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Evaluation of a Class of Definite Integrals Involving Generalized Hypergeometric Functions

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ABSTRACT: The major goal of this study is to develop eleven new classes of integrals involving generalised hypergeometric function by using highly intriguing and useful generalisations of several classical summation theorems for the ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$ generalized hypergeometric series obtained by Masjed- Jamei and Koepf and MacRobert integral. Several special cases have also been mentioned.

Key Words: Gauss's hypergeometric function, Kummer function, classical summation theorems, MacRobert integral.

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1. Introduction

Special functions, particularly the hypergeometric function ${}_2F_1$ and confluent hypergeometric function ${}_1F_1$ are well known for their importance in solving several problems in Mathematical Physics, Engineering, and Applied Mathematics. Many mathematicians have been encouraged by this fact to look into a number of new and fascinating results involving these functions.

The Gauss's hypergeometric function is defined by [1, 3, 14, 15, 17]

$$
{}_2F_1\left[\begin{matrix} u, & v \\ & w \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(u)_k (v)_n}{(w)_k} \frac{z^k}{k!}. \qquad (|z| < 1, w \neq 0, -1, -2, \dots)
$$

and confluent hypergeometric function is defined by [1, 3, 14, 15, 17]

$$
{}_1F_1\left[\begin{matrix} u \\ w \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(u)_k}{(w)_k} \frac{z^k}{k!}
$$

which converges everywhere.

Both of the aforementioned functions are special cases of the generalised hypergeometric function with p numerator and q denominator parameters defined by [1, 3, 17, 19, 20, 21]

$$
{}_pF_q\left[\begin{matrix} u_1, & \cdots, & u_p \\ v_1, & \cdots, & v_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (u_i)_k}{\prod_{i=1}^{q} (v_i)_k} \frac{z^k}{k!} \tag{1.1}
$$

where $(u)_k$ denotes the well known Pochhammer symbol [8] for any complex number u defined as

$$
\begin{aligned}
(u)_k &= \frac{\Gamma(u+k)}{\Gamma(u)} \\
&= \begin{cases} 1 & (k = 0, u \in \mathbb{C} \setminus \{0\}) \\ u(u+1)\cdots(u+k-1) & (k \in \mathbb{N}, u \in \mathbb{C}), \end{cases}
\end{aligned} \tag{1.2}
$$

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where $\Gamma(z)$ is the well known gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$$

provided $\mathrm{Re}(z) > 0$.

Furthermore, the ratio test may simply verify [1,2] that the series (1.1) is convergent for all $p \leq q$. Also for $p = q+1$ it converges in $|z| < 1$, for $p < q+1$ converges everywhere and for $p > q+1$ converges nowhere $(z \neq 0)$.

Further, if $p = q+1$, it converges absolutely for $|z| = 1$ provided

$$\delta = \Re\left(\sum_{j=1}^{q} v_j - \sum_{j=1}^{p} u_j\right) > 0$$

holds and is conditionally convergent for $|z| = 1$ and $z \neq 1$ if $-1 < \delta \leq 0$ and diverges for $|z| = 1$ and $z \neq 1$ if $\delta \leq -1$. For more details, we refer [17].

It's worth noting that anytime a generalised hypergeometric function reduces to the gamma function, the results are quite relevant in terms of applications.

Furthermore, in order for the work to be self-contained, we will state the following summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_7F_6$ [18].

$$\textit{Gauss's Theorem}: \quad \textit{For} \quad \Re(w-u-v) > 0 \tag{1.3}$$

$${}_2F_1\left[\begin{matrix} u, \quad v \\ w \end{matrix}; 1\right] = \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}$$

$$\textit{Kummer's Theorem}: \tag{1.4}$$

$${}_2F_1\left[\begin{matrix} u, \quad v \\ 1+u-v \end{matrix}; -1\right] = \frac{\Gamma(1+u-v)\Gamma(1+\frac{1}{2}u)}{\Gamma(1-v+\frac{1}{2}u)\Gamma(1+u)}$$

$$\textit{Second Gauss's Theorem}: \tag{1.5}$$

$${}_2F_1\left[\begin{matrix} u, \quad v \\ \frac{1}{2}(u+v+1) \end{matrix}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(u+v+1))}{\Gamma(\frac{1}{2}(u+1))\Gamma(\frac{1}{2}(v+1))}$$

$$\textit{Bailey's Theorem}: \tag{1.6}$$

$${}_2F_1\left[\begin{matrix} u, \quad 1-u \\ v \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}(v+1))}{\Gamma(\frac{1}{2}(u+v))\Gamma(\frac{1}{2}(v-u+1))}$$

$$\textit{Dixon's Theorem}: \textit{For} \quad \Re(u-2v-2w) > -2 \tag{1.7}$$

$$\begin{aligned}{}_3F_2&\left[\begin{matrix} u, & v, & w \\ 1+u-v, & 1+u-w \end{matrix}; 1\right] \\ &= \frac{\Gamma(1+\frac{1}{2}u)\Gamma(1+u-v)\Gamma(1+u-w)\Gamma(1-v-w+\frac{1}{2}u)}{\Gamma(1+u)\Gamma(1-v+\frac{1}{2}u)\Gamma(1-w+\frac{1}{2}u)\Gamma(1+u-v-w)}\end{aligned}$$

$$Watson's \quad Theorem: \quad For \ \Re(2w-u-v)>1 \tag{1.8}$$

$${}_3F_2\left[\begin{matrix} u, & v, & w \\ \frac{1}{2}(u+v+1), & 2w \end{matrix};1\right] = \frac{\sqrt{\pi}\,\Gamma(w+\frac{1}{2})\Gamma(\frac{1}{2}(u+v+1))\Gamma(w-\frac{1}{2}(u+v-1))}{\Gamma(\frac{1}{2}(u+1))\Gamma(\frac{1}{2}(v+1))\Gamma(w-\frac{1}{2}(u-1))\Gamma(w-\frac{1}{2}(v-1))}$$

$$Whipple's \quad Theorem: \quad For \ \Re(v)>0 \tag{1.9}$$

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} u, & 1-u, & v \\ w, & 2v-w+1 \end{matrix};1\right] \\ &= \frac{\pi\, 2^{1-2v}\,\Gamma(w)\Gamma(2v-w+1)}{\Gamma(\frac{1}{2}(u+w))\Gamma(v+\frac{1}{2}(u-w+1))\Gamma(\frac{1}{2}(1-u+w))\Gamma(v+1-\frac{1}{2}(u+w))} \end{aligned}$$

