.,t;q^2))(\lambda)=e_{q^2}(-q^{-2\alpha-1}t\lambda^2).$$

(P_5) *q-Young condition* [3]. Let $p,q,r\in[1,\infty]$ satisfies $1/p+1/q-1/r=1$. Then the map $(f,g)\to f*_qg$ extends to a continuous map from $L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)\times L^q(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$ to $L^r(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$ and we have

$$\|f*_qg\|_{L^r(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}\leq\|f\|_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}\|g\|_{L^q(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}.$$

Specially, we need the positivity of the q-even translation operator [6] for proving the following inequality for $f\in L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$,

$$\|T_{q,\alpha,x}f\|_{L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}\leq\|f\|_{L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}.$$

This positivity property holds if $q\in(0,q_0]$, where q_0 is first zero of the function $q\mapsto{}_1\varphi_1(0;q;q,q)$.

3. q-Poisson kernel and q-Poisson integral

Definition 3.1 Consider the functions $P_t(x;q^2)$ called α-Poisson kernel given for $t\in\mathbb{R}_{q,+}$ by

$$P_t^\alpha(x;q^2) := d_{\alpha,q}\,\frac{(q^{2\alpha}x^2+t^2)}{t^{2\alpha}(x^2+t^2)(x^2+q^2t^2)W_{\alpha+1/2}(ix/t;q^2)},\quad \text{where}\quad d_{\alpha,q}=\frac{(q^2;q^2)_\infty}{q^{2\alpha}(1+q)^{\alpha+1}(q^{-2\alpha-2};q^2)_\infty}, \tag{3.1}$$

$i=\sqrt{-1}$ and $W_\alpha(x;q^2) := \dfrac{(x^2q^2;q^2)_\infty}{(x^2q^{2\alpha+1};q^2)_\infty}$, is the q-binomial function which tends to $(1-x^2)^{\alpha-1/2}$ when $q\uparrow 1^-$.

Proposition 3.1 *[14] For $x,t\in\mathbb{R}_{q,+}$, we have*

(i) $P_t^\alpha(x;q^2)=\displaystyle\int_0^{+\infty}\frac{E_{q^2}(-q^2u)}{u^{\alpha+1}}G_\alpha(x,\frac{t^2}{q(1+q)^2u};q^2)\,d_{q^2}\mu_\alpha(u).$

(ii) $\parallel G_\alpha(.,t;q^2)\parallel_{L^1(\mathbb{R}_{q,+},d_q\mu_\alpha)}=1.$

Furthermore, from (2.1), the Poisson kernel satisfies the following.

Lemma 3.1

(i) $D_{q,x}P_t(x;q^2)=d_{\alpha,q}\dfrac{(t^2+q^{2\alpha+2}x^2)(t^2+q^{2\alpha+3}x^2)(q^2t^2+x^2)-q^2(t^2+q^{2\alpha}x^2)(t^2+q^2x^2)^2}{(1-q)q^2xt^{2\alpha}(t^2+x^2)(q^2t^2+x^2)(t^2+q^2x^2)^2W_{\alpha+1/2}(ix/t;q^2)}$

(ii) $D_{q,t}P_t^\alpha(x;q^2)=d_{\alpha,q}\dfrac{(q^{2\alpha}-q^{2\alpha+2})x^4+(q^{2\alpha+4}-q^4)t^2x^2+(q^{2\alpha+6}-q^4)t^4}{(1-q)t^{2\alpha+1}(t^2+x^2)(q^2t^2+x^2)(q^4t^2+x^2)W_{\alpha+1/2}(ix/qt;q^2)}.$

(iii) For all $k\in\mathbb{N}$, $\|D_{q,t}^kP_t(.;q^2)\|_{L^\infty(\mathbb{R}_{q,+},d_q\mu_\alpha)}\le C_{\alpha,q}t^{-(2\alpha+k+2)}$

Proof. (i) and (ii) are obvious. (iii) follows from (2.1) and property (P_4). □

Definition 3.2 Let $f\in L^p(\mathbb{R}_{q,+})$, $p\in[1,+\infty]$, the α-Poisson integral of f denoted $u(f)(x,t;q^2)$ is defined by

$$u^\alpha(f)(x,t;q^2)=(P_t^\alpha(.;q^2)*_{q,\alpha}f)(x)=\int_0^\infty f(y)T_{q,\alpha,x}P_t^\alpha(y;q^2)\mathrm{d}_q\mu_\alpha(y). \tag{3.2}$$

Noting that from Proposition 3.1(i), $u^\alpha(f)(x,t;q^2)$ can be written us

$$u^\alpha(f)(x,t;q^2)=\int_0^{+\infty}\frac{E_{q^2}(-q^2u)}{u^{\alpha+1}}T_\alpha^{\frac{t^2}{u}}f(x)\mathrm{d}_{q^2}\mu_\alpha(u),$$

where

$$T_\alpha^tf(x)=G_\alpha(.,t/q(1+q)^2;q^2)*_{q,\alpha}f(x) \tag{3.3}$$

satisfying from property (P_5) and Proposition 3.1(ii)

$$\parallel T_\alpha^tf(x)\parallel_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}\le\parallel f\parallel_{L^p(\mathbb{R}_{q,+}),\mathrm{d}_q\mu_\alpha)}. \tag{3.4}$$

