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A New Characterization of Groups $\mathbf{B_4(q)}$

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ABSTRACT: One of an important problems in finite groups theory is, characterizable of groups by specific property. In this paper, we prove that groups $B_4(q)$, where $3 < q$ and $\frac{q^4+1}{2}$ is a prime number, can be uniquely determined by the largest elements order and the order of group.

Key Words: Element order, the largest elements order, Frobenius group, prime graph.

Contents

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. We denote a set of primes by π. A natural number n with $\pi(n) \subseteq \pi$, is called a π-number, while a group G with $\pi(G) \subseteq \pi$ is called a π-group. We denote the largest elements order of G by $k(G)$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$. One of an important problems in finite groups theory is, the characterization of groups by specific property. Properties, such as elements order, the elements with the same order, etc.

One of the methods is group characterization by using the order of the group and the largest element order. In fact, we say the group G is characterizable by using the order of group and the largest elements of group if there is the group H, so that, $k(G) = k(H)$, $|G| = |H|$, then $G \cong H$. However, in the way the authors try to characterize some finite simple groups by using less quantities and have successfully characterized simple K_3-groups, sporadic simple groups, $PSL_2(q)$, $PSL_3(q)$ and $PSU_3(q)$ where q is some special power of prime, by using three numbers: the order of group, the largest and the second largest element orders, of which some results can be seen in [12]. Also in [3], Li-Guan He and Gui-Yun Chen proved a group $PSL_2(q)$ where $q = p^n < 125$ by largest element order and group order can be characterized. In [4], Li-Guan He and Gui-Yun proved characterization K_4 -group of type $PSL_2(p)$ only by using the order of a group and the largest element order, where p is a prime but not 2^n-1. Next, Chen and etal in [2] proved that the sporadic groups are characterizable by using the largest element order and second largest element order. Also the follow, Ebrahimzadeh and etal in [5,6,7,8,9,10] proved that groups as the suzuki groups $Sz(q)$, where $q-1$, $q \pm \sqrt{2q}+1$ are prime number, the projective special unitary groups $PSU_3(3^n)$, the simple groups ${}^2D_n(3)$, where $(n = 2^e + 2, e \geq 4)$, the projective special linear groups $PSL(5,2)$ and $PSL(4,5)$, the symplectic groups $PSP(8,q)$, where q be odd prime number and the symplectic group $C_4(q)$, where $q > 2$ and $\frac{q^4+1}{2}$ are prime numbers by this method can be characterized. Next, in this paper, we prove that groups $B_4(q)$, where $3 < q$ and $\frac{q^4+1}{2}$ is a prime number, can be uniquely determined by the largest elements order and the order of the group. In fact, we prove the following main theorem.

Main Theorem. Let G be a group with $|G| = |B_4(q)|$ and $k(G) = k(B_4(q))$, where $3 < q$ and $\frac{q^4+1}{2}$ is a prime number. Then $G \cong B_4(q)$.

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2. Title Material

In this section, we denote the several lemmas and defintion where we for proving the main theorem need them. Hence, we have the following Lemmas.

Lemma 2.1. *[13] Let H be a finite soluble group all of whose elements are of a power prime order. Then* $|\pi(H)| \leq 2$.

Lemma 2.2. *[11] Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.3. *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.4. *[1] Let G be a 2-Frobenius group of even order. Then*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.5. *[18] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group. In particular, a 2-Frobenius group is soluble.*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Lemma 2.6. *[19] Let q, k, l be natural numbers. Then*

1. $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.
2. $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
3. $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3. Mathematics

In this section, we prove that the groups $B_4(q)$ are characterizable by the order of the group and the largest element order. In fact, we prove that if G is a group with $|G| = |B|$ and $k(G) = k(B_4(q))$, where $3 < q$ and $\frac{q^4+1}{2}$ is a prime number, then $G \cong B_4(q)$. We divide the proof to several lemmas. From now on, we denote the group $B_4(q)$ and $\frac{q^4+1}{2}$ by B, p respectively. Recall that G is a group with $|G| = |B| = \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$ and $k(G) = k(B) = \frac{q(q+1)(q^2+1)}{2}$.

