Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–5.
ISSN-0037-8712
doi:10.5269/bspm.63567

A note on (I,J)-continuous functions

Ennis Rosas, Carlos Carpintero* and José Sanabria

ABSTRACT: The purpose of the present paper is to introduce, study and characterize the (I,J)-continuous functions. Also, we investigate their relationships with other types of continuous functions.

Key Words: I-open set, $\theta_{(I,J)}$-continuous function, (I,J)-continuous function

Contents

1. Introduction

It is well known today that the notion of continuous function is playing a very important role in general topology. Generalizations of the notions of continuity have been extensively studied on functions $f:(X,\tau)\to(Y,\sigma)$. Recently using the notion of topological ideal, different types of continuous functions have been introduced and studied. The notion of ideal topological spaces has been introduced and studied by Kuratowski [5] and the local function of a subset A of a topological space (X,τ) was introduced by Vaidyanathaswamy [9] as follows: given a topological space (X,τ) with an ideal I on X and $P(X)$ the set of all subsets of X, a set operator $(.)^*: P(X)\to P(X)$, defined as for each $A\subseteq X$, $A^*(\tau,I)=\{x\in X \ / \ U\cap A\notin I \text{ for every } U\in\tau_x\}$, where $\tau_x=\{U\in\tau : x\in U\}$ is called the local function of A with respect to τ and I. The Kuratowski closure operator $cl^*(,)$ for a topology $\tau^*(\tau,I)$ called the *-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\tau,I)$. We will denote $A^*(\tau,I)$ by A^* and $\tau^*(\tau,I)$ by τ^*. Note that when $I=\{\emptyset\}$, (respectively $I=P(X)$) $A^*=cl(A)$, (respectively $A^*=\emptyset$). In 1961, Levine [4] introduced and studied the weakly J- continuous. In 1990, Jankovic and Hamlett [3] introduced the notion of I-open set in a topological space (X,τ) with an ideal I on X. In 2005, Yüksel et al. [10] introduced, studied and investigated the $\theta_{(I,J)}$-continuous functions. In 2012, Al-Omari and Noiri [1] investigated some properties of $\theta_{(I,J)}$-continuous functions in ideal topological spaces and its relations with another functions. In this article, we continue with the study of the (I,J)-continuous functions on ideal topological spaces. Additionally, the relationships with others related functions are discussed.

2. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) are topological spaces without separation axioms are assumed, unless explicitly stated. If I is and ideal on X, (X,τ,I) means an ideal topological space. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively, for any subset A of (X,τ). A subset A of (X,τ,I) is an I-open [3], if $A\subseteq\mathrm{int}(A^*)$. A subset A of $(X\tau,I)$ is called I-closed if its complement of an I-open set. In the case that $I=\{\emptyset\}$, (respectively, $I=P(X)$) the I-open sets form the collection of all preopen sets of X (respectively, the only I-open set is $\emptyset$). The notion of I-closure and the I-interior of a subset A of an ideal topological space, can be defined in the same form as $\mathrm{Cl}(A)$ and $\mathrm{int}(A)$, respectively, will be denoted by $I\,\mathrm{Cl}(A)$ and $I\mathrm{int}(A)$, respectively. The collection of all I-open (resp. I-closed) subsets of

* Corresponding author

Submitted May 15, 2022. Published April 14, 2025
2010 *Mathematics Subject Classification:* 54C05, 54C08, 54C60

(X,τ,I) is denoted by $IO(X)$ (resp. $IC(X)$). We set $IO(X,x) = \{A : A \in IO(X) \text{ and } x \in A\}$. Given (X,τ,I), if the ideal is τ-acotado, that is $\tau \cap I = \emptyset$, then $X^* = X$. Otherwise, $X^* \subseteq X$.

Definition 2.1 *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be θ-continuous [2] if for each $x \in X$ and each open set V in Y, such that $f(x) \in V$, there exists an open set U, such that $x \in U$ and $f(cl(U)) \subseteq cl(V)$.*

Definition 2.2 *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be precontinuous [6] (resp. preirresolute [8]) if $f^{-1}(G)$ is preopen in X for every open (resp. preopen) set G of Y.*

Definition 2.3 *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be p-continuous [7] if $f^{-1}(G)$ is open in X for every preopen set G of Y.*

Definition 2.4 *A function $f : (X,\tau,I) \to (Y,\sigma,J)$ is said to be weakly J-continuous [4] (resp. $\theta_{(I,J)}$-continuous [10]) if for each $x \in X$ and each open set V in Y, such that $f(x) \in V$, there exists an open set U, such that $x \in U$ and $f(U) \subseteq cl^*(V)$ (resp. $f(cl^*(U)) \subseteq cl^*(V)$).*

The following was proved in [1].

