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New Create of Julia Sets and Some Properties

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ABSTRACT: In fractals, Picard iterations have been widely employed to study Julia set and its generalizations. This paper is dedicated to examine superior Julia set, construct Julia sets by iterative procedure $w_{i+1} = \frac{1}{2}\left[\xi_i h_i + (1 - \xi_i T h_i)\right], h_i = \mu_i g_i + (1 - \mu_i) T g_i, g_i = \gamma_i w_i + (1 - \gamma_i) T w_i$ for polynomials $Q_c = w^n + c$, where $\{\xi_i\}$, $\{\mu_i\}$, $\{\gamma_i\}$ are reals. We produce some superior Julia set for various values of $(i \geq 2)$.

Key Words: Superior Julia set, Superior escape criterions (SEC), Multistep Halpern scheme (MSHS), complex polynomials.

Contents

1. Introduction

During our study of mathematics, and in particular geometry, we noticed that the interest was in regular and smooth shapes, which have simple representation, are differentiable and continuous. There is a lot to be said about these functions, but when looking outside we see so many natural details that it is difficult to include under classical geometry, we see tree branches, cloud shapes, mountains, lightning formation, streams branching from the river, etc [1], [2], [3] and [4]. Many of these shapes have properties in common. If you look at an edge of a large cloud, you will find it difficult to identify their characteristics, and you will need additional information to be able to do so [5], [6] and [7]. But it would be almost the same if you focus on the whole cloud or part of it.

As will be seen later, there are many properties such as self-symmetry that can be see more of them in [8], [9], [10] and [11]. The study of such shapes resulted in Fractal geometry which is an important branch of mathematics.

Fractal geometry's history is attributed to mathematicians Julia and Fatou. At first, they focused on the complex functions and expounded all the properties of Julia and Fatou. At that time computer had not detected yet, so it was difficult to study the Julia set with its applications. Its graphic behavior was studied in 1975 by Mandelbrot [12]. He plotted a Julia diagram for $w^2 + c$ [13].

While studying the Julia sets, he showed that Julia had a lot of number of properties. The authors after him made several modifications to his work, the Julia set for the general complex function was presented by Lakhtakia et al. [14]. It has also been used in [15] rational complex functions and supreme complex function. The use of quaternions binary and triple complex numbers was the most obvious generalization for Julia sets and Mendelbort sets see [16], [17] and [18]. The best ways to generate fractals are iterative methods, which make fractal geometry more attractive and general, the authors use different iterations

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to generate more beautiful shapes. Rani et al. [19] introduced some generalizations for Julia sets with some non-standard iterations. After using the iterations from the fixed point theorem, in [20], [21], [22] and [23] the researchers generalized most of the studied fractals from fractal geometry by studying the different iterations, particularly Julia sets.
Our aim is to introduce new Superior Julia sets by multi-step Halpern scheme (MSHS), that is an case of a 4-step feedback procedure. This work is divided into five sections interspersed giving some basis concepts, processors superior escape criterions (SEC) for quadratic, cubic and n^{th} degree polynomials and generating superior Julia sets. As well as apply these results in creating beautiful patterns fractals

2. Preliminaries

Let $\mathbb{C}$ be the complex plane, for a point $w_0 \in mathbbC$ the orbit of a mapping $T : \mathbb{C} \to \mathbb{C}$ is the set of all iterates of w_0

$$O(t, w_0) = \{w_n : w_n = Tw_{n-1}, n = 1, 2, \cdots\} \tag{2.1}$$

The orbit $O(F, w_0)$ of F at the initial point w_0 is the sequence $\{Fw_n\}$.
Recall the original construction of Julia set, if Q is a function and $Q^k(w)$ is k^{th} iterate of Q. The set $K(Q) = \{w \in C : Q^k(w) \nrightarrow \infty\}$ is Julia's filled set and Julia set of Q is $J(Q) = \partial K(Q)$, where $J(Q)$ is the Julia set the boundary of $\partial K(Q)$ [24].

Definition 2.1 *[25] Julia set which is contains all points w_0 whose orbits of $Q_c(w) = w^2 + c$ (by (2.1)) are bounded.*

Choosing an initial point will be 0, it will be the only critical point for Q_c is equal to zero i.e., only element that $Q'_c(w) = 0$ and (2.1) called critical orbit. To study whether orbit is bounded or not, we recommend to see the following known theorem.