Lemma 3.2 *The q-bessel Fourier transform of the q-Poisson kernel $P_t^\alpha(x;q^2)$ is given by*

$$\mathcal{F}_q(P_t^\alpha(.;q^2))(\lambda)=d_{\alpha,q}\theta_\alpha\mathcal{E}_{q^2}(-q^{-2\alpha-2}\lambda t),$$

where $\mathcal{E}_{q^2}(x)$ is given by (2.10) and θ_α defined by

$$\theta_\alpha=(1-q)\sum_{m=-\infty}^{\infty}\frac{(1+q^{2\alpha+2m})q^{2m\alpha}}{(1+q^{-2m})(q^2+q^{2m})W_{\alpha+1/2}(iq^m)}.$$

Proof. Noting that from (2.5), $\theta_\alpha = \displaystyle\int_0^\infty \frac{1+q^{2\alpha+1}x^2}{(1+x^2)(q^2+x^2)W_{\alpha+1/2}(ix;q^2)} d_{q^2}\mu_\alpha(x)$. To prove the lemma, we need to prove first that the function $x \mapsto \theta_\alpha \mathcal{E}_{q^2}(-q^{-2\alpha-2}x)$ is the unique solution of the problem (P) given by

$$(P)\begin{cases} \Delta_{q,\alpha,x} u = u \\ \lim\limits_{n\to\infty} u(q^n x) = \theta_\alpha \\ \lim\limits_{n\to-\infty} u(q^n x) = 0. \end{cases}$$

In fact, by proceeding by the same way in [12,13]. For $f_1(x) = \mathcal{E}_{q^2}(-q^{-2\alpha-2}x)$ and $f_2(x) = \mathcal{E}_{q^2}(q^{-2\alpha-2}x)$, we can easily verified from (2.11) that they are solutions of the problem (P). Thus, any solution can be written in the form $u(x) = p_1(x)f_1(x) + p_2(x)f_2(x)$, where p_1 and p_2 are two periodic functions. Moreover, by (2.12) and the second initial condition, we obtain that $u(x) = p_1(x)\mathcal{E}_{q^2}(-q^{-2\alpha-2}x)$. So, by the first initial condition, we deduce that $p_1(x) = \theta_\alpha$. Now, applying (2.15) and replacing $P_t^\alpha(x;q^2)$ by its expansion (3.1), we have by the substitution $x = yt$

$$\begin{aligned} \mathcal{F}_{q,\alpha}(P_t^\alpha(.;q^2))(\lambda) &= d_{\alpha,q}\int_0^\infty \frac{(q^{2\alpha}x^2+t^2)}{t^{2\alpha}(x^2+t^2)(x^2+q^2t^2)W_{\alpha+1/2}(ix/t;q^2)} j_\alpha(\lambda x;q^2) d_q\mu_\alpha(x) \\ &= d_{\alpha,q}\int_0^\infty \frac{1+q^{2\alpha+1}x^2}{(1+x^2)(q^2+x^2)W_{\alpha+1/2}(ix;q^2)} j_\alpha(\lambda t y;q^2) d_q\mu_\alpha(y). \end{aligned}$$

The result follows from the fact that $\lambda t \mapsto \displaystyle\int_0^\infty \frac{1+q^{2\alpha+1}x^2}{(1+x^2)(q^2+x^2)W_{\alpha+1/2}(ix;q^2)} j_\alpha(\lambda t y;q^2) d_q\mu_\alpha(y)$ verifies the problem (P). $\square$

Proposition 3.2 *For all $f \in L^p(\mathbb{R}_{q,+})$, we have*

$$u^\alpha(f)(x,t;q^2) = \theta_\alpha \int_0^\infty \mathcal{E}_{q^2}(-q^{-2\alpha}\lambda t) j_\alpha(\lambda x;q^2)\mathcal{F}_q(f)(\lambda) d_q\mu(\lambda) = d_q\theta_\alpha \mathcal{F}_{\alpha,q}\Big(\mathcal{E}_{q^2}(-q^{-2\alpha}.t)\mathcal{F}_{\alpha,q}(f)(.)\Big)(x)$$

Proof. Applying the q-Bessel Fourier transform to both sides of the formula (3.2), we get

$$\mathcal{F}_{\alpha,q}(u^\alpha(f)(.,t;q^2))(\lambda) = \mathcal{F}_{\alpha,q}(P_t^\alpha(.;q^2))(\lambda)\mathcal{F}_{\alpha,q}(f)(\lambda).$$

So the result follows from property (P_2) and Lemma 3.2. $\square$

In the following, we give some technical lemmas concerning some properties of the Poisson integral $u(f)(x,t;q^2)$ and its q-derivatives.

Lemma 3.3 *Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ be a positive function and $p \in (1,\infty)$. Then*

(i) $u^\alpha(f)(x,t;q^2) \geq 0$.

(ii) $\Delta_{q,\alpha} u^\alpha(f)(x,t;q^2) = \Delta_{q,\alpha,x} u^\alpha(f)(x,t;q^2) + \Delta_{q,t} u^\alpha(f)(x,t;q^2) = 0$

(iii) For all $k \in \mathbb{N}$, there exist $C_q > 0$ such that $\left|D_{q,t}^k u^\alpha(f)(x,t;q^2)\right| \leq C_q t^{-(2\alpha+k+2)}$.

Proof. The lemma follows directly from property (P_5) and Lemma 3.1, (iii). $\square$

Lemma 3.4 *Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ be a positive function and $p \in (1,\infty)$. Then for x large, there exist respectively $C_{1,q}, C_{2,q} > 0$ such that*

(i) $u^\alpha(f)(x,t;q^2) \leq C_{1,\alpha,q}(t^2+x^2)^{-\alpha-1}$.