Lemma 3.1. *p is an isolated vertex of $\Gamma(G)$.*

Proof. we prove that p is an isolated vertex of Γ(G). Assume the contrary, thus there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geq 2p = 2(\frac{q^4+1}{2}) > \frac{q(q+1)(q^2+1)}{2}$. As a result $k(G) > \frac{q(q+1)(q^2+1)}{2}$, which is a contradiction. So $t(G) \geq 2$. □

Lemma 3.2. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Now by Lemma 2.2, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $\frac{q^4+1}{2} \mid \frac{\frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}}{\frac{q^4+1}{2}} - 1$. So we conclude that $(q^4 + 1) \mid ((q^4 + 1)(2q^{28} - 2q^{26} - 6q^{24} + 4q^{22} + 10q^{20} - 2q^{18} - 14q^{16} + 16q^{12} - 16q^8 + 16q^4 - 16) + 15$. Therefore $q^4 + 1 \mid 15$ where is a impossible. Hence G is not a Frobenius group. □

Lemma 3.3. *The group G is not a 2-Frobenius group.*

Proof. We prove that that G is not a 2-Frobenius group. For this purpose, by Lemma 2.5 we prove that G is not a soluble group. On the contrary, we assume, G be a soluble group. Also r be prime divisor of $\frac{q^4+1}{2}$, $s \neq 3$. Then there would exist a $\{p, r, s\}$-Hall subgroup H of G. Since B does not contain any elements of orders pr, ps, rs. Thus all of elements of H would be of prime power order. But this contradicts by Lemma 2.1. Hence, G is not a 2-Frobenius group.
□

Here, we prove the main theorem. In otherwords, the following isomorphic.

Lemma 3.4. *The group G is isomorphic to the group B.*

Proof. By Lemmas(2.2, 3.3), we have that third case of Lemma 2.5 is satisfies, as G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and also K/H is a non-abelian simple group. On the other hand every odd order components of G are the odd ordercomponents of K/H. Since $p \mid K/H$ so $t(k/H) \geq 2$. In conclusion according to the classification of the finite simple groups we know that the possibilites for K/H are alternating group A_m, $m \geq 5$, 26 sporadic groups, simple groups Lie type. First, since K/H is a non-abelian simple group, so K/H is isomorphic one of the following groups, for this purpose, first we suppose K/H be isomorphic alternating groups, in otherwords, we have the following isomorphic:

Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = r,\ r+1,\ r+2$. Then by [18] $\pi(A_m) = r,\ r-2$ and $|A_m| \mid |G|$. So we consider $\frac{q^4+1}{2} = r,\ r-2$. In the way if $\frac{q^4+1}{2} = r$, then $\frac{q^4+3}{2} = r+1$, now since $r+1 \mid |A_m|||G|$ hence $\frac{q^4+3}{2} \mid |G|$, which is a contradiction. Now we consider $\frac{q^4+1}{2} = r-2$, then $\frac{q^4+5}{2} = r$, again since $r \mid |A_m|||G|$, hence $\frac{q^4+5}{2} \mid |G|$, which is a contradiction.

Step 2. If K/H be isomorphic sporadic groups, then by [14], $k((S) = \{11, 13, 17, 19, 23, 29, 31, 43, 47, 67, 71\}$, where S be a sporadic groups and also this number are the largest elements order of sporadic groups. Now we consider $\frac{q(q+1)(q^2+1)}{2} = 11, 13, 17, 19, 23, 29, 31, 43, 47, 67, 71$. For this purpose, for example if $\frac{q(q+1)(q^2+1)}{2} = 11$, then $q^4 + q^3 + q^2 + q - 22 = 0$, which is impossible. If $\frac{q(q+1)(q^2+1)}{2} = 13$, then we can see easily a contradiction. For other groups we have a contradiction, similarly.

Step 3. In this case we consider K/H is isomorphic to a the group of Lie-type.