$$Saalschütz's \quad Theorem: \tag{1.10}$$

$${}_3F_2\left[\begin{matrix} u, & v, & -n \\ w, & 1+u+v-w-n \end{matrix};1\right] = \frac{(w-u)_n(w-v)_n}{(w)_n(w-u-v)_n}$$

$$Second \quad Whipple's \quad Theorem: \tag{1.11}$$

$${}_4F_3\left[\begin{matrix} u, & 1+\frac{1}{2}u, & v, & w \\ \frac{1}{2}u, & u-v+1, & u-w+1 \end{matrix};-1\right] = \frac{\Gamma(u-v+1)\Gamma(u-w+1)}{\Gamma(u+1)\Gamma(u-v-w+1)}$$

$$Dougall's \quad Theorem: \quad For \ \Re(u-w-d-e)>-1 \tag{1.12}$$

$$\begin{aligned} &{}_5F_4\left[\begin{matrix} u, & 1+\frac{1}{2}u, & w, & d, & e \\ \frac{1}{2}u, & u-w+1, & u-d+1, & u-e+1 \end{matrix};1\right] \\ &\quad = \frac{\Gamma(u-w+1)\Gamma(u-d+1)\Gamma(u-e+1)\Gamma(u-w-d-e+1)}{\Gamma(u+1)\Gamma(u-d-e+1)\Gamma(u-w-e+1)\Gamma(u-w-d+1)} \end{aligned}$$

$$Second \quad Dougall's \quad Theorem: \tag{1.13}$$

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} u, & 1+\frac{1}{2}u, & v, & w, & d, & 1+2u-v-w-d+n, & -n \\ \frac{1}{2}u, & u-v+1, & u-w+1, & u-d+1, & v+w+d-u-n, & u+1+n \end{matrix};1\right] \\ &\quad = \frac{(u+1)_n(u-v-w+1)_n(u-v-d+1)_n(u-w-d+1)_n}{(u+1-v)_n(u+1-w)_n(u+1-d)_n(u+1-v-w-d)_n} \end{aligned}$$

For finite sums of hypergeometric series, we will use the following symbol

$${}_pF_q^{(m)}\left[\begin{matrix} u_1, & \cdots, & u_p \\ v_1, & \cdots, & v_q \end{matrix};z\right] = \sum_{n=0}^{m} \frac{\prod_{i=1}^{p}(u_i)_n}{\prod_{i=1}^{q}(v_i)_n}\,\frac{z^n}{n!},$$

where for instance

$$\overset{(-1)}{{}_pF_q}(z)=0,\quad \overset{(0)}{{}_pF_q}(z)=1,\quad \overset{(1)}{{}_pF_q}(z)=1+\frac{u_1\cdots u_p}{v_1\cdots v_q}z.$$

By using the following relation [16]

$$\begin{aligned}
&{}_pF_q\left[\begin{matrix}u_1,\cdots,\ u_{p-1}, & 1\\ v_1,\cdots,\ v_{q-1}, & m\end{matrix};z\right] \qquad (1.14)\\
&=\frac{\Gamma(v_1)\cdots\Gamma(v_{q-1})}{\Gamma(u_1)\cdots\Gamma(u_{p-1})}\,\frac{\Gamma(u_1-m+1)\cdots\Gamma(u_{p-1}-m+1)}{\Gamma(v_1-m+1)\cdots\Gamma(v_{q-1}-m+1)}\,\frac{(m-1)!}{z^{m-1}}\\
&\times\left\{{}_{p-1}F_{q-1}\left[\begin{matrix}u_1-m+1,\cdots,\ u_{p-1}-m+1\\ v_1-m+1,\cdots,\ v_{q-1}-m+1\end{matrix};z\right]\right.\\
&\left.-\overset{(m-2)}{{}_{p-1}F_{q-1}}\left[\begin{matrix}u_1-m+1,\cdots,\ u_{p-1}-m+1\\ v_1-m+1,\cdots,\ v_{q-1}-m+1\end{matrix};z\right]\right\},
\end{aligned}$$

Masjed-Jamei and Koepf [13] have recently established generalizations of the classical summation theorems (1.3) to (1.13) in the following form:

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}u, & v, & 1\\ & w, & m\end{matrix};1\right] \qquad (1.15)\\
&=\frac{\Gamma(m)\Gamma(w)\Gamma(u-m+1)\Gamma(v-m+1)}{\Gamma(u)\Gamma(v)\Gamma(w-m+1)}\\
&\times\left\{\frac{\Gamma(w-m+1)\Gamma(w-u-v+m-1)}{\Gamma(w-u)\Gamma(w-v)}-\overset{(m-2)}{{}_2F_1}\left[\begin{matrix}u-m+1,\ v-m+1\\ w-m+1\end{matrix};1\right]\right\}\\
&=\chi_1
\end{aligned}$$

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}u, & v, & 1\\ u-v+m, & m\end{matrix};-1\right] \qquad (1.16)\\
&=(-1)^{m-1}\frac{\Gamma(m)\Gamma(u-v+m)\Gamma(u-m+1)\Gamma(v-m+1)}{\Gamma(u)\Gamma(v)\Gamma(u-v+1)}\\
&\times\left\{\frac{\Gamma(u-v+1)\Gamma(1+\frac{1}{2}(u-m+1))}{\Gamma(2+u-m)\Gamma(m-v+\frac{1}{2}(u-m+1))}-\overset{(m-2)}{{}_2F_1}\left[\begin{matrix}u-m+1,\ v-m+1\\ u-v+1\end{matrix};-1\right]\right\}\\
&=\chi_2
\end{aligned}$$

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}u, & v, & 1\\ \frac{1}{2}(u+v+1), & m\end{matrix};\frac{1}{2}\right] \qquad (1.17)\\
&=2^{m-1}\frac{\Gamma(m)\Gamma(\frac{1}{2}(u+v+1))\Gamma(u-m+1)\Gamma(v-m+1)}{\Gamma(u)\Gamma(v)\Gamma(-m+1+\frac{1}{2}(u+v+1))}\\
&\times\left\{\frac{\sqrt{\pi}\,\Gamma(-m+1+\frac{1}{2}(u+v+1))}{\Gamma(1+\frac{1}{2}(u-m))\Gamma(1+\frac{1}{2}(v-m))}-\overset{(m-2)}{{}_2F_1}\left[\begin{matrix}u-m+1,\ v-m+1\\ -m+1+\frac{1}{2}(u+v+1)\end{matrix};\frac{1}{2}\right]\right\}\\
&=\chi_3
\end{aligned}$$