(ii) $\left|D_{q,x}u^{\alpha}(f)(x,t;q^2)\right| \leq C_{2,\alpha,q}(t^2+x^2)^{-\alpha-3/2}$.

Proof. (i) Since $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, there exist $a \in \mathbb{R}_{q,+}$, such that $\mathrm{supp}(f) \subset [0,a]$. Then from (2.13) and (2.14), we can write

$$\begin{aligned}
& T_{q,\alpha,y}P_t^{\alpha}(x;q^2) \\
= & \sum_{n=0}^{\infty} b_{n,\alpha}(y;q^2)\frac{q^{-n(n+1)}}{x^{2n}}\sum_{k=-n}^{n}(-1)^{k+n}\frac{q^{(k+n)(k+n+1)/2-2kn}(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}P_t^{\alpha}(q^kx;q^2) \\
= & d_{\alpha,q}\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(q^{2\alpha+2};q)_n}y^{2n}\sum_{k=-n}^{n}\frac{(-1)^{n-k}q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}}\frac{(q^{2(\alpha+k)}x^2+t^2)}{t^{2\alpha}(q^{2k}x^2+t^2)(q^{2k}x^2+q^2t^2)W_{\alpha+1/2}(iq^kx/t;q^2)} \\
= & d_{\alpha,q}\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(q^{2\alpha+2};q)_n}y^{2n}\Big[\sum_{k=1}^{n}\frac{(-1)^{n-k}q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}}\frac{(q^{2(\alpha+k)}x^2+t^2)}{t^{2\alpha}(q^{2k}x^2+t^2)(q^{2k}x^2+q^2t^2)W_{\alpha+1/2}(iq^kx/t;q^2)} \\
+ & \sum_{k=0}^{n}\frac{(-1)^{n+k}q^{(n+k)(n+k-1)/2}}{(q;q)_{n+k}(q;q)_{n-k}}\frac{(q^{2(\alpha-k)}x^2+t^2)}{t^{2\alpha}(q^{-2k}x^2+t^2)(q^{-2k}x^2+q^2t^2)W_{\alpha+1/2}(iq^{-k}x/t;q^2)}\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& T_{q,y}P_t^{\alpha}(x;q^2) \\
& \leq d_{\alpha,q}t^{-2\alpha}\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(q^{2\alpha+2};q)_n}y^{2n}\sum_{k=1}^{n}\left[\frac{1}{(q^{2k}x^2+t^2)W_{\alpha+1/2}(iq^kx/t;q^2)}+\frac{1}{(q^{-2k}x^2+t^2)W_{\alpha+1/2}(iq^{-k}x/t;q^2)}\right] \\
& \leq d_{\alpha,q}t^{-2\alpha}\sum_{n=0}^{\infty}q^ny^{2n}\sum_{k=1}^{n}\frac{[W_{\alpha+1/2}(iq^kx/t;q^2)+W_{\alpha+1/2}(iq^{-k}x/t;q^2)]t^2+q^{2k}[W_{\alpha+1/2}(iq^kx/t;q^2)+q^{-4k}W_{\alpha+1/2}(iq^{-k}x/t;q^2)]x^2}{(q^{2k}x^2+t^2)W_{\alpha+1/2}(iq^kx/t;q^2)(q^{-2k}x^2+t^2)W_{\alpha+1/2}(iq^{-k}x/t;q^2)} \\
& \leq d_{\alpha,q}t^{-2\alpha}\sum_{n=0}^{\infty}q^ny^{2n}\sum_{k=1}^{n}\frac{2W_{\alpha+1/2}(iq^{-k}x/t;q^2)t^2+2q^{2k}q^{-4k}W_{\alpha+1/2}(iq^{-k}x/t;q^2)x^2}{(q^{2k}x^2+t^2)W_{\alpha+1/2}(iq^kx/t;q^2)(q^{-2k}x^2+t^2)W_{\alpha+1/2}(iq^{-k}x/t;q^2)} \\
& \leq d_{\alpha,q}\sum_{n=0}^{\infty}q^ny^{2n}\sum_{k=1}^{n}\frac{2q^{-4k}}{(q^{-2k}x^2+t^2)W_{\alpha+1/2}(iq^{-k}x/t;q^2)} \\
& \leq \frac{2}{\mid q^4-1\mid\mid 1-qy^2\mid}(t^2+x^2)^{-\alpha-1},
\end{aligned}$$

by using the fact that $W_{\alpha+1/2}(iq^{-k}x/t;q^2) \geq t^{-2\alpha}(x^2+t^2)^{\alpha}$, we obtain

$$\begin{aligned}
u^{\alpha}(f)(x,t;q^2) &= (P_t(.;q^2) *_{q,\alpha} f)(x) = \int_0^a f(y)T_{q,\alpha,x}P_t^{\alpha}(y;q^2)\mathrm{d}_q\mu_{\alpha}(y) \\
&\leq \frac{2a}{\mid q^4-1\mid}\sup_{y\in[0,a]}\frac{\mid f(y)\mid}{\mid 1-qy^2\mid}(t^2+x^2)^{-\alpha-1} \\
&\leq C_{1,\alpha,q}(t^2+x^2)^{-\alpha-1}.
\end{aligned}$$

Thus the first inequality is proved.