3.1. If $K/H \cong^2 G_2(3^{2m+1})$, where $m \geq 1$ then by [14], $k(^2G_2(3^{2m+1}))) = 3^{2m+1} + 3^{m+1} + 1$. Now we consider $\frac{q(q+1)(q^2+1)}{2} = 3^{2m+1} + 3^{m+1} + 1$, so $q(q^3 + q^2 + q + 1) = 2(3^{2m+1} + 3^{m+1} + 1)$. As a result $2 \mid q$ and $q^3 + q^2 + q + 1 = 3^{2m+1} + 3^{m+1} + 1$. Now if 2 divide q, then there is a contradiction. Now if $q^3+q^2+q+1 = 3^{2m+1}+3^{m+1}+1$, then $q(q^2+q+1) = 3^{m+1}(3^m+1)$. Hence, $q = 3^{m+1}$, $q^2+q+1 = 3^m+1$. Since $q > 3$, so $3^{m+1} > 3$, as a result $m = 1$, $q = 9$, since $|^2G_2(27)|||B_4(9)|$, which is contradiction.

3.2. If $K/H \cong^2 F_4(q')$, where $q' = 2^{2m+1} > 2$ then by [14], $k(^2F_4(q') = 2^{4m+2} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. So we consider $\frac{q(q+1)(q^2+1)}{2} = 2^{4m+2)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. As a result, $2(2^{4m+2)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1) = q(q^3 + q^2 + q + 1)$, similarily there is a contradiction.

3.3. If $K/H \cong^2 B_2(2^{2m+1})$, where $m \geq 1$ then by [14], $k(^2B_2(2^{2m+1}))) = 2^{2m+1} + 2^{m+1} + 1$, also $|^2B_2(2^{2m+1})| = q'^2(q'^2+1)(q'-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. For this purpose, we consider $\frac{q(q+1)(q^2+1)}{2} =$

$2^{2m+1}+2^{m+1}+1$. As a result, $2(2^{2m+1}+2^{m+1}+1)=q(q^3+q^2+q+1)$. Hence $2 \mid q$, which is impossible and $2^{2m+1}+2^{m+1}=(q^3+q^2+q)$. So $2^{m+1}(2^m+1)=q(q^2+q+1)$, which is a contradiction.

3.4. If $K/H \cong G_2(q')$, then by [14], $k(G_2(q')=q'^2+q'+1$ and also $|G_2(q')|=q'^6(q'^6-1)(q'^2-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. For this purpose, we consider $\frac{q(q+1)(q^2+1)}{2}=q'^2+q'+1$. As a result $q(q^3+q^2+q+1)=2(q'^2+q'+1)$, it follows $2 \mid q$ and $q^3+q^2+q+1)=(q'^2+q'+1)$. Now if $q^3+q^2+q=q'(q'+1)$, then $q(q^2+q+1)=q'(q'+1)$. On the other hand, we have $(q',q+1)=1$, so $q=q'$, $q'+1=q^2+q+1$. Since $|G_2(q')| \nmid |G|$, which is a contradiction.

3.5. If $K/H \cong^2 A_n(q')$, where $n>1$.Then by [14], $k(^2A_n(q'))=\frac{q'^{2n}-1}{(n+1,q'+1)}$. Also we know $|^2A_n(q')| \mid |G|$. First if $n=2$, then we have $\frac{1}{(n+1,q'+1)}q'^{n(n+1)/2}\prod_{i=2}^{n+1}(q'^i-(-1)^i \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. For this purpose, we consider $\frac{q(q+1)(q^2+1)}{2}=\frac{q'^4-1}{(3,q'+1)}$, now if $(3,q'+1)=1$, then $\frac{q(q+1)(q^2+1)}{2}=q'^4-1$, as a result $(q'^2-1)(q'^2+1)=\frac{q(q+1)(q^2+1)}{2}$. Since $2 \mid (q'^2-1)(q'^2+1)$, so $2 \mid \frac{q(q+1)(q^2+1)}{2}$. It follows that $4 \mid q^4+q^3+q+q$, which is impossible. Thus we have a contradiction. Another case is impossible, similarly.