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} u,\ 2m-u-1,\ 1 \\ v,\quad m \end{matrix};\frac{1}{2}\right] \\
&= 2^{m-1}\frac{\Gamma(m)\Gamma(v)\Gamma(u-m+1)\Gamma(m-u)}{\Gamma(u)\Gamma(2m-u-1)\Gamma(v-m+1)} \\
&\times\left\{\frac{\Gamma(\frac{1}{2}(v-m+1))\Gamma(\frac{1}{2}(v-m+2))}{\Gamma(-m+1+\frac{1}{2}(u+v))\Gamma(\frac{1}{2}(v-u+1))} - {}_2^{(m-2)}F_1\left[\begin{matrix} u-m+1,\ m-u \\ v-m+1 \end{matrix};\frac{1}{2}\right]\right\} \\
&= \chi_4
\end{aligned}
\tag{1.18}
$$

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} u, & v, & w, & 1 \\ u-v+m, & u-w+m, & m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(u-v+m)\Gamma(u-w+m)\Gamma(u+1-m)\Gamma(v+1-m)\Gamma(w+1-m)}{\Gamma(u)\Gamma(v)\Gamma(w)\Gamma(u-v+1)\Gamma(u-w+1)} \\
&\times\left\{\frac{\Gamma(\frac{1}{2}(u+3-m))\Gamma(u-v+1)\Gamma(u-w+1)\Gamma(-v-w+\frac{1}{2}(u+3m-1))}{\Gamma(u+2-m)\Gamma(-v+\frac{1}{2}(u+m+1))\Gamma(-w+\frac{1}{2}(u+m+1))\Gamma(u-v-w+m)}\right. \\
&\quad\left. - {}_3^{(m-2)}F_2\left[\begin{matrix} u-m+1,\ v-m+1,\ w-m+1 \\ u-v+1,\quad u-w+1 \end{matrix};1\right]\right\} \\
&= \chi_5
\end{aligned}
\tag{1.19}
$$

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} u, & v, & w, & 1 \\ \frac{1}{2}(u+v+1), & 2w+1-m, & m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(\frac{1}{2}(u+v+1))\Gamma(2w+1-m)\Gamma(u+1-m)\Gamma(v+1-m)\Gamma(w+1-m)}{\Gamma(u)\Gamma(v)\Gamma(w)\Gamma(-m+\frac{1}{2}(u+v+3))\Gamma(2w-2m+2)} \\
&\times\left\{\frac{\sqrt{\pi}\,\Gamma(w-m+\frac{3}{2})\Gamma(-m+\frac{1}{2}(u+v+3))\Gamma(w-\frac{1}{2}(u+v-1))}{\Gamma(1+\frac{1}{2}(u-m))\Gamma(1+\frac{1}{2}(v-m))\Gamma(w+1-\frac{1}{2}(u+m))\Gamma(w+1-\frac{1}{2}(v+m))}\right. \\
&\quad\left. - {}_3^{(m-2)}F_2\left[\begin{matrix} u-m+1,\ v-m+1,\ w-m+1 \\ -m+1+\frac{1}{2}(u+v+1),\ 2w-2m+2 \end{matrix};1\right]\right\} \\
&= \chi_6
\end{aligned}
\tag{1.20}
$$

.

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} u, & 2m-1-u, & v, & 1 \\ w, & 2v-w+1, & m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(w)\Gamma(2v-w+1)\Gamma(m-u)\Gamma(u+1-m)\Gamma(v+1-m)}{\Gamma(u)\Gamma(v)\Gamma(2m-1-u)\Gamma(w+1-m)\Gamma(2v-w-m+2)} \\
&\times\left\{\frac{\pi\, 2^{2m-2v-1}\,\Gamma(w-m+1)}{\Gamma(-m+1+\frac{1}{2}(u+w))\Gamma(-m+1+v+\frac{1}{2}(u-w+1))\Gamma(\frac{1}{2}(1-u+w))}\right. \\
&\quad\left.\times\frac{\Gamma(2v-w-m+2)}{\Gamma(v+1-\frac{1}{2}(u+w))} - {}_3^{(m-2)}F_2\left[\begin{matrix} u-m+1,\ v-m+1,\ m-u \\ w-m+1,\ 2v-w-m+2 \end{matrix};1\right]\right\} \\
&= \chi_7
\end{aligned}
\tag{1.21}
$$

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} u,\ v,\ -n+m-1,\ 1 \\ w,\ 1+u+v-w-n,\ m \end{matrix};1\right] = \frac{(m-1)!\,(1-w)_{m-1}}{(1-u)_{m-1}(1-v)_{m-1}} \\
&\quad\times \frac{(w-u-v+n)_{m-1}}{(n+2-m)_{m-1}} \times \left\{ \frac{(w-u)_n(w-v)_n}{(w+1-m)_n(w-u-v+m-1)_n} \right. \\
&\quad\left. - {}^{(m-2)}_{\ \ 3}F_2\left[\begin{matrix} u-m+1,\ v-m+1,\ -n \\ w-m+1,\ 2+u+v-w-m-n \end{matrix};1\right] \right\} \\
&= \chi_8
\end{aligned} \tag{1.22}
$$

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} u,\ \frac{1}{2}(u+m+1),\ v,\ w,\ 1 \\ \frac{1}{2}(u+m-1),\ u-v+m,\ u-w+m,\ m \end{matrix};-1\right] = (-1)^{m-1}\Gamma(m) \\
&\times \frac{\Gamma(\frac{1}{2}(u+m-1))\Gamma(u-v+m)\Gamma(u-w+m)\Gamma(\frac{1}{2}(u-m+3))\Gamma(u-m+1)}{\Gamma(u)\Gamma(v)\Gamma(w)\Gamma(\frac{1}{2}(u+m+1))\Gamma(\frac{1}{2}(u-m+1))} \\
&\times \frac{\Gamma(v+1-m)\Gamma(w+1-m)}{\Gamma(u-v+1)\Gamma(u-w+1)} \times \left\{ \frac{\Gamma(1+u-v)\Gamma(1+u-w)}{\Gamma(2-m+u)\Gamma(m+u-v-w)} \right. \\
&\quad\left. - {}^{(m-2)}_{\ \ 4}F_3\left[\begin{matrix} u-m+1,\ v-m+1,\ \frac{1}{2}(u-m+3),\ w-m+1 \\ \frac{1}{2}(u-m+1),\ u-v+1,\ u-w+1 \end{matrix};-1\right] \right\} \\
&= \chi_9
\end{aligned} \tag{1.23}
$$