Now, we will prove the second inequality. By derivation under the q-integral sign

$$D_{q,x}u^{\alpha}(f)(x,t;q^2) = D_{q,x}(P_t^{\alpha}(.;q^2) *_{q,\alpha} f)(x) = \int_0^a f(y)D_{q,x}T_{q,\alpha,x}P_t^{\alpha}(y;q^2)\mathrm{d}_q\mu_{\alpha}(y).$$

But from Lemma 3.1(i)

$$\begin{aligned}
&D_{q,x}T_{q,\alpha,x}P_t^{\alpha}(y;q^2)\\
&=\sum_{n=0}^{\infty} b_{n,\alpha}(y;q^2)D_{q,x}\Delta_{q,\alpha,x}^n(P_t^{\alpha}(y;q^2))\\
&=\sum_{n=0}^{\infty} b_{n,\alpha}(y;q^2)q^{-n(n+1)}\sum_{k=-n}^{n}(-1)^{k+n}\frac{q^{(k+n)(k+n+1)/2-2kn}(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}D_{q,x}(x^{-2n}P_t^{\alpha}(q^kx;q^2))\\
&=\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n(q;q)_{2n}}{(q^{2\alpha+2};q)_n}y^{2n}\sum_{k=-n}^{n}(-1)^{k+n}\frac{q^{(k+n)(k+n+1)/2-2kn}}{(q;q)_{n+k}(q;q)_{n-k}}\Big([-2n]_qx^{-2n-1}P_t^{\alpha}(q^kx;q^2)\\
&+q^{-2n}x^{-2n}D_{q,x}P_t^{\alpha}(q^kx;q^2)\Big)\\
&=\frac{d_{\alpha,q}}{(1-q)xt^{2\alpha}}\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n(q;q)_{2n}}{(q^{2\alpha+2};q)_n}\frac{y^{2n}}{x^{2n}}\sum_{k=-n}^{n}(-1)^{k+n}\frac{q^{(k+n)(k+n+1)/2-2kn-2n}}{(q;q)_{n+k}(q;q)_{n-k}}\mathbb{K}_n^{\alpha}(t,q^kx;q^2),
\end{aligned}$$

where

$$\mathbb{K}_n^{\alpha}(t,x;q^2)=\frac{(q^{2n}-1)(t^2+q^{2\alpha}x^2)(t^2+q^{2\alpha+2}x^2)-(t^2+q^{2\alpha+2}x^2)(t^2+q^{2\alpha+3}x^2)(q^2t^2+x^2)+q^2(t^2+q^{2\alpha}x^2)(t^2+q^2x^2)^2}{q^2(t^2+x^2)(q^2t^2+x^2)(t^2+q^2x^2)W_{\alpha+1/2}(ix/t;q^2)}.$$

Now by proceeding as the same way in the first inequality (i) and using the fact that there exist $c_{\alpha,q}>0$ such that $\left|\mathbb{K}_n^{\alpha}(t,q^kx;q^2)+\mathbb{K}_n^{\alpha}(t,q^{-k}x;q^2)\right|\le\dfrac{c_{\alpha,q}}{W_{\alpha+1/2}(iq^{-k}x/t;q^2)}$.
Therefore,

$$\begin{aligned}
\left|D_{q,x}T_{q,\alpha,x}P_t^{\alpha}(y;q^2)\right| &\le \frac{d_{\alpha,q}}{(1-q)\mid x\mid t^{2\alpha}}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(q;q)_{2n}}{(q^{2\alpha+2};q)_n}\frac{y^{2n}}{x^{2n}}\sum_{k=1}^{n}\frac{q^{-k(n+1)}}{(q;q)_{n-k}(q;q)_{n+k}}\left[\mathbb{K}_n^{\alpha}(t,q^{-k}x;q^2)+\mathbb{K}_n^{\alpha}(t,q^kx;q^2)\right]\\
&\le \frac{c_{\alpha,q}d_{\alpha,q}}{(1-q)\mid x\mid t^{2\alpha}}\sum_{n=0}^{\infty}q^n\frac{y^{2n}}{x^{2n}}\sum_{k=1}^{n}\frac{q^{-2k}}{W_{\alpha+1/2}(iq^{-k}x/t;q^2)}\\
&\le \frac{c_{\alpha,q}d_{\alpha,q}}{(1-q)\mid x\mid t^{2\alpha}}\sum_{n=0}^{\infty}q^n\frac{y^{2n}}{x^{2n}}\sum_{k=1}^{n}q^{-2k}t^{2\alpha}(x^2+t^2)^{-\alpha}\\
&\le \frac{c_{\alpha,q}d_{\alpha,q}}{(1-q)\mid x\mid}(t^2+x^2)^{-\alpha}\sum_{n=0}^{\infty}q^n(y/x)^{2n}\sum_{k=1}^{n}q^{-2k}\\
&\le \frac{q^{-2}c_{\alpha,q}d_{\alpha,q}}{\mid 1-q^{-2}\mid\mid 1-qy^2\mid}\frac{\mid x\mid}{(t^2+x^2)^{\alpha}}.
\end{aligned}$$

In the same manner, by the fact that $\mid x\mid\le(t^2+x^2)^{1/2}\le(t^2+x^2)^{3/2}$

$$\begin{aligned}
D_{q,x}u(f)(x,t;q^2) &\le \frac{q^{-2}c_{\alpha,q}d_{\alpha,q}}{\mid 1-q^{-2}\mid}\sup_{y\in[0,a]}\frac{\mid f(y)\mid}{\mid 1-qy^2\mid}(t^2+x^2)^{-\alpha-3/2}\\
&\le C_{2,q}(t^2+x^2)^{-\alpha-3/2},
\end{aligned}$$

which gives (ii). □

Lemma 3.5 *Let $f\in\mathcal{D}_{*,q}(\mathbb{R}_q)$ be a positive function and $p\in(1,\infty)$. Then*