Theorem 2.1 *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces and $f : (X,\tau,I) \to (Y,\sigma,J)$ be a function. The following statements hold:*

1. *If f is continuous then f is weakly J-continuous;*
2. *If f is $\theta_{(I,J)}$-continuous then f is weakly J-continuous.*

Remark 2.1 None of the implications of Theorem 2.1 is reversible. Furthermore, the notions of continuous function and $\theta_{(I,J)}$-continuous function are independent.

3. (I,J)-continuous functions

In this section, we define and characterize the (I,J)-continuous functions for any ideal topological spaces (X,τ,I) and (Y,σ,J).

Definition 3.1 *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces. A function $f : (X,\tau,I) \to (Y,\sigma,J)$ is said to be (I,J)-continuous if for each $x \in X$ and each J-open set V such that $f(x) \in V$, there exists an I-open set U such that $x \in U$ and $f(U) \subseteq V$.*

Theorem 3.1 *Let (X,τ,I) and (Y,σ,J) be two ideal topological spaces and $f : (X,\tau,I) \to (Y,\sigma,J)$, be a function. The following statements are equivalent:*

1. *f is (I,J)-continuous;*
2. *$f^{-1}(V)$ is I-open in X for each J-open set V in Y;*
3. *$f^{-1}(K)$ is I-closed in X for each J-closed set K in Y;*
4. *$f(I\,\mathrm{Cl}(A)) \subseteq J\,\mathrm{Cl}(f(A))$ for every subset A of X;*
5. *$I\,\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J\,\mathrm{Cl}(B))$ for every subset B of Y.*

Proof: (1) $\Longrightarrow$ (2). Let V be any J-open set in Y and x be any point in $f^{-1}(V)$. From (1), there exists an I-open U_x containing x such that $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} U_x$. Since the union of I-open sets is I-open, $f^{-1}(V)$ is I-open in X.

(2) $\Rightarrow$ (3) If K is a J-closed set in Y, then $f^{-1}(Y \setminus K)$ is an I-open set in X. But $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$, then $f^{-1}(K)$ is J-closed in X.

(3) $\Rightarrow$ (4) Let K be any J-closed set in Y containing $f(A)$. It follows that $A \subseteq f^{-1}(K)$. By (3), $f^{-1}(K)$ is a I-closed set in X containing A. In consequence, $I\,\mathrm{Cl}(A) \subseteq f^{-1}(K)$, and then $f(I\,\mathrm{Cl}(A)) \subseteq K$. Since this is true for all J-closed set K in Y containing $f(A)$, it follows that $f(I\,\mathrm{Cl}(A)) \subseteq J\,\mathrm{Cl}(f(A))$.

(4) $\Rightarrow$ (5) Let B any subset of Y. Taking $A = f^{-1}(B)$ and applying (4), we have $f(I\,\mathrm{Cl}(f^{-1}(B))) \subseteq J\,\mathrm{Cl}(f(f^{-1}(B))) \subseteq J\,\mathrm{Cl}(B)$. Therefore, $I\,\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J\,\mathrm{Cl}(B))$.

(5) $\Rightarrow$ (1) Suppose that $x \in X$ and V is a J-open set in Y containing $f(x)$. Then $V \cap (Y - V) = \emptyset$ and $f(x) \notin J\,\mathrm{Cl}(Y - V)$. Thus $x \notin f^{-1}(J\,\mathrm{Cl}(Y - V))$ and by (5), we have $x \notin I\,\mathrm{Cl}\left(f^{-1}(Y - V)\right)$, hence there exists an I-open set U in X such that $x \in U$ and $U \cap f^{-1}(Y - V) = \emptyset$, which implies that $f(U) \cap (Y - V) = \emptyset$. Therefore, $f(U) \subseteq V$ and so, f is (I,J)-continuous.

□

The following two example shows that the notions of (I,J)-continuous function and continuous function, are independent.

