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Ergodicity for a Family of Operators

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ABSTRACT: The aim of this paper is to introduce the notions of power boundedness, Cesàro boundedness, mean ergodicity, and uniform ergodicity for a family of bounded linear operators on a Banach space. The authors present some elementary results in this setting and show that some main results about power bounded, Cesàro bounded, mean ergodic, and the uniform ergodic operator can be extended from the case of a linear bounded operator to the case of a family of bounded linear operators acting on a Banach space. Also, we show that the Yosida theorem can be extended from the case of a bounded linear operator to the case of a family of bounded linear operators acting on a Banach space.

Key Words: Power boundedness, Cesàro boundedness, ergodic theorem, family of operators, Banach space.

Contents

1. Introduction

Let T be a bounded linear operator, on a complex Banach space $\mathcal{X}$. The ergodicity for T was already developed in different directions, (see, e.g. [2,3,4,6,7,8,9,12]). For example, in [3], it was shown that when $\frac{1}{n}\|T^n\| \longrightarrow 0$, $n \to \infty$, T is uniformly ergodic if and only if $(I-T)^2\mathcal{X}$ is closed. In [9], Lin showed that if $\frac{1}{n}\|T^n\| \longrightarrow 0$, $n \to \infty$, T is uniformly ergodic if and only if $(I-T)\mathcal{X}$ is closed. Hence $(I-T)^k\mathcal{X}$ is closed for each integer $k \geq 1$. In [1], the authors extended this result to the case of a family of bounded linear operators. It is well known that the Cesàro boundedness of the operator T, as well as the condition, $\lim_{n\longrightarrow\infty} \frac{1}{n}\|T^n x\| = 0$, for every $x \in \mathcal{X}$, which are, in general, independent; see [5, Remark 4], are necessary for the mean ergodicity of T. In [13], Yosida has established the following theorem for an operator acting on a locally convex space.

Theorem 1.1. *Let $\mathcal{X}$ be a locally convex linear topological space, and T a continuous linear operator on $\mathcal{X}$ into $\mathcal{X}$. We assume that the family of operators $\{T^n : n = 1, 2, \ldots\}$ is equi-continuous in the sense that, for any continuous semi-norm q on $\mathcal{X}$, there exists a continuous semi-norm q' on $\mathcal{X}$ such that $\sup_{n\geq 1} q(T^n x) \leq q'(x)$ for all $x \in \mathcal{X}$. Then the closure $R(I-T)^a$ of the range $R(I-T)$ satisfies*

$$R(I-T)^a = \left\{ x \in \mathcal{X} \; : \; \lim_{n\longrightarrow\infty} M_n(T)x = 0, \; M_n(T) = \frac{1}{n}\sum_{m=1}^{n} T^m \right\}.$$

In particular,

$$R(I-T)^a \cap \ker(I-T) = \{0\}.$$

In this paper we introduce the notions of power boundedness, Cesàro boundedness, mean ergodicity, and uniform ergodicity for a family of bounded linear operators from the Banach algebra $C_b\left((0,1],\ \mathcal{B}(\mathcal{X})\right)$ (respectively from $\mathcal{B}_\infty$), this class was introduced and studied by S. Macovei in [10,11]. The notion of uniform ergodicity was introduced in [1], see below for the definitions. We deal with giving relations between these definitions. In Theorem 3.10 we show that the restriction of a mean (respectively uniformly) ergodic family of bounded linear operators on an invariant subspace is also mean (respectively uniformly)

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ergodic. Also, we extend the Theorem 1.1 for a family of bounded linear operators acting on a Banach space, see Theorem 3.13 below.

Papers dedicated to the study of the class of a family of bounded linear operators acting on a Banach space have been elaborated in [10,11].

2. Preliminaries

Let $\mathcal{X}$ be an infinite-dimensional Banach space and $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operators on $\mathcal{X}$. We denote by I the identity operator on $\mathcal{X}$.

In [10], Macovei showed that the set

$$C_b\left((0,1],\ \mathcal{B}(\mathcal{X})\right) = \left\{ \{T_h\}_{h\in(0,1]} \subset \mathcal{B}(\mathcal{X}) \ :\ \{T_h\}_{h\in(0,1]} \text{ is a bounded family, i.e. } \sup_{h\in(0,1]} \|T_h\| < \infty \right\},$$

is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = \sup_{h\in(0,1]} \|T_h\|.$$