Theorem 2.1 *[25] If $|c| \geq 2$ and $|w| \geq c$ then $\{Q^k(w)\}$ escapes to ∞. In particular, the point $c \in J(Q)$.*

Now, we present MSHS which is a function of five tuples $(T, w_0, \xi_i, \mu_i, \gamma_i)$.

Definition 2.2 *The sequence MSHS $\{w_i\}$ is defined as initial point $w_0 \in X$ and*

$$w_{i+1} = \frac{1}{2}\left[\xi_i h_i + (1 - \xi_i T h_i)\right], h_i = \mu_i g_i + (1 - \mu_i)T g_i, g_i = \gamma_i w_i + (1 - \gamma_i)T w_i \tag{2.2}$$

where $\{\xi_i\}$,$\{\mu_i\}$, $\{\gamma_i\}$ are sequences in $[0, 1]$.

The MSHS is a good example of four-step feedback processes:

$$\begin{aligned} w_{i+1} =& \frac{1}{2}\left[\xi_i(\mu_i(\gamma_i w_i) + (1 - \gamma_i)T w_i) + (1 - \mu_i)T(w_i + (1 - \gamma_i)T w_i)\right. \\ &+ (1 - \xi_i)T(\mu_i(\gamma_i w_i + (1 - \gamma_i)T w_i)). \end{aligned}$$

Definition 2.3 *The filled of Julia set in MSHS is the filled in superior Julia set of the function Q, i.e., $SK(Q) = \{w \in \mathbb{C} : Q^k(w) \nrightarrow \infty\}$, $Q^k(w)$ is k^{th} iterate of Q and $Sk(Q)$ indicates the filled Superior set. The superior Julia set of Q is defined to be $SJ(Q) = \partial SK(Q)$, where $SJ(Q) :=$ superior Julia set.*

3. SEC for Complex Polynomials in MSHS

The SEC is very significant to the calculation and analysis of $SJ(Q)$ and its variants. Here, will be used SEC for generating $SJ(Q)$ in the quadratics, cubic and higher degree polynomials.

3.1. ESC for Quadratic Polynomials

Theorem 3.1 *Assume that $|w| \geq |c|\frac{2}{1-\xi}$, $|w| \geq |c|\frac{2}{1-\mu}$ and $|w| \geq |c|\frac{2}{1-\gamma}$, where $\xi, \mu, \gamma \in [0, 1]$ and $c \in \mathbb{C}$ (set of complex numbers). Define*
$w_1 = \frac{1}{2}[\xi h + (1 - \xi)Q_c(h)]$

$w_2 = \dfrac{1}{2}[\xi h_1 + (1-\xi)Q_c(h)]$
$\cdots$
$w_n = \dfrac{1}{2}[\xi h_{i-1} + (1-\xi)Q_c(h_{i-1})]$
where $Q_c(h)$ can be any polynomial in terms of μ and $i = 1, 2, 3, \cdots$, then $|w_i| \to \infty$ as $i \to \infty$.

Proof: Consider

$$\begin{aligned}
|g| &= |\gamma w| + (1-\gamma)Q_c(w)| \\
&= |\gamma w + (1-\gamma)(w^2 + c)| \\
&= |\gamma w + (1-\gamma)w^2 + (1-\gamma)c| \\
&\geq (1-\gamma)|w^2| - (1-\gamma)|c| - \gamma|w| \\
&\geq (1-\gamma)|w^2| - (1-\gamma)|w| - \gamma|w|, |w| \geq |c| \\
&= |w|[(1-\gamma)|w| - 1].
\end{aligned} \tag{3.1}$$

Also

$$\begin{aligned}
|h| &= |\mu g + (1-\mu)g| \\
&\geq |\mu g + (1-\mu)(g^2 + c)| \\
&\geq |\mu[|w|\,|\{(1-\gamma)|\,|w| - 1\}] + (1-\mu)[|w|\{(1-\gamma)|w| - 1\}^2 + c].
\end{aligned} \tag{3.2}$$