Example 3.1 Let $X = Y = \{a,b,c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, $J = \{\emptyset, \{b\}\}$. Define a function $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = b$, $f(b) = a$ and $f(c) = c$. It is easy to see that the collection of all I-open sets is $\{\emptyset\}$ and the collection of all J-open sets is $\{\emptyset, Y, \{a\}, \{a,b\}, \{a,c\}\}$. In consequence, f is continuous but is not (I,J)-continuous.

Example 3.2 Let $X = Y = \{a,b,c\}$ with two topologies $\tau = \{\emptyset, X, \{a,c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and two ideals $I = \{\emptyset\}$, $J = \{\emptyset, \{a\}\}$. Define a function $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = b$, $f(b) = a$ and $f(c) = c$. It is easy to see that the collection of all I-open sets is $\{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$ and the collection of all J-open sets is $\{\emptyset, Y, \{b\}, \{a,b\}, \{b,c\}\}$. In consequence, f is (I,J)-continuous but is not continuous.

The following two example shows that the notions of (I,J)-continuous function and $\theta_{(I,J)}$-continuous function, are independent.

Example 3.3 Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b,c\}\}$ and $I = \{\emptyset, \{a\}\}$, $Y = \{b,c\}$, $\sigma = \{\emptyset, Y, \{c\}\}$ with $J = \{\emptyset, \{b\}\}$. Define a function $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = c$, $f(b) = b$ and $f(c) = b$. It is easy to see that the collection of all I-open sets is $\{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and the collection of all J-open sets is $\{\emptyset, \{c\}, \{b,c\}, Y\}$. Observe that in (X,τ,I), we have $Cl^*(X) = Cl^*(\{b\}) = Cl^*(\{c\}) = Cl^*(\{a,b\}) = Cl^*(\{a,c\}) = Cl^*(\{b,c\}) = X$, $Cl^*(\emptyset) = \emptyset$ and $Cl^*(\{a\}) = \{a\}$. In the same form in (Y,σ,J), we have $Cl^*(Y) = Cl^*(\{c\}) = Cl^*(\{a,b\}) = Cl^*(\{a,c\}) = Cl^*(\{b,c\}) = Y$, $Cl^*(\emptyset) = \emptyset$, $Cl^*(\{a\}) = \{a,b\}$ and $Cl^*(\{b\}) = \{b\}$. Hence, we conclude that f is not (I,J)-continuous but is $\theta_{(I,J)}$-continuous and weakly J-continuous.

Example 3.4 Let $X = Y = \{a,b,c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a function $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(x) = x$, for each $x \in X$. It is easy to see that the collection of all I-open sets is $\{\emptyset, X, \{b\}, \{a,b\}, \{b,c\}\}$ and the collection of all J-open sets is $\{\emptyset, Y\}$. Observe that in (X,τ,I), $Cl^*(X) = Cl^*(\{b\}) = Cl^*(\{a,b\}) = Cl^*(\{b,c\}) = X$, $Cl^*(\emptyset) = \emptyset$ and $Cl^*(\{a\}) = \{a\}$, $Cl^*(\{c\}) = \{a,c\} = Cl^*(\{a,c\})$. In the same form in (Y,σ,J), $Cl^*(Y) = Cl^*(\{a,b\}) = Cl^*(\{b,c\}) = Y$, $Cl^*(\{b\}) = Cl^*(\{c\}) = Cl^*(\{b,c\}) = \{b,c\}$, $Cl^*(\emptyset) = \emptyset$, and $Cl^*(\{a\}) = \{a\}$. Hence, we conclude that f is (I,J)-continuous but is neither $\theta_{(I,J)}$-continuous nor weakly J-continuous.

4. Relations between (I,J) continuity with other type of functions

In this section, we consider a function $f : (X,\tau,I) \to (Y,\sigma,J)$, where (X,τ,I) and (Y,σ,J) are two ideal topological spaces. Our purpose is to find the relationships between continuity, $\theta_{(I,J)}$-continuity and (I,J)-continuity, when $I = J = \{\emptyset\}$ or $I = P(X)$ and $J = P(Y)$.

Example 4.1 Let $X = Y = \{a,b,c\}$, $\tau = \{X, \emptyset, \{a\}\}$, $\sigma = \{X, \emptyset\}$ and $I = J = \{\emptyset\}$. Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as the identity function. It is easy to see that f is continuous but is not (I,J)-continuous.