And

$$C_0\left((0,1],\ \mathcal{B}(\mathcal{X})\right) = \left\{ \{T_h\}_{h\in(0,1]} \in C_b\left((0,1],\ \mathcal{B}(\mathcal{X})\right) \ :\ \lim_{h\to 0} \|T_h\| = 0 \right\},$$

is a closed bilateral ideal of $C_b\left((0,1],\ \mathcal{B}(\mathcal{X})\right)$. The quotient algebra $C_b\left((0,1],\ \mathcal{B}(\mathcal{X})\right)/C_0\left((0,1],\ \mathcal{B}(\mathcal{X})\right)$, which will be denoted $\mathcal{B}_\infty$, is also a Banach algebra with quotient norm

$$\left\|\{\dot{T_h}\}\right\| = \inf_{\{U_h\}_{h\in(0,1]}\in C_0((0,1],\ \mathcal{B}(\mathcal{X}))} \|\{T_h\}+\{U_h\}\| = \inf_{\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}} \|\{S_h\}\| \le \|\{S_h\}\|,$$

for any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$. On the other hand,

$$\limsup_{h\to 0} \|S_h\| \le \left\|\{\dot{T_h}\}\right\|,$$

for any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$.
In [11], Macovei showed that the set

$$\mathcal{X}_b\left((0,1],\ \mathcal{X}\right) = \left\{ \{x_h\}_{h\in(0,1]} \subset \mathcal{X} \ :\ \{x_h\}_{h\in(0,1]} \text{ is a bounded sequence, i.e. } \sup_{h\in(0,1]} \|x_h\| < \infty \right\},$$

is a Banach space in rapport with norm

$$\|\{x_h\}\| = \sup_{h\in(0,1]} \|x_h\|.$$

And

$$\mathcal{X}_0\left((0,1],\ \mathcal{B}(\mathcal{X})\right) = \left\{ \{x_h\}_{h\in(0,1]} \in \mathcal{X}_b\left((0,1],\ \mathcal{X}\right) \ :\ \lim_{h\to 0} \|x_h\| = 0 \right\},$$

is a closed subspace of $\mathcal{X}_b\left((0,1],\ \mathcal{X}\right)$. The quotient space $\mathcal{X}_b\left((0,1],\ \mathcal{X}\right)/\mathcal{X}_0\left((0,1],\ \mathcal{X}\right)$, which will be denoted $\mathcal{X}_\infty$, is a Banach space in rapport with quotient norm

$$\left\|\{\dot{x_h}\}\right\| = \inf_{\{u_h\}_{h\in(0,1]}\in\mathcal{X}_0((0,1],\ \mathcal{X})} \|\{x_h\}+\{u_h\}\| = \inf_{\{y_h\}_{h\in(0,1]}\in\{\dot{x_h}\}} \|\{y_h\}\| = \inf_{\{y_h\}_{h\in(0,1]}\in\{\dot{x_h}\}} \sup_{h\in(0,1]} \|y_h\|.$$

In [11], it has shown that $\mathcal{B}_\infty \subset \mathcal{B}(\mathcal{X}_\infty)$, where $\mathcal{B}(\mathcal{X}_\infty)$ is the algebra of linear bounded operators on $\mathcal{X}_\infty$.

Let $T \in \mathcal{B}(\mathcal{X})$, we denote the Cesàro means by

$$M_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k,\ n = 1, 2 \ldots .$$

We say that T is Cesàro bounded if

$$\sup_{n\in\mathbb{N}^*} \|M_n(T)\| < \infty.$$

We say that the operator T is mean ergodic if there exists $P \in \mathcal{B}(\mathcal{X})$ such that

$$\|M_n(T)x - Px\| \longrightarrow 0 \text{ as } n \to \infty,$$

for each $x \in \mathcal{X}$.
We say that the operator T is uniformly ergodic if there exists $P \in \mathcal{B}(\mathcal{X})$ such that

$$\|M_n(T) - P\| \longrightarrow 0 \text{ as } n \to \infty.$$

In the following definitions, we introduce the notions of power boundedness , Cesàro boundedness, mean ergodicity and uniform ergodicity for family of operators of $C_b((0,1],\ \mathcal{B}(\mathcal{X}))$.

Definition 2.1. *We say that a family of operators $\{T_h\}_{h\in(0,1]} \in C_b((0,1],\ \mathcal{B}(\mathcal{X}))$ is power bounded if*

$$\sup_{n\in\mathbb{N}} \limsup_{h\to 0} \|T_h^n\| < \infty.$$

We say that it is Cesàro bounded if

$$\sup_{n\in\mathbb{N}^*} \limsup_{h\to 0} \|M_n(T_h)\| < \infty.$$

Remark 2.2. *If $T_h = T$ for each $h \in (0,1]$. Then, T is power bounded (respectively Cesàro bounded) if and only if $\{T_h\}_{h\in(0,1]}$ power bounded (respectively Cesàro bounded).*

Definition 2.3. *We say that a family of operators $\{T_h\}_{h\in(0,1]} \in C_b((0,1],\ \mathcal{B}(\mathcal{X}))$ is mean ergodic if there exists $\{P_h\}_{h\in(0,1]} \in C_b((0,1],\ \mathcal{B}(\mathcal{X}))$ such that*

$$\lim_{n\to\infty} \limsup_{h\to 0} \|M_n(T_h)x_h - P_h x_h\| = 0,$$

for each $\{x_h\}_{h\in(0,1]} \in \mathcal{X}_b((0,1],\ \mathcal{X})$.
We say that it is uniformly ergodic if there exists $\{P_h\}_{h\in(0,1]} \in C_b((0,1],\ \mathcal{B}(\mathcal{X}))$ such that

$$\lim_{n\to\infty} \limsup_{h\to 0} \|M_n(T_h) - P_h\| = 0.$$

Remark 2.4. *(i) If $T_h = T$ for each $h \in (0,1]$. Then, T is mean (respectively uniformly) ergodic if and only if $\{T_h\}_{h\in(0,1]}$ is mean (respectively uniformly) ergodic.*

(ii) The operator T_h is mean ergodic for any $h \in (0,1]$, does not imply that the family $\{T_h\}_{h\in(0,1]}$ is mean ergodic.