Since $|w| \geq \dfrac{2}{1-\lambda}$ implies $(1-\lambda)|w| - 1 > 1$

$$|w|[(1-\gamma)|w| - 1] > |w|. \tag{3.3}$$

Using (3.3) in (3.2), we have

$$\begin{aligned}
|h| &\geq |\mu|w\left|(1-\mu)(|w|^2 + c)\right| \\
&\geq \left|(1-\mu)|w|^2 + \mu|w|\right| - (1-\gamma)|c| \\
&\geq \left|(1-\mu)w^2 + (1-\mu)|w|\right| - \mu|w|, \text{since} |w| \geq |c| \\
&\geq |w|\left((1-\mu)|w| - 1\right).
\end{aligned} \tag{3.4}$$

For $w_i = \dfrac{1}{2}\left[\xi h_{i-1} + (1-\xi)Q(h_{i-1})\right]$

$$\begin{aligned}
|w_1| &= \frac{1}{2}\left|\xi h + (1-\xi)Q(h)\right| \\
&= \left|\xi h + (1-\xi)(h^2 + c)\right| \\
&\geq \frac{1}{2}\left|\xi|w|\right|\left((1-\mu)|w| - 1\right) + (1-\mu)\left[\left(|w|(1-\mu)|w| - 1\right)^2 + c\right].
\end{aligned} \tag{3.5}$$

Since $|w| \geq \dfrac{2}{1-\mu} \Rightarrow (1-\mu)|w| - 1 > 1$

$$\Rightarrow |w|(1-\mu)|w| - 1 > |w|. \tag{3.6}$$

Using (3.6) in (3.5), to have

$$\begin{aligned}
|w_1| &\geq \frac{1}{2}\left|\xi|w| + (1-\xi)(|w|^2 - 1)\right| \\
&\geq \frac{1}{2}|w|\left((1-\xi)|w| - 1\right)
\end{aligned}$$

$\Rightarrow |w_1| \geq \frac{1}{2}|w|\,((1-\xi)|w|-1)$.

Since $|w| \geq |c| > \frac{2}{(1-\xi)}$, $|w| \geq |c| > \frac{2}{(1-\mu)}$, $|w| \geq |c| > \frac{2}{(1-\gamma)}$ exist.

Therefore, $(1-\xi)|w|-1 > 1$. So, $\exists \gamma > 0 \ni \xi|w| - 1 > \xi + 1$. $\Rightarrow |w_1| > \frac{1+\gamma}{2}|w|$.

In particular, $|w_i| > |w|$ and by repeating this argument i times to get $|w_i| > (1+\gamma)^i|w| \to \infty$. □

Theorem (3.1) implies the following corollaries:

Corollary 3.1 *Suppose that* $|c| > \frac{1}{(1-\xi)}|c| > \frac{2}{(1-\mu)}$ *and* $|c| > \frac{2}{(1-\gamma)}$, *then the MSHS* $(Q_c, 0, \xi, \mu, \gamma)$ *escapes to* ∞.

The using of SEC, in the proof of Theorem (3.1), gives a little more information and we used only $|w| > |c| > \frac{2}{(1-\xi)}|c| > \frac{2}{(1-\mu)}$ and $|c| > \frac{2}{(1-\gamma)}$. After that we give improvement of the SEC as follow:

Corollary 3.2 *(SEC): Suppose* $|w| > max\left\{|c|, \frac{2}{(1-\xi)}, \frac{2}{(1-\mu)}, \frac{2}{(1-\gamma)}\right\}$, *then* $|w_i| > (1+\gamma)^i|w|$ *and* $|w_i| \to \infty$ *as* $i \to \infty$.

To have the next result, applying Corollary (3.2) to $|w_r|$ for some $r \geq 0$.

Corollary 3.3 *Suppose* $|w_r| > max\left\{|c|, \frac{2}{(1-\xi)}, \frac{2}{(1-\mu)}, \frac{2}{(1-\gamma)}\right\}$, *for some* $r \geq 0$, *then* $|w_{r+1}| > (1+\gamma)^i|w_r|$ *and* $|w_i| \to \infty$ *as* $i \to \infty$.