(i) $\displaystyle\lim_{R\to\infty}\int_0^R\int_0^R\Delta_{q,t}((u^{\alpha}(f)(x,t))^p)td_qtd_q\mu_{\alpha}(x)=\int_0^{\infty}f^p(x)d_q\mu_{\alpha}(x).$

(ii) $\displaystyle\lim_{R\to\infty}\int_0^R\int_0^R\Delta_{q,\alpha,x}((u^{\alpha}(f)(x,t))^p)td_q\mu_{\alpha}(x)d_qt=0.$

(iii) $$\int_0^\infty \int_0^\infty \Delta_{q,\alpha}((u^\alpha(f)(x,t))^p) t d_q t d_q\mu_\alpha(x) = \| f \|_{L^p(\mathbb{R}_{q,+}, d_q\mu_\alpha)} .$$

Proof. Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ be an even positive function and $p \in (1,\infty)$. To prove (i), from (2.2) and q-integration by part formula (2.6), we get

$$\begin{aligned}
\int_0^R \Delta_{q,t}((u^\alpha(f)(x,t))^p) t d_q t &= \int_0^R D^2_{q,t}((u^\alpha(f)(x,q^{-1}t))^p) t d_q t \\
&= \left[t D_{q,t}((u^\alpha(f)(x,q^{-1}t))^p)\right]_0^R - \int_0^R D_{q,t}((u^\alpha(f)(x,q^{-1}t))^p) d_q t \\
&= R\Big(\sum_{k=0}^{p-1}((u^\alpha(f)(x,q^{-1}R))^k)(u^\alpha(f)(x,R))^{p-k-1}\Big) D_{q,t} u^\alpha(f)(x,q^{-1}R) \\
&- (u^\alpha(f)(x,q^{-1}R))^p + f^p(x).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^R \int_0^R \Delta_{q,t}((u^\alpha(f)(x,t))^p) t d_q t \mathrm{d}_q\mu(x) &= R \int_0^R \Big(\sum_{k=0}^{p-1}((u^\alpha(f)(x,q^{-1}R))^k)(u^\alpha(f)(x,R))^{p-k-1}\Big) D_{q,t} u^\alpha(f)(x,q^{-1}R) \mathrm{d}_q\mu_\alpha(x) \\
&- \int_0^R (u^\alpha(f)(x,q^{-1}R))^p \mathrm{d}_q\mu(x) + \int_0^R f^p(x) \mathrm{d}_q\mu_\alpha(x).
\end{aligned}$$

From Lemma 3.3(iii), we get easily respectively

$$\Big| \int_0^R (u^\alpha(f)(x,q^{-1}R))^p \mathrm{d}_q\mu(x) \Big| \leq C_{1,q} R^{-p(2\alpha+2)} R^{2\alpha+2} = C_{1,q} R^{-(p-1)(2\alpha+2)} \underset{R\to\infty}{\longrightarrow} 0,$$

and

$$\Big| R \int_0^R \Big(\sum_{k=0}^{p-1}((u^\alpha(f)(x,q^{-1}R))^k)(u^\alpha(f)(x,R))^{p-k-1}\Big) D_{q,t} u^\alpha(f)(x,q^{-1}R) \mathrm{d}_q\mu_\alpha(x) \Big| \leq$$

$$C_{2,q} R^{2\alpha+2} \Big(\sum_{k=0}^{p-1} R^{-k(2\alpha+2)} R^{-(p-k-1)(2\alpha+2)}\Big) R^{-(2\alpha+3)} = p C_{2,q} R^{-(p-1)(2\alpha+2)} \underset{R\to\infty}{\longrightarrow} 0 .$$

Which leads to the result.

(ii) From the fact that

$$\begin{aligned}
\int_0^R \Delta_{q,\alpha,x}((u^\alpha(f)(x,t))^p) \mathrm{d}_q\mu_\alpha(x) &= \int_0^R D_{q,\alpha,x}(x^{2\alpha+1} D_{q,\alpha,x}(u^\alpha(f)(q^{-1}x,t))^p) d_q x = \left[x^{2\alpha+1} D_{q,\alpha,x}((u^\alpha(f)(q^{-1}x,t))^p)\right]_0^R \\
&= R^{2\alpha+1} \Big(\sum_{k=0}^{p-1}(u^\alpha(f)(q^{-1}R,t))^k(f)(u^\alpha(f)(q^{-1}R,t))^{p-k-1}\Big) D_{q,\alpha,x} u^\alpha(f)(q^{-1}R,t) \\
&- \Big(\sum_{k=0}^{p-1} u^\alpha(f)(0,t))^k(f) u^\alpha(f)(q^{-1}R,t))^{p-k-1}\Big) D_{q,\alpha,x} u^\alpha(f)(0,t).
\end{aligned}$$