Example 2.5. *To see Remark 2.4,(ii), consider the Banach space l_∞ of bounded sequences $\mathbf{x} = (x_n)_{n=1,\ldots}$, with the supremum norm.*

For every integer j define the operator T_j by $T_j(\mathbf{x}) = (y_n)_{n=1,2\ldots.}$ where $y_n = x_{n+1}$ for $n \le j-1$; $y_j = 0$; $y_n = x_n$ for $n > j$. The operator T_j acts as a shift to the left on the j first values of $\mathbf{x}$, but acts as the identity on the infinite part of $\mathbf{x}$ after j.

One can show that each T_j is mean ergodic with the corresponding projector P_j defined by $P_j(\mathbf{x}) = (z_n)_{n=1,2\ldots}$, where $z_n = 0$ for $n \le j$; $z_n = x_n$ for $n > j$. By taking $n < j$ and the constant sequence $\mathbf{1}$ we get $\|M_n(T_j)\mathbf{1} - P_j\mathbf{1}\|_\infty = 1$.

Now for $h \in (0,1]$ put $S_h = T_j$ when $\frac{1}{j+1} < h \le \frac{1}{j}$. Each S_h is mean ergodic with the projector $P_h = P_j$ for $\frac{1}{j+1} < h \le \frac{1}{j}$. But with the constant sequence $\mathbf{1}$, from the property of the operators T_j, we have $\limsup_{h\to 0} \|M_n(S_h)\mathbf{1} - P_h\mathbf{1}\|_\infty = 1$ for every integer n, and the condition of Definition 2.3 does not hold.

In the following definitions, we introduce the notions of power boundedness, Cesàro boundedness, mean ergodicity, and uniform ergodicity for a family of operators of $\mathcal{B}_\infty$.

Definition 2.6. *We say that* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *is power bounded if*

$$\sup_{n\in\mathbb{N}} \left\| \{\dot{T_h}\}^n \right\| < \infty,$$

where $\{\dot{T_h}\}^n = \{\dot{T_h^n}\}$.
We say that it is Cesàro bounded if

$$\sup_{n\in\mathbb{N}^*} \left\| M_n(\{\dot{T_h}\}) \right\| < \infty,$$

where $M_n(\{\dot{T_h}\}) = \{M_n \dot{(T_h)}\}$.

Definition 2.7. *We say that* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *is mean ergodic if there exists* $\{\dot{P_h}\} \in \mathcal{B}_\infty$ *such that*

$$\lim_{n\to\infty} \left\| M_n(\{\dot{T_h}\})\{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} \right\| = 0,$$

for each $\{\dot{x_h}\}_{h\in(0,1]} \in \mathcal{X}_\infty$, *where*

$$M_n(\{\dot{T_h}\})\{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} = \{M_n \dot{(T_h)}\}\{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} = \{M_n \dot{(T_h)} - P_h\}\{\dot{x_h}\}.$$

We say that it is uniformly ergodic if there exists $\{\dot{P_h}\} \in \mathcal{B}_\infty$ *such that*

$$\lim_{n\to\infty} \left\| M_n(\{\dot{T_h}\}) - \{\dot{P_h}\} \right\| = 0,$$

where

$$M_n(\{\dot{T_h}\}) - \{\dot{P_h}\} = \{M_n \dot{(T_h)}\} - \{\dot{P_h}\} = \{M_n \dot{(T_h)} - P_h\}.$$

3. Main results

We start this section with some propositions relating power boundedness, Cesàro boundedness, mean ergodicity, and uniform ergodicity of a family of bounded linear operators $\{\dot{T_h}\} \in \mathcal{B}_\infty$.

Proposition 3.1. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *be power bounded. Then any* $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ *is also power bounded.*

Proof. Suppose that $\{\dot{T_h}\}$ is power bounded, then

$$\sup_{n\in\mathbb{N}} \left\| \{\dot{T_h}\}^n \right\| < \infty.$$

Let $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ be arbitrary. Then

$$\sup_{n\in\mathbb{N}} \limsup_{h\to 0} \|S_h^n\| \leq \sup_{n\in\mathbb{N}} \left\| \{\dot{T_h^n}\} \right\| = \sup_{n\in\mathbb{N}} \left\| \{\dot{T_h}\}^n \right\| < \infty.$$

Therefore, $\{S_h\}_{h\in(0,1]}$ is power bounded.