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} u,\ \frac{1}{2}(u+m+1),\ w,\ d,\ e,\ 1 \\ \frac{1}{2}(u+m-1),\ u-w+m,\ u-d+m,\ u-e+m,\ m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(\frac{1}{2}(u+m-1))\Gamma(u-w+m)\Gamma(u-d+m)\Gamma(u-e+m)}{\Gamma(u-w+1)\Gamma(u-d+1)\Gamma(u-e+1)} \\
&\times \frac{\Gamma(u-m+1)\Gamma(\frac{1}{2}(u-m+3))\Gamma(w+1-m)\Gamma(d+1-m)\Gamma(e+1-m)}{\Gamma(u)\Gamma(w)\Gamma(d)\Gamma(e)\Gamma(\frac{1}{2}(u+m+1))\Gamma(\frac{1}{2}(u-m+1))} \\
&\times \left\{ \frac{\Gamma(u-w+1)\Gamma(u-d+1)\Gamma(u-e+1)\Gamma(u-w-d-e+2m-1)}{\Gamma(2-m+u)\Gamma(u-w-e+m)\Gamma(u-d-e+m)\Gamma(u-w-d+m)} \right. \\
&\quad\left. - {}^{(m-2)}_{\ \ 5}F_4\left[\begin{matrix} u-m+1,\ w-m+1,\ \frac{1}{2}(u-m+3),\ d-m+1,\ e-m+1 \\ \frac{1}{2}(u-m+1),\ u-w+1,\ u-d+1,\ u-e+1 \end{matrix};1\right] \right\} \\
&= \chi_{10}
\end{aligned} \tag{1.24}
$$

$$
\begin{aligned}
&{}_8F_7\left[\begin{matrix} u,\ \frac{1}{2}(u+m+1),\ v,\ w,\ d,\ 2u-v-w-d+2m-1+n,\ m-n-1,\ 1 \\ \frac{1}{2}(u+m-1),\ u-v+m,\ u-w+m,\ u-d+m,\ v+w+d-u+1-m-n,\ u+n+1,\ m \end{matrix};1\right] \\
&= (-1)^{m-1}(m-1)! \times \frac{(\frac{1}{2}(3-u-m))_{m-1}(1-u+v-m)_{m-1}}{(\frac{1}{2}(1-u-m))_{m-1}(1-u)_{m-1}} \\
&\times \frac{(1-u+w-m)_{m-1}(1-u+d-m)_{m-1}(m+n+u-v-w-d)_{m-1}(-u-n)_{m-1}}{(1-v)_{m-1}(1-w)_{m-1}(1-d)_{m-1}(v+w+d-2u+2-2m-n)_{m-1}(n+2-m)_{m-1}} \\
&\times \left\{ \frac{(u-m+2)_n(u-v-w+m)_n(u-v-d+m)_n(u-w-d+m)_n}{(u-v+1)_n(u-w+1)_n(u-d+1)_n(u-v-w-d+2m-1)_n} \right. \\
&\left. - {}^{(m-2)}_{\ \ 7}F_6\left[\begin{matrix} u-m+1,\ \frac{1}{2}(u-m+3),\ v-m+1,\ w-m+1,\ d-m+1,\ 2u-v-w-d+m+n,\ -n \\ \frac{1}{2}(u-m+1),\ u-v+1,\ u-w+1,\ u-d+1,\ v+w+d+-u+2-2m-n,\ u-m+n+2 \end{matrix};1\right] \right\} \\
&= \chi_{11}
\end{aligned} \tag{1.25}
$$

It's worth noting that when m=1, the results (1.15) to (1.25) become the results (1.3) to (1.13) respectively. We refer to [7, 9, 10, 11, 19] for further generalizations and extensions of the findings (1.4) to (1.9).

Using the summation theorems (1.15) to (1.25) in the following integral credited to MacRobert[12], the goal of this work is to develop eleven new classes of integrals involving generalised hypergeometric functions,

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}d\theta = e^{\frac{i\pi\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.26}$$

provided $\Re(\alpha)>0$ and $\Re(\beta)>0$.

2. Main Results

The following theorems define the eleven new classes of integrals involving generalized hypergeometric functions that will be established in the paper.

Theorem 2.1 *For $m \in \mathbb{N}$, $Re(v) > 0$, $Re(w-v) > 0$ and $Re(w-u-v+m) > 1$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-v-1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & m\end{matrix}; e^{i\theta}cos\theta\right]d\theta = e^{i\frac{\pi}{2}(w-v)}\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}\chi_1, \tag{2.1}$$

where χ_1 is the same as given in (1.15).

Proof: In order to prove the result (2.1) asserted in theorem 1, we proceed as follows.
Denoting the left-hand side of (2.1) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-v-1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & m\end{matrix}; e^{i\theta}cos\theta\right]d\theta$$

Now, change the order of integration and summation after expressing ${}_2F_1$ as a series, (which are easily seen to be justified due to uniform convergence of the series in the interval $(0, \frac{\pi}{2})$), we have

$$I = \sum_{n=0}^{\infty}\frac{(u)_n(1)_n}{(m)_n n!}\int_0^{\frac{\pi}{2}} e^{i(w+n)\theta}(sin\theta)^{w-v-1}(cos\theta)^{v+n-1}d\theta$$

Now evaluating the integral with the help of the MacRobert's integral (1.26), we have

$$I = \sum_{n=0}^{\infty}\frac{(u)_n(1)_n}{(m)_n n!}\, e^{\frac{i\pi(w-v)}{2}}\frac{\Gamma(v+n)\Gamma(w-v)}{\Gamma(w+n)}.$$

Using the identity $(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)}$, we have

$$I = e^{\frac{i\pi}{2}(w-v)}\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}\sum_{n=0}^{\infty}\frac{(u)_n(v)_n(1)_n}{(w)_n(m)_n}\,\frac{1}{n!}.$$

Finally summing up the series, we have

$$I = e^{\frac{i\pi(w-v)}{2}}\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}{}_3F_2\left[\begin{matrix}u, & v, & 1 \\ & w, & m\end{matrix}; 1\right].$$

Now, we can see that the ${}_3F_2$ appearing may be assessed using the result (1.15) and we can easily get to the right side of the equation (2.1).