Thus, from Lemma 3.4 we get

$$\Big|\int_0^R\int_0^R \Delta_{q,x}(u^\alpha(f)(x,t))t\mathrm{d}_q\mu(x)d_qt\Big| \leq R^{2\alpha+1}\int_0^R\Big|\Big(\sum_{k=0}^{p-1}(u^\alpha(f)(q^{-1}R,t))^k(f)(u^\alpha(f)(q^{-1}R,t))^{p-k-1}\Big)D_{q,\alpha,x}u^\alpha(f)(q^{-1}R,t)$$
$$-\Big(\sum_{k=0}^{p-1}u^\alpha(f)(0,t))^k(f)u^\alpha(f)(q^{-1}R,t))^{p-k-1}\Big)D_{q,\alpha,x}u^\alpha(f)(0,t)\Big|td_qt$$
$$\leq 2pC_{2,q}R^{2\alpha+1}\int_0^R(t^2+R^2)^{-\alpha(p+1)-1/2}td_qt$$
$$\leq 2pC_{2,q}R^{-2p(\alpha+1)+2\alpha}\int_0^R td_qt$$
$$= C_qR^{-(p-1)(2\alpha+2)} \underset{R\to\infty}{\longrightarrow} 0\,.$$

Which completes the proof of (ii). □

4. The q-Littlewood-Paley g-function

In this section, we define and study the so called L^p-boundedness of the q-Littlewood-Paley g-function when $p\in(1,2]$. For this, we need first use the q-Hardy-Littlewood maximal $\mathcal{M}_q^\alpha(f)$ function.

Definition 4.1 Let $f\in\mathcal{D}_{*,q}(\mathbb{R}_q)$. the q-Hardy-Littlewood maximal $\mathcal{M}_q^\alpha(f)$ function is defined by

$$\mathcal{M}_q^\alpha(f)(x) := \sup_{t\in\mathbb{R}_{q,+}} \mid u^\alpha(f)(x,t)\mid,\quad x\in\mathbb{R}_q.$$

Proposition 4.1 *Let $f\in\mathcal{D}_{*,q}(\mathbb{R}_q)$ and $p\in(1,\infty)$. Then there exist $C_{p,q}>0$ such that*

$$\parallel \mathcal{M}_q^\alpha(f)(x)\parallel_{L^p(\mathbb{R}_{q,+},d_q\mu_\alpha)}\leq C_{p,q}\parallel f\parallel_{L^p(\mathbb{R}_{q,+},d_q\mu_\alpha)}.$$

Proof. By changing variable $y=t^2/u$ in Proposition 3.1(i), we obtain

$$P_t^\alpha(x;q^2) = t^{-2}\int_0^{+\infty}\phi(y/t^2)T_\alpha^y f(x)\mathrm{d}_{q^2}\mu_\alpha(y). \tag{4.1}$$

where $\phi(y)=y^{-3/2}E_{q^2}(-q^2y^{-1})$ and $T_\alpha^y f(x)$ is given by (3.3).
We verify easily that $\phi(y)$ and $yD_{q,y}\phi(y)$ belong to $L^1(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)$. Then by q-integration by part in (4.1) and relation (3.4) shows that

$$\begin{aligned}
\mid P_t(x;q^2)\mid &= \Big|-\int_0^\infty\Big(\int_0^y T^tf(x)d_qt\Big)D_{q,\alpha,y}\phi(qy/t^2)\mathrm{d}_q\mu_\alpha(y)\Big| \\
&= \Big|\int_0^\infty\Big(1/y\int_0^y T_\alpha^tf(x)d_qt\Big).\Big(yD_{q,\alpha,y}\phi(qy/t^2)\Big)\mathrm{d}_q\mu_\alpha(y)\Big| \\
&\leq \sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T_\alpha^tf(x)d_qt\Big|\Big(t^{-2}\int_0^\infty\Big|yD_{q,\alpha,y}\phi(qy/t^2)\Big|\Big)\mathrm{d}_q\mu_\alpha(y)\mid \\
&\leq C_q\sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T_\alpha^tf(x)d_qt\Big| \\
&\leq C_q\mathcal{M}_q^{T,\alpha}(f)(x),
\end{aligned}$$

where $\mathcal{M}_q^{T,\alpha}(f)(x)=\sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T_\alpha^tf(x)d_qt\Big|$.
So that

$$\mathcal{M}_q^\alpha(f)(x)\leq\mathcal{M}_q^{T,\alpha}(f)(x).$$

The result follows by the Hopf-Dunford-Schwartz ergodic theorem [7, Theorem 7, p.693]. □

Definition 4.2 The q-Littlewood-Paley g-function for $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ is given by

$$g^{\alpha}(f)(x;q^2) := \Big(\int_0^{\infty} \Big|\nabla_q u^{\alpha}(f)(x,t;q^2)\Big|^2 t d_q t\Big)^{1/2},$$

where $u(f)(x,t)$ is the q-Poisson integral and ∇_q is the q-gradient, defined by

$$\Big|\nabla_q u^{\alpha}(f)(x,t;q^2)\Big|^2 := (D_{q,x}u^{\alpha}(f)(x,t;q^2))^2 + (D_{q,t}u^{\alpha}(f)(x,\alpha,t;q^2))^2$$

Theorem 4.1 *For $p \in (1,2]$, there exit two constants $A_{p,q,\alpha} > 0$ and $B_{p,q,\alpha} > 0$ such that for $f \in L^p(\mathbb{R}_{q,+}, d_q\mu_{\alpha})$*

$$B_{p,q,\alpha} \parallel f \parallel_{L^p(\mathbb{R}_{q,+},d_q\mu_{\alpha})} \leq \parallel g^{\alpha}(f)(x;q^2) \parallel_{L^p(\mathbb{R}_{q,+})} \leq A_{p,q,\alpha} \parallel f \parallel_{L^p(\mathbb{R}_{q,+},d_q\mu_{\alpha})}.$$