□

In particular, we obtain the following results.

Corollary 3.2. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *be power bounded. Then* $\{T_h\}_{h\in(0,1]}$ *is also power bounded.*

Proposition 3.3. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *be Cesàro bounded. Then any* $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ *is also Cesàro bounded.*

Proof. Suppose that $\{\dot{T_h}\}$ is Cesàro bounded, then

$$\sup_{n\in\mathbb{N}^*} \left\| M_n(\{\dot{T_h}\}) \right\| < \infty.$$

Let $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ be arbitrary. Then

$$\sup_{n\in\mathbb{N}^*} \limsup_{h\to 0} \|M_n(S_h)\| \leq \sup_{n\in\mathbb{N}^*} \left\| M_n(\dot{\{T_h\}}) \right\| = \sup_{n\in\mathbb{N}^*} \left\| M_n(\{\dot{T_h}\}) \right\| < \infty.$$

Therefore, $\{S_h\}_{h\in(0,1]}$ is Cesàro bounded. □

In particular, we obtain the following results.

Corollary 3.4. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$ be Cesàro bounded. Then $\{T_h\}_{h\in(0,1]}$ is also Cesàro bounded.*

Proposition 3.5. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$ be mean ergodic. Then any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ is also mean ergodic.*

Proof. Suppose that $\{\dot{T_h}\}$ is mean ergodic, then there exists $\{\dot{P_h}\} \in \mathcal{B}_\infty$ such that

$$\lim_{n\to\infty} \left\| M_n(\{\dot{T_h}\})\{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} \right\| = 0,$$

for each $\{\dot{x_h}\}_{h\in(0,1]} \subset \mathcal{X}_\infty$.

Let $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ be arbitrary. Then for $\{P_h\}_{h\in(0,1]} \in \{\dot{P_h}\}$, we have

$$\begin{aligned} \lim_{n\to\infty} \limsup_{h\to 0} \|M_n(S_h)\, x_h - P_h x_h\| \leq{}& \lim_{n\to\infty} \left\| M_n(\{T_h\})\{x_h\} \dot{-} \{P_h\}\{x_h\} \right\| \\ ={}& \lim_{n\to\infty} \left\| M_n(\{\dot{T_h}\})\{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} \right\| = 0. \end{aligned}$$

Therefore, $\{S_h\}_{h\in(0,1]}$ is mean ergodic. □

In particular, we obtain the following results.

Corollary 3.6. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$ be mean ergodic. Then $\{T_h\}_{h\in(0,1]}$ is also mean ergodic.*

Proposition 3.7. *[1] Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$ be uniformly ergodic. Then any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ is also uniformly ergodic. In particular, $\{T_h\}_{h\in(0,1]}$ is uniformly ergodic.*

It is easy to show that Propositions 3.8 and 3.9 below holds.

Proposition 3.8. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$.*

(1) If $\{\dot{T_h}\}$ is mean ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \{\dot{T_h}\}^n \{\dot{x_h}\} = \{\dot{0}\},$$

for each $\{\dot{x_h}\} \in \mathcal{X}_\infty$.

(2) If $\{\dot{T_h}\}$ is uniformly ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \left\| \{\dot{T_h}\}^n \right\| = 0.$$

Proof. It suffices to show that

$$\frac{1}{n}\{\dot{T_h}\}^n = M_n(\{\dot{T_h}\}) - \frac{n-1}{n} M_{n-1}(\{\dot{T_h}\}). \tag{3.1}$$

□

Proposition 3.9. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$.

(1) If $\{\dot{T_h}\}$ *is power bounded, then it is Cesàro bounded.*

(2) If $\{\dot{T_h}\}$ *is mean ergodic, then it is Cesàro bounded.*

Proof. It suffices to use equality (3.1).

□

In the following Theorem, we prove that the restriction of a mean (respectively uniformly) ergodic family of bounded linear operators on an invariant subspace is also mean (respectively uniformly) ergodic.

Theorem 3.10. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *be mean (respectively uniformly) ergodic and let* $\mathcal{Y}_\infty \subset \mathcal{X}_\infty$ *be a closed subspace, which is* $\{\dot{T_h}\}$*-invariant. Then, the restriction* $\{\dot{S_h}\} = \{\dot{T_h}\}_{|\mathcal{Y}_\infty}$ *is mean (respectively uniformly) ergodic.*

Proof. Since $\mathcal{Y}_\infty$ is $\{\dot{T_h}\}$-invariant, then it is is $\{\dot{T_h}\}^n$-invariant, for $n = 1, 2, \ldots$. Thus it is $M_n(\{\dot{S_h}\}) = M_n(\{\dot{T_h}\})_{|\mathcal{Y}_\infty}$-invariant. Hence, the strong (respectively uniform) limit, if it exists, is in $\mathcal{Y}_\infty$. Therefore $\{\dot{S_h}\}$ is mean (respectively uniformly) ergodic. □

Theorem 3.11. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *such that* $\lim_n \frac{1}{n}\{\dot{T_h}\}^n\{\dot{x_h}\} = \{\dot{0}\}$*, for each* $\{\dot{x_h}\} \in \mathcal{X}_\infty$*. Then*

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^k \mathcal{X}_\infty \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\},$$

for each $k = 1, 2, \ldots$.