3.6. If $K/H \cong D_n(q')$, $C_n(q')$, where $n \geq 4$, $n \geq 3$, respectively. Then, we have a contradiction, similarily.

3.7. If $K/H \cong^3 D_4(q')$, then by [14], $k(^3D_4(q'))=(q'^3-1)(q'+1)$. Also we know $|^3D_4(q')| \mid |G|$, so $q'^{12}(q'^8+q'^4+1)(q'^6-1)(q'^2-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now we consider $(q'^3-1)(q'+1)=\frac{q(q+1)(q^2+1)}{2}$. As a result $q(q^3+q^2+q+1)=2(q'^4+q'^3-q'-1)$, so $q \mid 2$ and $q'^4+q'^3-q'-1 \mid q^3+q^2+q+1$. Now if $q \mid 2$, then this is impossible.The other case is impossible.

3.8. $K/H \cong E_6(q')$, $E_7(q')$, $E_8(q')$, $F_4(q')$. For example if $K/H \cong E_8(q')$, then by [14] $k(E_8(q')=(q'+1)(q'^2+q'+1)(q'^5-1)$. On the other hand, $|E_8(q')|=q'^{120}(q'^{30}-1)(q'^{24}-1)(q'^{20}-1)(q'^{18}-1)(q'^{14}-1)(q'^{12}-1)(q'^8-1)(q'^2-1) \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Hence, we consider $\frac{q(q+1)(q^2+1)}{2}=(q'+1)(q'^2+q'+1)(q'^5-1)$, so $(q'+1)(q'^2+q'+1)(q'^5-1)<q(q+1)(q^2+1)$. As a result, $q'^8 \leq q^4$, so $q'^{120} \mid q^{60}$, but $q^{60} \nmid |G|$, which is a contradiction.

For $K/H \ncong E_6(q')$,$E_7(q')$, $F_4(q')$,we have a contradiction, similarily.

3.9. If $K/H \cong^2 E_6(q')$, then by [14] $k(^2E_6(q')=\frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$ and also $|^2E_6(q')|=\frac{q'^{36}(q'^2-1)(q'^5+1)(q'^6-1)}{(3,q'+1)} \mid \frac{q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1)}{2}$. Now we consider $\frac{q(q+1)(q^2+1)}{2}=\frac{(q'+1)(q'^2+1)(q'^3-1)}{(3,q'+1)}$. First if $(3,q'+1)=1$, then $\frac{q(q+1)(q^2+1)}{2}=(q'+1)(q'^2+1)(q'^3-1)$. As a result $q'^6 \leq q^4$. Hence $q'^{36} \leq q^{24}$, but $q^{24} \nmid |G|$, which is a contradiction.

3.10. If $K/H \cong L_{n+1}(q')$, where $n \geq 1$. For this purpose, first we assume $n=1$, so we have $K/H \cong L_2(q')$. For this purpose, by [14], $k(L_2(q'))=q'+1, q'$, where q' be even, odd respectively. Now we consider $\frac{q(q+1)(q^2+1)}{2}=q', q'+1$. As a result, $q'=q^4+q^3+q^2+q$, $q'=q^4+q^3+q^2+q-1$. Since $|L_2(q')| \nmid |G|$, which is a contradiction. For $n>1$, $K/H \ncong L_{n+1}(q')$, similarily.

Hence, $K/H \cong B$. As a result $|K/H|=|B|$. On the other hand we know that $H \trianglelefteq K \trianglelefteq G$, where p is an isolated vertex of $\Gamma(G)$. Also by assumption we knowt that $k(K/H)$ divide $k(G)$. Hence $\frac{q(q+1)(q^2+1)}{2}=\frac{q'(q'+1)(q'^2+1)}{2}$. As a result $n=n'$. Now since $|K/H|=|B|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H=1$ and $G=K \cong B$. □

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