We will go to prove the theorem. For this purpose, we will use the function $g_1^{\alpha}(f)$ given in [15] by

$$g_1^{\alpha}f(x;q^2) := \left(\int_0^{\infty} t\Big|D_{q,t}u^{\alpha}(f)(x,t;q^2)\Big|^2 d_q t\right)^{1/2}, \quad f \in L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_{\alpha}).$$

Obviously, we have

$$g_1^{\alpha}(f)(x;q^2) \leq g^{\alpha}(f)(x;q^2). \tag{4.2}$$

To prove the theorem, we need the following lemma

Lemma 4.1 *For $f_1, f_2 \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, there exist $A_{\alpha,q} > 0$, such that*

$$\int_0^{\infty}\int_0^{\infty} tD_{q,t}u^{\alpha}(f_1)(x,t;q^2)D_{q,t}u^{\alpha}(f_2)(x,t;q^2)d_q t d_q\mu_{\alpha}(x) = A_{\alpha,q}\int_0^{\infty} f_1(x)f_2(x)d_q\mu_{\alpha}(x).$$

Proof. Let $f_1, f_2 \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, we have by applying twice Holder's inequality

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty} t\Big|D_{q,t}u^{\alpha}(f_1)(x,t;q^2)\Big|\Big|D_{q,t}u^{\alpha}(f_2)(x,t;q^2)\Big|d_q t d_q\mu_{\alpha}(x) &\leq \int_0^{\infty} g^{\alpha}(f_1)(x;q^2)g^{\alpha}(f_2)(x;q^2)d_q\mu_{\alpha}(x)\\ &\leq \parallel g^{\alpha}(f_1) \parallel_{L^2(\mathbb{R}_{q,+},\mathrm{d}_q\mu_{\alpha})}\parallel g^{\alpha}(f_2) \parallel_{L^2(\mathbb{R}_{q,+},\mathrm{d}_q\mu_{\alpha})}.\end{aligned}$$

Then using Fubini, property (P_3) and Proposition 3.2, we obtain

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty} tD_{q,t}u^{\alpha}(f_1)(x,t;q^2)\overline{D_{q,t}u^{\alpha}(f_2)(x,t;q^2)}d_q t\mathrm{d}_q\mu_{\alpha}(x)\\
=\;& \int_0^{\infty}\int_0^{\infty} t\mathcal{F}_{q,\alpha}\Big(D_{q,t}u^{\alpha}(f_1)(.,t;q^2)\Big)(\lambda)\overline{\mathcal{F}_{q,\alpha}\Big(D_{q,t}u^{\alpha}(f_2)(.,t;q^2)\Big)}(\lambda)\mathrm{d}_q\mu_{\alpha}(\lambda)d_q t\\
=\;& \int_0^{\infty}\int_0^{\infty} tq^{-4\alpha}\lambda^2\mathcal{E}_{q^2}^2(-q^{2\alpha}\lambda t)\mathcal{F}_{q,\alpha}(f_1)(\lambda)\overline{\mathcal{F}_{q,\alpha}(f_2)(\lambda)}d_q t\mathrm{d}_q\mu_{\alpha}(\lambda)\\
=\;& q^{-4\alpha}\int_0^{\infty} \lambda^2\mathcal{F}_{q,\alpha}(f_1)(\lambda)\overline{\mathcal{F}_{q,\alpha}(f_2)(\lambda)}\Big(\int_0^{\infty} t\mathcal{E}_{q^2}^2(-q^{2\alpha}\lambda t)d_q t\Big)\mathrm{d}_q\mu_{\alpha}(\lambda)\\
=\;& \int_0^{\infty} \mathcal{F}_{q,\alpha}(f_1)(\lambda)\overline{\mathcal{F}_{q,\alpha}(f_2)(\lambda)}\Big(\int_0^{\infty} u\;\mathcal{E}_{q^2}^2(-u)d_q u\Big)\mathrm{d}_q\mu(\lambda), \quad u = q^{-2\alpha}\lambda t\\
=\;& A_{\alpha,q}\int_0^{\infty} \mathcal{F}_{q,\alpha}(f_1)(\lambda)\overline{\mathcal{F}_{q,\alpha}(f_2)(\lambda)}\mathrm{d}_q\mu_{\alpha}(\lambda), \quad A_{\alpha,q} = \int_0^{\infty} u\;\mathcal{E}_{q^2}^2(-u)d_q u\\
=\;& A_{\alpha,q}\int_0^{\infty} f_1(x)\overline{f_2(x)}\mathrm{d}_q\mu(x).
\end{aligned}$$

$\square$

Proof of the Theorem 4.1. Let $p \in (1,2]$, it is clear that from the density of $\mathcal{D}_{*,q}(\mathbb{R}_q)$ in $L^p(\mathbb{R}_{q,+}, \mathrm{d}_q\mu_\alpha)$ ([7], Theorem 4.28), taken $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$. Note that from (2.3), we have

$$\Delta_{q,\alpha,x}(f^n(x)) = q\Big[\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(q^{-1}x)f^{-i}(x).\Big(D_{q,x}f(x)\Big)\Big(D_{q,x}f(q^{-1}x)\Big)\Big) + q\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-2} f^i(qx)f^{-i}(x).\Big(D_{q,x}f(x)\Big)^2\Big]$$
$$\times f^{n-2}(x) + \Big[\sum_{k=0}^{n-1} f^k(q^{-1}x)f^{-k}(x)\Big]f^{n-1}(x)\Delta_{q,\alpha,x}f(x).$$