Proof. It is easy to show that

$$M_n(\{\dot{T_h}\})\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \frac{1}{n}\left(\{\dot{T_h}\} - \{\dot{T_h}\}^{n+1}\right). \tag{3.2}$$

Now, let $\{\dot{y_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$. Then, there exists $\{\dot{x_h}\} \in \mathcal{X}_\infty$ such that $\{\dot{y_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{x_h}\}$. Thus, by (3.2), we get

$$\begin{aligned} M_n(\{\dot{T_h}\})\{\dot{y_h}\} = & \quad M_n(\{\dot{T_h}\})\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{x_h}\} = \frac{1}{n}\left(\{\dot{T_h}\}\{\dot{x_h}\} - \{\dot{T_h}\}^{n+1}\{\dot{x_h}\}\right) \\ = & \quad \frac{1}{n}\{\dot{T_h}\}\{\dot{x_h}\} - \frac{1}{n}\{\dot{T_h}\}^{n+1}\{\dot{x_h}\}, \end{aligned}$$

which converges to $\{\dot{0}\}$ as $n \longrightarrow \infty$ by hypothesis.
On the other hand, let $\{\dot{y_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right)$, then $\{\dot{y_h}\} = \{\dot{T_h}\}\{\dot{y_h}\}$. Thus, $\{\dot{y_h}\} = \{\dot{T_h}\}^k\{\dot{y_h}\}$, for each $k = 1, 2, \ldots$. Hence,

$$M_n(\{\dot{T_h}\})\{\dot{y_h}\} = \quad \frac{1}{n}\sum_{k=1}^{n}\{\dot{T_h}\}^k\{\dot{y_h}\} = \{\dot{y_h}\}.$$

Since $M_n(\{\dot{T_h}\})\{\dot{y_h}\} \longrightarrow \{\dot{0}\}$ as $n \longrightarrow \infty$, then $\{\dot{y_h}\} = \{\dot{0}\}$. Thus $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\}$. But, $\{\dot{0}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)^k\mathcal{X}_\infty \subseteq \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$, for each $k = 1, 2, \ldots$. Therefore,

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^k \mathcal{X}_\infty \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\}.$$

□

Corollary 3.12. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$. If $\{\dot{T_h}\}$ is mean ergodic, then*

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^k \mathcal{X}_\infty \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\},$$

for each $k = 1, 2, \ldots$.

Proof. It suffices to use Proposition 3.8 and Theorem 3.11. □

In the following Theorem, we will extend the known ergodic theorem of K. Yosida [13] from the case of a bounded linear operator to the case of a family of bounded linear operators on a Banach space.

Theorem 3.13. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$. If*

(i) $\lim_n \frac{1}{n}\{\dot{T_h}\}^n\{\dot{x_h}\} = \{\dot{0}\}$, *for each $\{\dot{x_h}\} \in \mathcal{X}_\infty$, and*

(ii) $\{\dot{T_h}\}$ *is Cesàro bounded.*

Then,

$$\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} = \left\{\{\dot{x_h}\} \in \mathcal{X}_\infty \ :\ M_n(\{\dot{T_h}\})\{\dot{x_h}\} \longrightarrow \{\dot{0}\} \text{ as } n \longrightarrow 0\right\},$$

and

$$\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\}.$$

Proof. Let $\mathcal{Y} = \left\{\{\dot{x_h}\} \in \mathcal{X}_\infty \ :\ M_n(\{\dot{T_h}\})\{\dot{x_h}\} \longrightarrow \{\dot{0}\} \text{ as } n \longrightarrow 0\right\}$. Let $\{\dot{y_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$, then, as in the proof of Theorem 3.11, $\{\dot{y_h}\} \in \mathcal{Y}$. Let $\{\dot{z_h}\} \in \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}$, thus, by (ii), there exists a constant $C > 0$ such that $\left\|M_n(\{\dot{T_h}\})\{\dot{y_h}\}\right\| \leq C\left\|\{\dot{y_h}\}\right\|$ for every $\{\dot{y_h}\} \in \mathcal{Y}$ and each $n = 1, 2, \ldots$. Now, fix $\epsilon > 0$, then there exists $\{\dot{u_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty \subseteq \mathcal{Y}$ such that $\left\|\{\dot{z_h}\} - \{\dot{u_h}\}\right\| < \frac{\epsilon}{C}$ (where $\{\dot{z_h}\} - \{\dot{u_h}\} = \{z_h \dot{-} u_h\}$). Then, we obtain

$$\begin{aligned}
\left\|M_n(\{\dot{T_h}\})\{\dot{z_h}\}\right\| &\leq \left\|M_n(\{\dot{T_h}\})\{\dot{u_h}\}\right\| + \left\|M_n(\{\dot{T_h}\})\left(\{\dot{z_h}\} - \{\dot{u_h}\}\right)\right\| \\
&\leq \left\|M_n(\{\dot{T_h}\})\{\dot{u_h}\}\right\| + C\left\|\{\dot{z_h}\} - \{\dot{u_h}\}\right\| \\
&< \left\|M_n(\{\dot{T_h}\})\{\dot{u_h}\}\right\| + \epsilon.
\end{aligned}$$