3.2. SEC for Cubic Polynomials

For Cubic Polynomials $Q_c(w) = w^3 + c$, where $c \in \mathbb{C}$ we obtain

Theorem 3.2 *Suppose that* $|w| \geq |c| > \left(\frac{2}{1-\xi}\right)^{1/2}$, $|w| \geq |c| > \left(\frac{2}{1-\mu}\right)^{1/2}$, $|w| \geq |c| > \left(\frac{2}{1-\gamma}\right)^{1/2}$, *where* $0 < \xi < 1$, $0 < \mu < 1$, $0 < \gamma < 1$ *and* $c \in \mathbb{C}$. *Define*

$w_1 = \frac{1}{2}[\xi h + (1-\xi)Q_c(h)]$

$w_2 = \frac{1}{2}[\xi h_1 + (1-\xi)Q_c(h)]$

$\cdots$

$w_n = \frac{1}{2}[\xi h_{i-1} + (1-\xi)Q_c(h_{i-1})]$, $i = 1, 2, 3, \cdots$

where $Q_c(h)$ *is the function of* μ, *then* $|w_i| \to \infty$ *as* $i \to \infty$.

Proof: Consider

$$\begin{aligned}
|g| &= |\gamma w + (1-\gamma)Q_c(w)| \\
&= |\gamma w + (1-\gamma)(w^3 + c)| \\
&= |\gamma w + (1-\gamma)w^3 + (1-\gamma)c| \\
&\geq |\gamma w + | - (1-\gamma)|c| \\
&\geq |\gamma w + (1-\gamma)w^3| - (1-\gamma)|w|, |w| \geq |c| \\
&\geq (1-\gamma)|w|^3 - \gamma|w| - |w| + \gamma|w| \\
&= |w|\left[(1-\gamma)|w|^2 - 1\right] \qquad (3.7)
\end{aligned}$$

Also

$$\begin{aligned}|h| &= |\mu g + (1-\mu)Q_c(g)| \\ &= |\mu g + (1-\mu)(g^3 + c)| \\ &= |\mu g + (1-\mu)g^3 + (1-\mu)c|\end{aligned}$$

Since $|w| \geq \left(\frac{2}{1-\gamma}\right)^{1/2} \Rightarrow (1-\gamma)|w|^2 - 1 > 1$

$$|h| \geq \left|\mu|w|(\gamma|w|^2 - 1) + (1-\mu)\left[|w|\left((1-\gamma)|w|^2 - 1\right)\right]^3 + (1-\mu)c\right| \tag{3.8}$$

$$|w|(\gamma|w|^2 - 1) > |w| \tag{3.9}$$

Using (3.9) in (3.8), we have

$$\begin{aligned}|h| &\geq |\mu|w| + (1-\mu)|w|^3 + (1-\mu)c| \\ &\geq \left|(1-\mu)|w|^3 + \mu|w|\right| - (1-\mu)|w| \\ &\geq (1-\mu)|w|^3 - \mu|w| - |w| + \mu|w| \\ &= |w|\left((1-\mu)|w^2| - 1\right)\end{aligned} \tag{3.10}$$

Now, for

$$\begin{aligned}w_n &= \frac{1}{2}\xi h_{n-1} + (1-\xi)Q_c(h_{n-1}) \\ w_1 &= \frac{1}{2}\left[\xi h + (1-\xi)Q_c(h)\right] \\ &= \frac{1}{2}\left|\xi h + (1-\gamma)(h^3 + c)\right| \\ &\geq \frac{1}{2}\left|\xi|w|(\mu|w^2| - 1) + (1-\xi)\left[\left\{|w|(\mu|w^2| - 1)^3 + c\right\}\right]\right|\end{aligned} \tag{3.11}$$

Since $|w| \geq \left(\frac{2}{1=\mu}\right)^{1/2} \Rightarrow (1-\mu)|w|^2 - 1 > 1$

$$|w|\left(((1-\mu)|w^2| - 1)\right) > |w| \tag{3.12}$$

Using (3.12) in (3.11), we get

$$\begin{aligned}|w_1| &\geq \frac{1}{2}|\xi|w| + (1-\xi)(|w^3| - 1)| \\ &\geq \frac{1}{2}(1-\xi)|w|^3 + \xi|w| + (1-\xi)|w| \\ &\geq \frac{1}{2}|w|\left((1-\xi)|w|^2 - 1\right) \\ \Rightarrow |w_1|&\frac{1}{2}|w|\left((1-\xi)|w|^2 - 1\right)\end{aligned}$$

Since $|w| \geq \left(\frac{2}{1-\xi}\right)^{1/2}$, $|w| \geq \left(\frac{2}{1-\mu}\right)^{1/2}$ and $|w| \geq \left(\frac{2}{1-\gamma}\right)^{1/2}$ exist.
Therefore, getting $(1-\xi)|w|^2 - 1 > 1$.
Hence, $\exists \gamma > 1 \ni |w_1| \geq \gamma|w|$. Repeating this inequality i times, to get $|w_i| \geq \gamma^i|w|$. Therefore, the orbit of w under the cubic polynomial $Q_c(w) \to \infty$. □

A consequence, we have the following corollaries.