This completes the proof of our first result (2.1) asserted by the theorem 2.1. □

Corollary 2.1 *In Theorem 2.1, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-v-1}(cos\theta)^{v-1}(1-e^{i\theta}cos\theta)^{-u}d\theta = e^{\frac{i\pi}{2}(w-v)}\frac{\Gamma(v)\Gamma(w-u-v)}{\Gamma(w-u)}$$

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-v-1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & 2\end{matrix}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-v)}\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}\frac{(w-1)}{(u-1)(v-1)}\left[\frac{\Gamma(w-1)\Gamma(w-u-v+1)}{\Gamma(w-u)\Gamma(w-v)}-1\right]$$

and

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-v-1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & 3\end{matrix}; e^{i\theta}cos\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}(w-v)}\frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}\frac{(w-2)_2}{(u-2)_2(v-2)_2}\left[\frac{\Gamma(w-2)\Gamma(w-u-v+2)}{\Gamma(w-u)\Gamma(w-v)}-\frac{uv+w-2u-2v+2}{(w-2)}\right]$$

Using the results (1.16) to (1.25), the following Theorems 2.2 to 2.11 and their associated Corollaries 2.2 to 2.11 can be produced in the same ay. Hence, they are presented here without proof.

Theorem 2.2 *For $m \in \mathbb{N}$, $Re(v) > 0$ and $Re(u - 2v + m) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+m)\theta}(sin\theta)^{u-2v+m-1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & m\end{matrix}; -e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2v+m)}\frac{\Gamma(u-2v+m)\Gamma(v)}{\Gamma(u-v+m)}\chi_2, \tag{2.2}$$

where χ_2 is the same as given in (1.16).

Corollary 2.2 *In Theorem 2.2, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+1)\theta}(sin\theta)^{u-2v}(cos\theta)^{v-1}(1+e^{i\theta}cos\theta)^{-u}d\theta = e^{i\frac{\pi}{2}(u-2v+1)}\frac{\Gamma(v)\Gamma(1+u-2v)\Gamma(1+\frac{1}{2}u)}{\Gamma(1+u)\Gamma(1+\frac{1}{2}(u-v))},$$

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+2)\theta}(sin\theta)^{u-2v+1}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & 2\end{matrix}; -e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2v+2)}\frac{(u-v+1)}{(u-1)(v-1)}\frac{\Gamma(v)\Gamma(u-2v+2)}{\Gamma(u-v+2)}\left\{1-\frac{\Gamma(1+u-v)\Gamma(\frac{1}{2}u+\frac{1}{2})}{\Gamma(u)\Gamma(\frac{1}{2}u-v+\frac{3}{2})}\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+3)\theta}(sin\theta)^{u-2v+2}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & 3\end{matrix}; -e^{i\theta}cos\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}(u-2v+3)}\frac{(u-v+1)_2}{(u-2)_2(v-2)_2}\times\frac{\Gamma(v)\Gamma(u-2v+3)}{\Gamma(u-v+3)}\left\{\frac{\Gamma(\frac{1}{2}u)\Gamma(1+u-v)}{\Gamma(u-1)\Gamma(\frac{1}{2}u-v+2)}-\frac{3u+v-uv-3}{1+u-v}\right\}.$$

Theorem 2.3 *For $m \in \mathbb{N}$, $Re(v) > 0$ and $Re(u - v + 1) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(u+v+1)\theta}(sin\theta)^{\frac{1}{2}(u-v-1)}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1 \\ & m\end{matrix}; \frac{1}{2}e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{4}(u-v+1)}\frac{\Gamma(v)\Gamma(\frac{1}{2}(u-v+1))}{\Gamma(\frac{1}{2}(u+v+1))}\chi_3, \tag{2.3}$$

where χ_3 is the same as given in (1.17).

Corollary 2.3 *In Theorem 2.3, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(u+v+1)\theta}(sin\theta)^{\frac{1}{2}(u-v-1)}(cos\theta)^{v-1}(1-\frac{1}{2}e^{i\theta}cos\theta)^{-u} = e^{i\frac{\pi}{4}(u-v+1)}\frac{\Gamma(\frac{1}{2})\Gamma(v)\Gamma(\frac{1}{2}(u-v+1))}{\Gamma(\frac{1}{2}(u+1))\Gamma(\frac{1}{2}(v+1))},$$

$$\int_0^{\frac{\pi}{2}} e^{i\frac{(u+v+1)}{2}\theta}(sin\theta)^{\frac{1}{2}(u-v-1)}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1\\ & 2\end{matrix};\frac{1}{2}e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{4}(u-v+1)}\frac{(u+v-1)}{(u-1)(v-1)}\frac{\Gamma(v)\Gamma(\frac{1}{2}(u-v+1))}{\Gamma(\frac{1}{2}(u+v+1))}\left\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(u+v-1))}{\Gamma(\frac{1}{2}u)\Gamma(\frac{1}{2}v)}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i\frac{1}{2}(u+v+1)\theta}(sin\theta)^{\frac{1}{2}(u-v-1)}(cos\theta)^{v-1}{}_2F_1\left[\begin{matrix}u, & 1\\ & 3\end{matrix};\frac{1}{2}e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{4}(u-v+1)}\frac{2(u+v-1)(u+v-3)}{(u-2)_2(v-2)_2}\frac{\Gamma(v)\Gamma(\frac{1}{2}(u-v+1))}{\Gamma(\frac{1}{2}(u+v+1))}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(u+v-3))}{\Gamma(\frac{1}{2}(u-1))\Gamma(\frac{1}{2}(v-1))}-\frac{uv-u-v+1}{u+v-3}\right\}.$$

Theorem 2.4 *For $m \in \mathbb{N}$, $Re(u) > 0$ and $Re(v-u) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{iv\theta}(sin\theta)^{v-u-1}(cos\theta)^{u-1}{}_2F_1\left[\begin{matrix}2m-u-1, & 1\\ m & \end{matrix};\frac{1}{2}e^{i\theta}cos\theta\right]d\theta = e^{i\frac{\pi}{2}(v-u)}\frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)}\chi_4, \quad (2.4)$$

where χ_4 is the same as given in (1.18).