Since $\{\dot{u_h}\} \in \mathcal{Y}$, then, for n sufficiently large, we have $\left\|M_n(\{\dot{T_h}\})\{\dot{z_h}\}\right\| < 2\epsilon$. Thus,

$$\lim_{n \longrightarrow 0}\left\|M_n(\{\dot{T_h}\})\{\dot{z_h}\}\right\| = 0.$$

Hence, $\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} \subseteq \mathcal{Y}$. To prove the converse, let $\{\dot{x_h}\} \in \mathcal{Y}$ and $n \in \mathbb{N}^*$. Then

$$\begin{aligned}
\{\dot{x_h}\} - M_n(\{\dot{T_h}\})\{\dot{x_h}\} &= \{\dot{x_h}\} - \frac{1}{n}\sum_{k=1}^{n}\{\dot{T_h}\}^k\{\dot{x_h}\} \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\{\dot{I}\} - \{\dot{T_h}\}^k\right)\{\dot{x_h}\} \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\left[\left(\{\dot{I}\} + \{\dot{T_h}\} + \ldots + \{\dot{T_h}\}^{k-1}\right)\{\dot{x_h}\}\right],
\end{aligned}$$

thus $\{\dot{x_h}\} - M_n(\{\dot{T_h}\})\{\dot{x_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$. Since, $\{\dot{x_h}\} \in \mathcal{Y}$, by passing to limit as $n \longrightarrow 0$, we check that $\{\dot{x_h}\} \in \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}$. The proof of the equality

$$\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\}$$

is similar to the proof of Theorem 3.11.

□

Corollary 3.14. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$. If $\{\dot{T_h}\}$ is mean ergodic, then*

$$\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} = \left\{\{\dot{x_h}\} \in \mathcal{X}_\infty \ : \ M_n(\{\dot{T_h}\})\{\dot{x_h}\} \longrightarrow \{\dot{0}\} \text{ as } n \longrightarrow 0\right\},$$

and

$$\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} \cap \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \{\dot{0}\}.$$

Proof. It suffices to use Propositions 3.8, 3.9 and Theorem 3.11.

□

Theorem 3.15. *Let $\{\dot{T_h}\} \in \mathcal{B}_\infty$ is mean ergodic. If we denote $\{\dot{P_h}\}\{\dot{x_h}\} = \lim_{n\longrightarrow\infty} M_n(\{\dot{T_h}\})\{\dot{x_h}\}$ for all $\{\dot{x_h}\} \in \mathcal{X}_\infty$. Then, $\{\dot{P_h}\} : \mathcal{X}_\infty \longrightarrow \mathcal{X}_\infty$ satisfies:*

(1.) $\{\dot{P_h}\}^2 = \{\dot{P_h}\} = \{\dot{T_h}\}\{\dot{P_h}\} = \{\dot{P_h}\}\{\dot{T_h}\}$, *($\{\dot{P_h}\}$ is a projection);*

(2.) $\{\dot{P_h}\}\mathcal{X}_\infty = \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right)$;

(3.) $\ker\left(\{\dot{P_h}\}\right) = \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty} = \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\mathcal{X}_\infty$.

Moreover,

$$\mathcal{X}_\infty = \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \oplus \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}.$$

Proof. (1.) Let $\{\dot{x_h}\} \in \mathcal{X}_\infty$, then

$$\begin{aligned}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{P_h}\}\{\dot{x_h}\} &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\lim_{n\longrightarrow\infty} M_n(\{\dot{T_h}\})\{\dot{x_h}\} \\ &= \lim_{n\longrightarrow\infty}\frac{1}{n}\left(\{\dot{T_h}\} - \{\dot{T_h}\}^{n+1}\right)\{\dot{x_h}\} = \{\dot{0}\}.\end{aligned}$$

Thus, $\{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{T_h}\}\{\dot{P_h}\}\{\dot{x_h}\}$ and $\{\dot{P_h}\} = \{\dot{T_h}\}\{\dot{P_h}\}$. Hence, $\{\dot{T_h}\}^n\{\dot{P_h}\} = \{\dot{P_h}\}$ for every $n = 1, 2, \ldots$. Therefore, $M_n(\{\dot{T_h}\})\{\dot{P_h}\} = \{\dot{P_h}\}$ for every $n = 1, 2, \ldots$. Then, we have

$$\{\dot{P_h}\}^2\{\dot{x_h}\} = \lim_{n\longrightarrow\infty} M_n(\{\dot{T_h}\})\{\dot{P_h}\}\{\dot{x_h}\} = \lim_{n\longrightarrow\infty}\{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{P_h}\}\{\dot{x_h}\},$$

which implies $\{\dot{P_h}\}^2 = \{\dot{P_h}\}$. Also,

$$\{\dot{P_h}\}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{x_h}\} = \lim_{n\longrightarrow\infty}\frac{1}{n}\left(\{\dot{T_h}\} - \{\dot{T_h}\}^{n+1}\right)\{\dot{x_h}\} = \{\dot{0}\},$$

thus, $\{\dot{P_h}\} = \{\dot{P_h}\}\{\dot{T_h}\}$.