Corollary 3.4 *Let $Q_c(w) = w^3 + c$, where c is any complex number.*
If $|w| > max\left\{|c|, \left(\frac{2}{1-\xi}\right)^{1/2}, \left(\frac{2}{1-\mu}\right)^{1/2}, \left(\frac{2}{1-\gamma}\right)^{1/2}\right\}$, *then $|w_i| \to \infty$ as $i \to \infty$. This gives SEC for a cubic polynomial*

Corollary 3.5 *For some $r \geq 0$, let us suppose*
$|w_r| > max\left\{|c|, \left(\frac{2}{1-\xi}\right)^{1/2}, \left(\frac{2}{1-\mu}\right)^{1/2}, \left(\frac{2}{1-\gamma}\right)^{1/2}\right\}$. *Then $|w_{r+1}| > \gamma|w_r|$ and $|w_i| \to \infty$ as $i \to \infty$.*

3.3. A General SEC

For polynomials of the form $M_c(w) = w^i + c$ we prove the following

Theorem 3.3 *For a general function $M_c(w) = w^i + c, i = 1, 2, 3,$ cdots where $0 < \xi < 1$, $0 < \mu < 1$, $0 < \gamma < 1$ and $c \in \mathbb{C}$. Define*
$w_1 = \xi h + (1-\xi)M_c(h)]$
$w_2 = \xi h_1 + (1-\xi)M_c(h_1)$
$\cdots$
$w_n = \xi h_{i-1} + (1-\xi)Q_c(h_{i-1}),\ i = 1, 2, 3, \cdots$
Then thee general SEC is max $\left\{|c|, \left(\frac{2}{1-\xi}\right)^{1/i-1}, \left(\frac{2}{1-\mu}\right)^{1/i-1}, \left(\frac{2}{1-\gamma}\right)^{1/i-1}\right\}$.

Proof: By indiction.
For $i = 1 \Rightarrow M_c(w) = w + c \Rightarrow |w| > \max\{|c|, 0, 0, 0\}$.
For $i = 2 \Rightarrow M_c(w) = w^2 + c$, so by Theorem (3.1) the SEC is $|w| > \max\left\{|c|, \frac{2}{1-\xi}, \frac{2}{1-\mu}, \frac{2}{1-\lambda}\right\}$.
Similarly, for $i = 3, \Rightarrow M_c(w) = w^3 + c$. Then the SEC from above theorem is
$|w| > \max\left\{|c|, \left(\frac{2}{1-\xi}\right)^{1/2}, \left(\frac{2}{1-\mu}\right)^{1/2}, \left(\frac{2}{1-\gamma}\right)^{1/2}\right\}$.
thus the theorem is true for $= 1, 2, 3, \cdots$.
Now, assume that theorem is true for any i. To prove that its true for $i+1$.
Let $M_c(w) = w^{i+1} + c$ and $|w| \geq |c| > \left(\frac{2}{1-\xi}\right)^{1/i}$, $|w| \geq |c| > \left(\frac{2}{1-\mu}\right)^{1/i}$ and $|w| \geq |c| > \left(\frac{2}{1-\gamma}\right)^{1/i}$.
Then,

$$\begin{aligned}
|q| &= |\gamma w + (1-\gamma)Q_c(w)| \\
&= |\gamma w + (1-\gamma)(w^{i+1} + c)| \\
&\geq \left|(1-\gamma)(w^{i+1} + c)\right| - \gamma|w| \\
&= \left|(1-\gamma)w^{i+1} + (1-\gamma)c\right| - \gamma|w| \\
&\geq \left|(1-\gamma)w^{i+1}\right| - |c| + \gamma|c| - \gamma|w|, |w| \geq |c| \\
&\geq \left|(1-\gamma)w^{i+1}\right| - |w| + \gamma|w| - \gamma|w| \\
&= |w|\left((1-\gamma)|w^i| - 1\right) \qquad (3.13)
\end{aligned}$$