Thus, using Lemma 3.3(ii) we obtain

$$\Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2)) = q^2\Big[q^{-1}\sum_{k=0}^{p-1}\sum_{i=0}^{k-1} u^\alpha(f)^i(q^{-1}x,t;q^2)f^{-i}(x,t;q^2).\Big(D_{q,x}u^\alpha(f)(x,t;q^2).D_{q,x}u^\alpha(f)(q^{-1}x,t;q^2)+$$

$$D_{q,t}u^\alpha(f)(x,t;q^2)D_{q,t}u^\alpha(f)(x,q^{-1}t;q^2)\Big) + \sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u^\alpha(f)^i(qx,t;q^2)u^\alpha(f)^{-i}(x,t;q^2).\Big(\nabla_q u^\alpha(f)(x,t;q^2)\Big)^2\Big]u^\alpha(f)^{p-2}(x,t;q^2).$$

So that from Lemma 3.3(i)

$$\Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2)) \geq q^2\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u^\alpha(f)^i(qx,t;q^2)u^\alpha(f)^{-i}(x,t;q^2)\Big]u^\alpha(f)^{p-2}(x,t;q^2)\Big(\nabla_{q,x}u^\alpha(f)(x,t;q^2)\Big)^2,$$

we deduce that

$$\Big(\nabla_q u^\alpha(f)(x,t;q^2)\Big)^2 \leq q^{-2}\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u^\alpha(f)^i(qx,t;q^2)u^\alpha(f)^{-i}(x,t;q^2)\Big]^{-1}u^\alpha(f)^{2-p}(x,t;q^2)\Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2)).$$

Now, from Lemma 3.4(i) and the fact that f is positive, and the function $x \to u^\alpha(f)(x,t;q^2)$ is decreasing we deduce that

$$\begin{aligned}\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u^\alpha(f)^i(qx,t;q^2)u^\alpha(f)^{-i}(x,t;q^2)\Big]^{-1} &\leq \Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u^\alpha(f)^i(qx,t;q^2)u^\alpha(f)^{-i}(x,t;q^2)\Big]^{-1}\\ &\leq \Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} 1\Big]^{-1} = \frac{2}{p(p-1)}.\end{aligned}$$

we deduce that

$$\Big|\nabla_{q,x}u^\alpha(f)(x,t;q^2)\Big|^2 \leq \frac{2q^{-2}}{p(p-1)}u^\alpha(f)^{2-p}(x,t;q^2)\Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2)). \tag{4.3}$$

Using (4.3), we obtain

$$\begin{aligned}\Big[g^\alpha(f)(x;q^2)\Big]^2 &\leq \frac{2q^{-2}}{p(p-1)}\int_0^\infty u^\alpha(f)^{2-p}(x,t;q^2)\Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2))td_qt\\ &\leq \frac{2q^{-2}}{p(p-1)}\mathcal{M}_q^\alpha(f)^{2-p}(x)\int_0^\infty \Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2))td_qt\end{aligned}$$

Thus observe that $(2-p)/2 + p/2 = 1$ and by Holder's inequality, we get

$$\begin{aligned}\| g^\alpha(f)(x;q^2) \|^p_{L^p(\mathbb{R}_{q,+})} &\leq \Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2}\int_0^\infty \mathcal{M}_q^\alpha(f)^{(2-p)(p/2)}(x)\Big[\int_0^\infty \Delta_q(u^\alpha(f)^p(x,t;q^2))td_qt\Big]^{p/2}\mathrm{d}_q\mu_\alpha(x)\\ &\leq \Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2} \| \mathcal{M}_q^\alpha(f) \|^{(2-p)(p/2)}_{L^p(\mathbb{R}_{q,+})}\ \Big[\int_0^\infty \Delta_{q,\alpha}(u^\alpha(f)^p(x,t;q^2))td_qt\mathrm{d}_q\mu_\alpha(x)\Big]^{p/2}\end{aligned}$$

Due to Lemma 3.4(iii), we obtain

$$\begin{aligned}
\| g^{\alpha}(f)(x;q^2) \|^{p}_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} &\leq \Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2} \| \mathcal{M}_q^{\alpha}(f) \|^{(2-p)(p/2)}_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} \| f \|^{p^2/2}_{L^p(\mathbb{R}_{q,+})} \\
&\leq C_{1,q}\Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2} \| f \|^{(2-p)(p/2)}_{L^p(\mathbb{R}_{q,+})} \| f \|^{p^2/2}_{L^p(\mathbb{R}_{q,+})} \\
&\leq A_{p,q,\alpha} \| f \|^{p}_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} .
\end{aligned}$$

Proving now the left inequality. Computing relations (2.8), (4.2), Lemma 4.1 and Holder inequality, we have

$$\begin{aligned}
\frac{1}{A_{q,\alpha}}\Big| \int_0^{\infty} f(x)h(x)\mathrm{d}_q\mu(x)\Big| &\leq \int_0^{\infty} g_1^{\alpha}(f)(x;q^2)g_1^{\alpha}(h)(x;q^2)\mathrm{d}_q\mu(x) \\
&\leq \| g_1^{\alpha}(f) \|_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} \| g_1^{\alpha}(h) \|_{L^m(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)}, \quad 1/p+1/m=1 \\
&\leq C_{q,m,\alpha} \| g_1^{\alpha}(f) \|_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} \\
&\leq C_{q,m,\alpha} \| g^{\alpha}(f) \|_{L^p(\mathbb{R}_{q,+},\mathrm{d}_q\mu_\alpha)} .
\end{aligned}$$

Which completes the proof of the theorem by taking the supremum. □

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