(2.) Let $\{\dot{x_h}\} \in \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right)$, then, it is easy to show that $M_n(\{\dot{T_h}\})\{\dot{x_h}\} = \{\dot{x_h}\}$. Thus $\{\dot{P_h}\}\{\dot{x_h}\} = \lim_{n\longrightarrow\infty} M_n(\{\dot{T_h}\})\{\dot{x_h}\} = \{\dot{x_h}\}$. Hence, $\{\dot{x_h}\} \in \{\dot{P_h}\}\mathcal{X}_\infty$. Conversely, Let $\{\dot{x_h}\} \in \{\dot{P_h}\}\mathcal{X}_\infty$. Then there exists $\{\dot{y_h}\} \in \mathcal{X}_\infty$ such that

$$\{\dot{x_h}\} = \{\dot{P_h}\}\{\dot{y_h}\} = \{\dot{P_h}\}^2\{\dot{y_h}\} = \{\dot{P_h}\}\left(\{\dot{P_h}\}\{\dot{y_h}\}\right) = \{\dot{P_h}\}\{\dot{x_h}\}.$$

We obtain

$$\{\dot{T_h}\}\{\dot{x_h}\} = \{\dot{T_h}\}\left(\{\dot{P_h}\}\{\dot{x_h}\}\right) = \{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{x_h}\}.$$

Therefore, $\{\dot{x_h}\} \in \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right)$.

(3.) Let $\{\dot{x_h}\} \in \ker\{\dot{P_h}\}$, then $\left(\{\dot{I}\} - \{\dot{P_h}\}\right)\{\dot{x_h}\} = \{\dot{x_h}\} - \{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{x_h}\}$. Thus

$$\{\dot{x_h}\} \in \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\mathcal{X}_\infty.$$

Conversely, let $\{\dot{x_h}\} \in \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\mathcal{X}_\infty$, then there exists $\{\dot{z_h}\} \in \mathcal{X}_\infty$ such that

$$\{\dot{x_h}\} = \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\{\dot{z_h}\}.$$

Thus

$$\{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{P_h}\}\{\dot{z_h}\} - \{\dot{P_h}\}^2\{\dot{z_h}\} = \{\dot{P_h}\}\{\dot{z_h}\} - \{\dot{P_h}\}\{\dot{z_h}\} = \{\dot{0}\}.$$

Hence, $\{\dot{x_h}\} \in \ker\{\dot{P_h}\}$. Therefore, $\ker\{\dot{P_h}\} = \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\mathcal{X}_\infty$. Now, we will show that both are equal to $\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}$. Let $\{\dot{x_h}\} \in \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\mathcal{X}_\infty$, then there exists $\{\dot{z_h}\} \in \mathcal{X}_\infty$ such that $\{\dot{x_h}\} = \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\{\dot{z_h}\}$. We obtain

$$\left(\{\dot{I}\} - M_n(\{\dot{T_h}\})\right)\{\dot{z_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\left[\frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{i}\{\dot{T_h}\}^j\{\dot{z_h}\}\right],$$

then $\left(\{\dot{I}\} - M_n(\{\dot{T_h}\})\right)\{\dot{z_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$. We have

$$\begin{aligned}\{\dot{x_h}\} &= \left(\{\dot{I}\} - \{\dot{P_h}\}\right)\{\dot{z_h}\} = \{\dot{z_h}\} - \lim_{n\longrightarrow\infty} M_n(\{\dot{T_h}\})\{\dot{z_h}\} \\ &= \lim_{n\longrightarrow\infty}\left(\{\dot{I}\} - M_n(\{\dot{T_h}\})\right)\{\dot{z_h}\} \in \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}.\end{aligned}$$

let $\{\dot{x_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$, then there exists $\{\dot{z_h}\} \in \mathcal{X}_\infty$ such that $\{\dot{x_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{z_h}\}$. Thus,

$$\{\dot{P_h}\}\{\dot{x_h}\} = \{\dot{P_h}\}\{\dot{z_h}\} - \{\dot{T_h}\}\{\dot{P_h}\}\{\dot{z_h}\} = \{\dot{P_h}\}\{\dot{z_h}\} - \{\dot{P_h}\}\{\dot{z_h}\} = \{\dot{0}\}.$$