Also,

$$\begin{aligned}
|h| &= |\mu g + (1-\mu)Q_c(g)| \\
&= \left|\mu g + (1-\mu)(g^{i+1} + c)\right| \\
&\geq \left|\mu|w|\left((1-\gamma)|w^i|\right) + (1-\mu)\left\{\left[|w|\left((1-\gamma)|w^i\right) - 1\right]^{i+1} + c\right\}\right| \qquad (3.14)
\end{aligned}$$

Since $|w| \geq \left(\frac{1}{1-\gamma}\right)^{1/i}$ implies $(1-\gamma)|w|^i - 1 > 1$ so

$$|w|\left((1-\gamma)|w|^i - 1\right) > |w| \tag{3.15}$$

Using (3.15) in (3.14), we have

$$\begin{aligned}|h| &\geq \left|\mu|w| + (1-\mu)\left(|g^{i+1}| + c\right)\right| \\ &\geq \left|(1-\mu)|g^{i+1}| + \mu|w|\right| - \mu|c| \\ &\geq \left|(1-\mu)|g^{i+1}| + \mu|w|\right| - \mu|w|, \text{since}|w| \geq |c| \\ &\geq |w|\left((1-\mu)|w|^i - 1\right)\end{aligned} \tag{3.16}$$

For $w_i = \xi h_i + (1-\xi)M_c(h_{i-1})$, we have

$$\begin{aligned}|w_1| &= |\xi h + (1-\xi)M_c(h)| \\ &= |\xi h + (1-\xi)(h^{i+1} + c)| \\ &\geq \left|\xi\left[|w|\left((1-\mu)|w^i|\right) - 1\right] + (1-\xi)\left[|w|\left((1-\mu)|w^i| - 1\right)^{i+1} + c\right]\right|\end{aligned} \tag{3.17}$$

Since $|w| > \left(\frac{2}{1-\mu}\right)^{1/i}$ implies $(1-\mu)|w^i| - 1 > 1$. So

$$|w|\left((1-\mu)|w^i| - 1\right) > |w| \tag{3.18}$$

Using (3.18)in (3.17), to have

$$\begin{aligned}|w_1| &= \left|\xi|w| + (1-\xi)|w|^{i+1} + c\right| \\ &\geq \left|\xi|w| + |w|^{i+1} - \xi|w^{i+1}| + (1-\xi)c\right| \\ &\geq |w|^{i+1} - \xi|w| - \xi|w^{i+1}| + (1-\xi)|c| \\ &\geq |w^{i+1}| - \xi|w| - \xi|w^{i+1}| - |w| + \xi|w|, \text{since}|w| \geq |c| \\ &\geq |w|\left((1-\xi)|w^i| - 1\right) \\ \Rightarrow |w_1| &\geq |w|\left((1-\xi)|w^i| - 1\right)\end{aligned}$$

Since $|w| > \left(\frac{2}{1-\xi}\right)^{1/i}$, $|w| > \left(\frac{2}{1-\mu}\right)^{1/i}$, and $|w| > \left(\frac{2}{1-\gamma}\right)^{1/i}$, exist.
We have $(1-\xi)|w^i| - 1 > 1 + \gamma > 1$. In particulate,

$$\begin{aligned}|w_1| &> (1+\gamma)|w| \\ &\vdots \\ |w_i| &> (1+\gamma)^i|w|\end{aligned}$$

Hence, $|w_i| \to \infty$ as $i \to \infty$ □

4. Superior Julia Sets in MSHS and Application

In this section, we have generated some superior Julia sets in (MSHS) for quadratic, cubic and higher polynomials. Now, we utilize the notion of Wang et, al. [26] about study escape lines around the filled superior Julia sets. This enabled to have desired purpose. In modern era, Julia sets present many applications in data transmission, image texture analysis, cryptography, image processing and other [27]. Researchers have presented various iterative procedures in the study of fractal theory and demonstrated

that it converges more quickly than the orbit in (2.1). [28], [29], [20], [31] and [32] showed there are some orbits that they converge faster. Therefore, these considerations are employed to form MSHS in this work.
For superior Julia Sets in the quadratic polynomial $Q_c(w) = w^2 + c$ the graphical representation of superior Julia sets illustrate the following observations:

1. When the value of c is fixed and the value of ξ, μ are decreasing, the quadratic superior Julia set will be disconnected as shown in figure 1.
2. With an increase in values ξ, μ and γ and a fixed in the value of c, we notice that the Julia set is more slimmer as shown in figure 2.