Example 4.2 Let $X = Y = \{a,b,c,d\}$, $\tau = \sigma = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ and $I = J = \{\emptyset\}$. Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = a$, $f(b) = d$, $f(c) = c$ and $f(d) = b$. It is easy to see that the collection of all preopen sets is $\{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$. Therefore, f is (I,J)-continuous but is neither continuous nor $\theta_{(I,J)}$-continuous.

Remark 4.1 Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ a function. If $I = J = \{\emptyset\}$, then:

1. f is $\theta_{(I,J)}$-continuous if and only if it is θ-continuous.
2. f is (I, J)-continuous if and only if it is preirresolute.
3. From Examples 4.1 and 4.2, we see that if $I = J = \{\emptyset\}$, then the notions of continuous function and (I, J)-continuous function are independent.

Remark 4.2 Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ a function.

1. If $I = P(X)$ and $J = P(Y)$, then f is $\theta_{(I,J)}$-continuous if and only if it is continuous.
2. If $J = P(Y)$, then f is (I, J)-continuous.

Theorem 4.1 *Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ a function.*

1. *If $I = \{\emptyset\}$ and f is preirresolute, then f is (I, J)-continuous.*
2. *If $J = \{\emptyset\}$ and f is (I, J)-continuous, then f is preirresolute.*

Proof: (1) Since $JO(Y) \subseteq PO(Y)$, the proof follows from the fact that $IO(X) = PO(X)$ when $I = \{\emptyset\}$.
(2) Since $IO(X) \subseteq PO(X)$, the proof follows from the fact that $JO(X) = PO(Y)$ when $J = \{\emptyset\}$. □

Theorem 4.2 *Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ a function.*

1. *If $I = \{\emptyset\}$, J is σ-bounded and f is (I, J)-continuous, then f is precontinuous.*
2. *If I is τ-bounded, $J = \{\emptyset\}$ and f is p-continuous, then f is (I, J)-continuous.*

Proof: (1) Follows from the fact that if J is σ-bounded, then $\sigma \subseteq JO(Y)$.
(2) Follows from the fact that if I is τ-bounded, then $\tau \subseteq IO(Y)$. □

References

1. Al-Omari, A.; Noiri, T. *On $\theta_{(I,J)}$-continuous functions.* Rend. Istit. Mat. Univ. Trieste 44, (2012), 399-411.
2. Fomin, S. *Extension of topological spaces.* Ann. of Math 44, (1943), 471-480.
3. Jankovic, D.S.; Hamlett, T. R. *New topologies from old via ideals.* Amer. Math. Monthly 97(4), (1990), 295–310.
4. Levine, N. *A decomposition of continuity in topological spaces.* Amer. Math. Monthly 68, (1961), 44-46.
5. Kuratowski, K. *Topology.* Academic Press, New York, USA, (1966).
6. Mashhour, A. S; Abd El-Monsef, M. E; El-Deep, S. N. *On precontinuous and weak precontinuous mappings.* Proced. Phys. Soc. Egyp 53, (1982), 47-53.
7. Navalagi, G. B. *Completely preirresolute functions and completely gp-irresolute functions.* Int. J. Math. Comput. Appl 3(1-2), (2011), 55-65.
8. Reilly, I. L; Vamanmurthy, M. K. *On α-continuity in topological spaces.* Acta Math. Hung 45(1-2), (1985), 27–32.
9. Vaidyanathaswamy, R. *The localisation theory in set-topology.* Proc. Indian Acad. Sci 20, (1944), 51-61.
10. Yüksel, S; Acikgöz, A; Noiri, T. *On δ-I-Continuous functions.* Turkish J. Math 29, (2005), 39-51.

Ennis Rosas,
Universidad de la Costa, Departamento de Ciencias Naturales y Exactas
Barranquilla, Colombia.
E-mail address: ennisrafael@gmail.com

and

Carlos Carpintero,
Corporación Universitaria del Caribe-CECAR
Departamento de Ciencias Básicas
Sincelejo, Colombia.
E-mail address: carpintero.carlos@gmail.com

and

José Sanabria,
Departamento de Matemáticas
Facultad de Educación y Ciencias
Universidad de Sucre
Sincelejo, Colombia.
E-mail address: jesanabri@gmail.com