Corollary 2.4 *In Theorem 2.4, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{iv\theta}(sin\theta)^{v-u-1}(cos\theta)^{u-1}\left(1-\frac{1}{2}e^{i\theta}cos\theta\right)^{u-1}d\theta = 2^{1-v}e^{i\frac{\pi}{2}(v-u)}\frac{\Gamma(\frac{1}{2})\Gamma(u)\Gamma(v-u)}{\Gamma(\frac{1}{2}(u+v))\Gamma(\frac{1}{2}(v-u+1))},$$

$$\int_0^{\frac{\pi}{2}} e^{iv\theta}(sin\theta)^{v-u-1}(cos\theta)^{u-1}{}_2F_1\left[\begin{matrix}3-u, & 1\\ 2 & \end{matrix};\frac{1}{2}e^{i\theta}cos\theta\right]d\theta$$
$$= 2e^{i\frac{\pi}{2}(v-u)}\frac{(1-v)}{(1-u)_2}\frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)}\left\{\frac{\Gamma(\frac{1}{2}(v-1))\Gamma(\frac{1}{2}v)}{\Gamma(\frac{1}{2}(u+v)-1)\Gamma(\frac{1}{2}(v-u+1))}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{iv\theta}(sin\theta)^{v-u-1}(cos\theta)^{u-1}{}_2F_1\left[\begin{matrix}5-u, & 1\\ 3 & \end{matrix};\frac{1}{2}e^{i\theta}cos\theta\right]d\theta$$
$$= 8e^{i\frac{\pi}{2}(v-u)}\frac{(v-2)_2}{(u-4)_4}\frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)}\left\{\frac{\Gamma(\frac{1}{2}(v-1))\Gamma(\frac{1}{2}(v-2))}{\Gamma(\frac{1}{2}(u+v)-2)\Gamma(\frac{1}{2}(v-u+1))}-\frac{5u-u^2+2v-10}{2(v-2)}\right\}.$$

Theorem 2.5 *For $m \in \mathbb{N}$, $Re(w) > 0$ and $Re(u-2w+m) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+m)\theta}(sin\theta)^{u-2w+m-1}(cos\theta)^{w-1}{}_3F_2\left[\begin{matrix}u, & v, & 1\\ u-v+m, & m & \end{matrix};e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2w+m)}\frac{\Gamma(w)\Gamma(u-2w+m)}{\Gamma(u-w+m)}\chi_5, \quad (2.5)$$

where χ_5 is the same as given in (1.19).

Corollary 2.5 *In Theorem 2.5, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+1)\theta}(sin\theta)^{u-2w}(cos\theta)^{w-1}{}_2F_1\left[\begin{matrix}u, & v, & 1\\ u-v+1 & & \end{matrix};e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2w+1)}\frac{\Gamma(w)\Gamma(1+u-2w)\Gamma(1+\frac{1}{2}u)\Gamma(1+u-v)\Gamma(1+\frac{1}{2}u-v-w)}{\Gamma(1+u)\Gamma(1+\frac{1}{2}u-v)\Gamma(1+\frac{1}{2}u-w)\Gamma(1+u-v-w)},$$

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+2)\theta}(sin\theta)^{u-2w+1}(cos\theta)^{w-1}{}_3F_2\left[\begin{array}{c} u, \quad v, \quad 1 \\ u-v+2, \quad 2 \end{array}; e^{i\theta}cos\theta\right] d\theta$$
$$= e^{i\frac{\pi}{2}(u-2w+2)}\frac{(1+u-v)(1+u-w)}{(u-1)(v-1)(w-1)}\frac{\Gamma(w)\Gamma(2+u-2w)}{\Gamma(u-w+2)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}(u+1))\Gamma(1+u-v)\Gamma(1+u-w)\Gamma(\frac{1}{2}u-v-w+\frac{5}{2})}{\Gamma(u)\Gamma(\frac{1}{2}u-v+\frac{3}{2})\Gamma(\frac{1}{2}u-w+\frac{3}{2})\Gamma(2+u-v-w)}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+3)\theta}(sin\theta)^{u-2w+2}(cos\theta)^{w-1}{}_3F_2\left[\begin{array}{c} u, \quad v, \quad w \\ u-v+3, \quad 3 \end{array}; \frac{1}{2}e^{i\theta}cos\theta\right] d\theta$$
$$= 2e^{i\frac{\pi}{2}(u-2w+3)}\frac{(u-v+1)_2(u-w+1)_2}{(u-2)_2(v-2)_2(w-2)_2}\frac{\Gamma(w)\Gamma(u-2w+3)}{\Gamma(u-w+3)}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}u)\Gamma(1+u-v)\Gamma(1+u-w)\Gamma(\frac{1}{2}u-v-w+4)}{\Gamma(u-1)\Gamma(\frac{1}{2}u-v+2)\Gamma(\frac{1}{2}u-w+2)\Gamma(3+u-v-w)}-\frac{(u-2)(v-2)(w-2)}{(u-v+1)(u-w+1)}-1\right\}.$$

Theorem 2.6 *For $m \in \mathbb{N}$ and $Re(w) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(2w-m+1)\theta}(sin\theta)^{w-m}(cos\theta)^{w-1}{}_3F_2\left[\begin{array}{c} u, \quad v, \quad 1 \\ \frac{1}{2}(u+v+1), \quad m \end{array}; \frac{1}{2}e^{i\theta}cos\theta\right] d\theta$$
$$= e^{i\frac{\pi}{2}(w-m+1)}\frac{\Gamma(w)\Gamma(w-m+1)}{\Gamma(2w-m+1)}\chi_6, \tag{2.6}$$

where χ_6 is the same as given in (1.20).

Corollary 2.6 *In Theorem 2.6, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{2iw\theta}(sin\theta)^{w-1}(cos\theta)^{w-1}{}_2F_1\left[\begin{array}{c} u, \quad v \\ \frac{1}{2}(u+v+1) \end{array}; \frac{1}{2}e^{i\theta}cos\theta\right] d\theta$$
$$= 2^{1-2w}e^{i\frac{\pi}{2}}\frac{\pi\Gamma(w)\Gamma(\frac{1}{2}(u+v+1))\Gamma(w-\frac{1}{2}(u+v-1))}{\Gamma(\frac{1}{2}(u+1))\Gamma(\frac{1}{2}(v+1))\Gamma(w-\frac{1}{2}(u-1))\Gamma(w-\frac{1}{2}(v-1))},$$