Hence, $\{\dot{x_h}\} \in \ker\{\dot{P_h}\}$. Now, let $\{\dot{u_h}\} \in \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty}$, then there exists a sequence $(\{\dot{v_h}\}_n)_n \subseteq \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathcal{X}_\infty$ such that $\lim_{n\longrightarrow\infty}\{\dot{v_h}\}_n = \{\dot{u_h}\}$. Therefore, by the continuity of $\{\dot{P_h}\}$, we conclude

$$\{\dot{P_h}\}\{\dot{u_h}\} = \{\dot{P_h}\}\lim_{n\longrightarrow\infty}\{\dot{v_h}\}_n = \lim_{n\longrightarrow\infty}\{\dot{P_h}\}\{\dot{v_h}\}_n = \lim_{n\longrightarrow\infty}\{\dot{0}\} = \{\dot{0}\}.$$

We have $\{\dot{u_h}\} \in \ker\{\dot{P_h}\}$ and thus the assertion (3.).

□

Theorem 3.16. *Let* $\{\dot{T_h}\} \in \mathcal{B}_\infty$ *be a Cesàro bounded linear operator on* $\mathcal{X}_\infty$ *which satisfies* $\lim_{n\longrightarrow\infty} \frac{\|\{\dot{T_h}\}^n\{\dot{x_h}\}\|}{n} = 0$ *for all* $\{\dot{x_h}\} \in \mathcal{X}_\infty$. *Then* $\{\dot{T_h}\}$ *is mean ergodic if and only if*

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty. \tag{3.3}$$

Proof. Assume that $\{\dot{T_h}\}$ is mean ergodic. Then

$$\mathcal{X}_\infty = \left\{\{\dot{x_h}\} \in \mathcal{X}_\infty \ : \ \{\dot{T_h}\}(\{\dot{x_h}\}) = \{\dot{x_h}\}\right\} \oplus \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty}.$$

Thus, the condition (3.3) holds.
Conversely, suppose that $\{\dot{T_h}\}$ is Cesàro bounded on $\mathcal{X}_\infty$ and $\lim_{n\longrightarrow\infty} \frac{\|\{\dot{T_h}\}^n\{\dot{x_h}\}\|}{n} = 0$ for all $\{\dot{x_h}\} \in \mathcal{X}_\infty$. Then, $\left\{M_n(\{\dot{T_h}\}\right\}$ converges strongly on $\ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \oplus \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty}$. Then, the condition

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty,$$

implies that, for $\{\dot{y_h}\} \in \mathcal{X}_\infty$ there exists $\{\dot{z_h}\} \in \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty}$ such that

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \{\dot{y_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right) \{\dot{z_h}\}.$$

Then

$$\{\dot{u_h}\} = \{\dot{y_h}\} - \{\dot{z_h}\} \in \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right),$$

thus

$$\{\dot{y_h}\} = \{\dot{u_h}\} + \{\dot{z_h}\} \in \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \oplus \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty}.$$

Hence,

$$\mathcal{X}_\infty = \ker\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \oplus \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right) \mathcal{X}_\infty}.$$

Therefore, $\left\{M_n(\{\dot{T_h}\}\right\}$ converges strongly on $\mathcal{X}_\infty$.

□

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References

1. A. Akrym, A. El Bakkali and A. Faouzi, *On ergodic theorem for a family of operators*, Oper. Matrices, Vol. 15, No. 3, 1161-1170, (2021).
2. Y. Derriennic, *On the mean ergodic theorem for Cesàro bounded operators*, Colloquium Math., 84/85 2, 443-445, (2000).
3. N. Dunford, *Spectral theory I. Convergence to projection*, Trans. Amer. Math. Soc. 54, 185-217, (1943).
4. R. Emilion, *Mean bounded operators and mean ergodic theorems*, J. Funct. Anal., 61, 1-14, (1985).
5. S. Grabiner and J. Zemànek, *Ascent, descent, and ergodic properties of linear operators*, J. Operator Theory 48, 69-81, (2002).
6. E. Hille, *Remarks on ergodic theorems*, Trans. Amer. Math. Soc., 57, 246-269, (1945).
7. U. Krengel, *Ergodic Theorems*, Walter de Gruyter Studies in Mathematics 6, Walter de Gruyter, Berlin - New York, (1985).
8. Z. Léka, *A note on the powers of Cesàro bounded operators*, Czechoslovak Math. J., 60 (135), 1091-1100, (2010).
9. M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. 43, No. 2, 337-340, (1974).
10. S. Macovei, *Spectrum of a Family of Operators*, Surv. Math. Appl. 6, 137-159, (2011).

11. S. Macovei, *Local Spectrum of a Family of Operators*, Ann. Funct. Anal. 4, no. 2, 131-143, (2013).

12. M. Mbekhta and J. Zemánek, *Sur le théorème ergodique uniforme et le spectre*, C. R. Acad. Sci. Paris série I Math., 317, 1155-1158, (1993).

13. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, (1980).

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