5. Conclusion

Throughout this work, superior Julia sets for complex *ith* degree polynomials (spatially, quadratic and cubic) have been generated via MSHS procedures. In our analysis, we find the following conclusions:

1. Using MSHS procedure, $20 - 35$ iterations are usually enough for a good approximation of superior Julia sets, while other iterative procedures Mann, Ishikawa, Noor etc. require more number of iterations for a good approximation of fractals
2. The graphical representation of superior Julia sets showed that their shapes depend upon the values of ξ, μ and γ.
3. For higher degree polynomials the superior Julia sets has the rotational symmetry about the center and their shapes.
4. The results obtained in this paper may provoke further research in the field of Fractal theory.

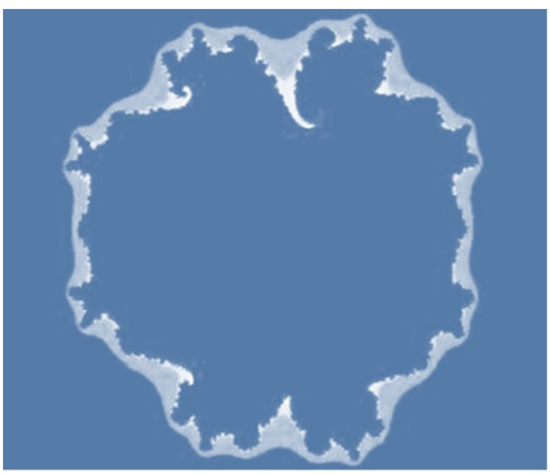

Figure 1: Quadratic Julia set when $\xi = 1.0$, $\mu = 1.0$, $\gamma = 1.0$,and $c = -1.37$

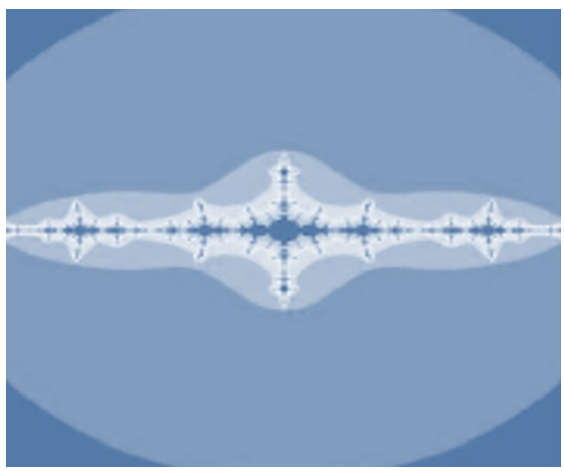

Figure 2: Quadratic Julia set when $\xi = 0.1$, $\mu = 0.1$, $\gamma = 0.5$,and $c = -0.5 - 0.5i$

As for superior Julia sets in the cubic polynomial $Q_c(w) = w^3 + c$, the following observations are obtained from the graphical representations of the cube:

1. When any two parameters out of ξ, μ and γ are fixed then the number of loops in Cubic superior Julia sets increase with the increase in the third parameter and become fattier when the second parameter increases as shown in figure 3.
2. We notice that the Cubic Superior Julia sets become more decorated when we set equal values for all parameters and increase them monotonously as shown in figure 4.

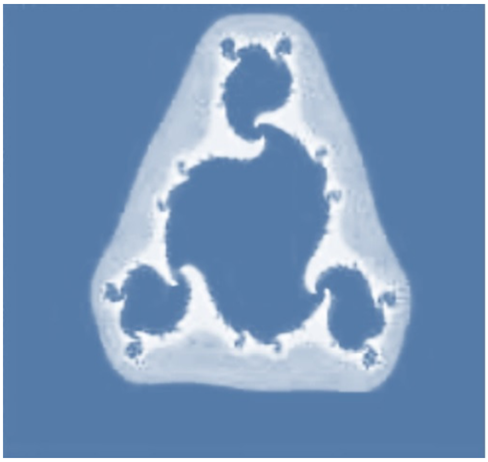

Figure 3: Cubic Julia set when $\xi = 0.3$, $\mu = 0.1$, $\gamma = 0.1$, and $c = 0.1 + 0.9i$

Figure 4: Cubic Julia set when $\xi = 0.7 - 0.4i$, $\mu = 0.7 - 0.4i$, $\gamma = 0.7 - 0.4i$, and $c = 0.7$

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