$$\int_0^{\frac{\pi}{2}} e^{i(2w-1)\theta}(sin\theta)^{w-2}(cos\theta)^{w-1}{}_3F_2\left[\begin{array}{c} u, \quad v, \quad 1 \\ \frac{1}{2}(u+v+1), \quad 2 \end{array}; \frac{1}{2}e^{i\theta}cos\theta\right] d\theta$$
$$= e^{i\frac{\pi}{2}(w-1)}\frac{(u+v-1)}{(u-1)(v-1)}\frac{\Gamma(w)\Gamma(w-1)}{\Gamma(2w-1)}\left\{\frac{\sqrt{\pi}\,\Gamma(w-\frac{1}{2})\Gamma(\frac{1}{2}(u+v-1))\Gamma(w-\frac{1}{2}(u+v-1))}{\Gamma(\frac{1}{2}u)\Gamma(\frac{1}{2}v)\Gamma(w-\frac{1}{2}u)\Gamma(w-\frac{1}{2}v)}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(2w-2)\theta}(sin\theta)^{w-3}(cos\theta)^{w-1}{}_3F_2\left[\begin{array}{c} u, \quad v, \quad 1 \\ \frac{1}{2}(u+v+1), \quad 3 \end{array}; \frac{1}{2}e^{i\theta}cos\theta\right] d\theta$$
$$= e^{i\frac{\pi}{2}(w-2)}\frac{(2w-3)(u+v-1)(u+v-3)}{(w-1)(u-2)_2(v-2)_2}\frac{\Gamma(w)\Gamma(w-2)}{\Gamma(2w-2)}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(w-\frac{3}{2})\Gamma(\frac{1}{2}(u+v-3))\Gamma(w-\frac{1}{2}(u+v-1))}{\Gamma(\frac{1}{2}(u-1))\Gamma(\frac{1}{2}(v-1))\Gamma(w-\frac{1}{2}(u+1))\Gamma(w-\frac{1}{2}(v+1))}-\frac{(u-2)(v-2)}{u+v-3}-1\right\}.$$

Theorem 2.7 *For $m \in \mathbb{N}$, $Re(u) > 0$ and $Re(w-u) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{c} 2m-1-u, \quad v, \quad 1 \\ 2v-w+1, \quad m \end{array}; e^{i\theta}cos\theta\right] d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}\chi_7, \tag{2.7}$$

where χ_7 *is the same as given in* (1.21).

Corollary 2.7 *In Theorem 2.7, if we take* $m = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_2F_1\left[\begin{array}{c} 1-u, \quad v \\ 2v-w+1 \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{\pi\, 2^{1-2v}\,\Gamma(u)\Gamma(w-u)\Gamma(2v-w+1)}{\Gamma(\frac{1}{2}(u+w))\Gamma(v+\frac{1}{2}(u-w+1))\Gamma(\frac{1}{2}(1-u+w))\Gamma(v+1-\frac{1}{2}(u+w))},$$

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{ccc} 3-u, & v, & 1 \\ 2v-w+1, & 2 & \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{(w-1)(w-2v)}{(u-2)_2(v-1)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}$$
$$\times\left\{\frac{\pi 2^{3-2v}\Gamma(w-1)\Gamma(2v-w)}{\Gamma(\frac{1}{2}(u+w)-1)\Gamma(v+\frac{1}{2}(u-w-1))\Gamma(\frac{1}{2}(1-u+w))\Gamma(v+1-\frac{1}{2}(u+w))}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{ccc} 5-u, & v, & 1 \\ 2v-w+1, & 3 & \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}\left\{\frac{\pi\, 2^{5-2v}\,\Gamma(w-2)\Gamma(2v-w+1)}{\Gamma(\frac{1}{2}(u+w)-2)\Gamma(v+\frac{1}{2}(u-w-3))\Gamma(\frac{1}{2}(1-u+w))\Gamma(v+1-\frac{1}{2}(u+w))}\right.$$
$$\left.-\frac{(u-2)(3-u)(v-2)}{(w-2)(2v-w-1)}-1\right\}$$

Theorem 2.8 *For* $m \in \mathbb{N}$, $Re(u) > 0$ *and* $Re(w-u) > 0$, *the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{ic\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{ccc} -n+m-1, & v, & 1 \\ 1+u+v-w-n, & m & \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}\chi_8, \tag{2.8}$$

where χ_8 *is the same as given in* (1.22).

Corollary 2.8 *In Theorem 2.8, if we take* $m = 1, 2, 3$, *we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_2F_1\left[\begin{array}{c} -n, \quad v \\ 1+u+v-w-n \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}\frac{(w-u)_n(w-v)_n}{(w)_n(w-u-v)_n},$$

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{ccc} -n+1, & v, & 1 \\ 1+u+v-w-n, & 2 & \end{array}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(w-u)}\frac{(1-w)(w-u-v+n)}{n(1-u)(1-v)}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}\times\left\{\frac{(w-u)_n(w-v)_n}{(w)_n(w-u-v+1)_n}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{iw\theta}(sin\theta)^{w-u-1}(cos\theta)^{u-1}{}_3F_2\left[\begin{array}{c} -n+2, \quad v, \quad 1 \\ 1+u+v-w-n, \quad 3 \end{array}; e^{i\theta}cos\theta\right]d\theta$$

$$= 2e^{i\frac{\pi}{2}(w-u)}\frac{(1-w)_2(w-u-v+n)_2}{(1-u)_2(1-v)_2}\frac{\Gamma(u)\Gamma(w-u)}{\Gamma(w)}$$

$$\times\left\{\frac{(w-u)_n(w-v)_n}{(w-2)_n(w-u-v+2)_n}+\frac{n(u-2)(v-2)}{(w-2)(u+v-w-n-1)}-1\right\}.$$

Theorem 2.9 *For $m \in \mathbb{N}$, $Re(v) > 0$ and $Re(u-2v+m) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+m)\theta}(sin\theta)^{u-2v+m-1}(cos\theta)^{v-1}{}_4F_3\left[\begin{array}{c} u, \quad \frac{1}{2}(u+m+1), \quad w, \quad 1 \\ \frac{1}{2}(u+m-1), \quad u-w+m, \quad m \end{array}; -e^{i\theta}cos\theta\right]d\theta \tag{2.9}$$

$$= e^{i\frac{\pi}{2}(u-2v+m)}\frac{\Gamma(v)\Gamma(u-2v+m)}{\Gamma(u-v+m)}\chi_9,$$

where χ_9 is the same as given in (1.23).

Corollary 2.9 *In Theorem 2.9, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+1)\theta}(sin\theta)^{u-2v}(cos\theta)^{v-1}{}_3F_2\left[\begin{array}{c} u, \quad \frac{1}{2}(u+2), \quad w \\ \frac{1}{2}u, \quad u-w+1 \end{array}; -e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2v+1)}\frac{\Gamma(v)\Gamma(1+u-w)\Gamma(1+u-2v)}{\Gamma(1+u)\Gamma(1+u-v-w)},$$

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+2)\theta}(sin\theta)^{u-2v+1}(cos\theta)^{v-1}{}_4F_3\left[\begin{array}{c} u, \quad \frac{1}{2}(u+3), \quad w, \quad 1 \\ \frac{1}{2}(u+1), \quad u-w+2, \quad 2 \end{array}; -e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2v+2)}\frac{(1+u-v)(1+u-w)}{(u+1)(v-1)(w-1)}\frac{\Gamma(v)\Gamma(u-2v+2)}{\Gamma(2+u-v)}\times\left\{1-\frac{\Gamma(1+u-v)\Gamma(1+u-w)}{\Gamma(u)\Gamma(2+u-v-w)}\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+3)\theta}(sin\theta)^{u-2v+2}(cos\theta)^{v-1}{}_4F_3\left[\begin{array}{c} u, \quad \frac{1}{2}(u+4), \quad w, \quad 1 \\ \frac{1}{2}(u+3), \quad u-w+3, \quad 3 \end{array}; -e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2v+3)}\frac{2(1+u-v)_2(1+u-w)_2}{(u+2)(u-1)(v-2)_2(w-2)_2}\frac{\Gamma(v)\Gamma(u-2v+3)}{\Gamma(3+u-v)}$$

$$\times\left\{\frac{\Gamma(1+u-v)\Gamma(1+u-w)}{\Gamma(u-1)\Gamma(3+u-v-w)}+\frac{u(v-2)(w-2)}{(1+u-v)(1+u-w)}-1\right\}.$$

Theorem 2.10 *For $m \in \mathbb{N}$, $Re(v) > 0$ and $Re(u-2w+m) > 0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+m)\theta}(sin\theta)^{u-2w+m-1}(cos\theta)^{w-1} \tag{2.10}$$

$$\times{}_5F_4\left[\begin{array}{c} u, \quad \frac{1}{2}(u+m+1), \quad d, \quad e, \quad 1 \\ \frac{1}{2}(u+m-1), \ u-d+m, \ u-e+m, \ m \end{array}; e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2w+m)}\frac{\Gamma(w)\Gamma(u-2w+m)}{\Gamma(u-w+m)}\chi_{10},$$

where χ_{10} is the same as given in (1.24).

Corollary 2.10 *In Theorem 2.10, if we take $m = 1, 2, 3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+1)\theta}(sin\theta)^{u-2w}(cos\theta)^{w-1}{}_4F_3\left[\begin{array}{c} u, \quad \frac{1}{2}(u+2), \quad d, \quad e \\ \frac{1}{2}u, \quad u-d+1, \quad u-e+1 \end{array}; e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2w+1)}\frac{\Gamma(w)\Gamma(u-2w+1)\Gamma(1+u-d)\Gamma(1+u-e)\Gamma(1+u-w-d-e)}{\Gamma(1+u)\Gamma(1+u-d-e)\Gamma(1+u-w-e)\Gamma(1+u-w-d)},$$

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+2)\theta}(sin\theta)^{u-2w+1}(cos\theta)^{w-1}{}_5F_4\left[\begin{matrix} u, & \frac{1}{2}(u+3), & d, & e, & 1 \\ \frac{1}{2}(u+1), & u-d+2, & u-e+2, & 2 \end{matrix}; e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2w+2)}\frac{(1+u-w)(1+u-d)(1+u-e)}{(1+u)(w-1)(d-1)(e-1)}\frac{\Gamma(w)\Gamma(u-2w+2)}{\Gamma(u-w+2)}$$

$$\times\left\{\frac{\Gamma(1+u-w)\Gamma(1+u-d)\Gamma(1+u-e)\Gamma(3+u-w-d-e)}{\Gamma(u)\Gamma(2+u-d-e)\Gamma(2+u-w-e)\Gamma(2+u-w-d)}-1\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(u-w+3)\theta}(sin\theta)^{u-2w+2}(cos\theta)^{w-1}{}_5F_4\left[\begin{matrix} u, & \frac{1}{2}(u+4), & d, & e, & 1 \\ \frac{1}{2}(u+2), & u-d+3, & u-e+3, & 3 \end{matrix}; e^{i\theta}cos\theta\right]d\theta$$

$$= 2e^{i\frac{\pi}{2}(u-2w+3)}\frac{(1+u-w)_2(1+u-d)_2(1+u-e)_2}{(u-1)(u+2)(w-2)_2(d-2)_2(e-2)_2}\frac{\Gamma(w)\Gamma(u-2w+3)}{\Gamma(u-w+3)}$$
$$\times\left\{\frac{\Gamma(1+u-w)\Gamma(1+u-d)\Gamma(1+u-e)\Gamma(5+u-w-d-e)}{\Gamma(u-1)\Gamma(3+u-d-e)\Gamma(3+u-w-e)\Gamma(3+u-w-d)}-\frac{u(w-2)(d-2)(e-2)}{(1+u-w)(1+u-d)(1+u-e)}\right\}.$$

Theorem 2.11 *For $m\in\mathbb{N}$, $Re(v)>0$ and $Re(u-2v+m)>0$, the following result holds true.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+m)\theta}(sin\theta)^{u-2v+m-1}(cos\theta)^{v-1}$$
$$\times{}_7F_6\left[\begin{matrix} u, & \frac{1}{2}(u+m+1), & w, & d, & 2u-v-w-d+2m+n-1, & m-n-1, 1 \\ \frac{1}{2}(u+m-1), & u-w+m, & u-d+m, & v+w+d-u-m-n+1, & u+n+1, & m \end{matrix}; e^{i\theta}cos\theta\right]d\theta \tag{2.11}$$
$$= e^{i\frac{\pi}{2}(u-2v+m)}\frac{\Gamma(v)\Gamma(u-2v+m)}{\Gamma(u-v+m)}\chi_{11},$$

where χ_{11} is the same as given in (1.25).

Corollary 2.11 *In Theorem 2.11, if we take $m=1,2,3$, we respectively get the following integrals.*

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+1)\theta}(sin\theta)^{u-2v}(cos\theta)^{v-1}$$
$$\times{}_6F_5\left[\begin{matrix} u, & \frac{1}{2}(u+2), & w, & d, & 2u-v-w-d+n+1, & -n \\ \frac{1}{2}u, & 1+u-w,, & 1+u-d, & v+w+d-u-n, & u+n+1 \end{matrix}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2v+1)}\frac{\Gamma(v)\Gamma(u-2v+1)}{\Gamma(u-v+1)}\frac{(1+u)_n(u-v-w+1)_n(u-v-d+1)_n(u-w-d+1)_n}{(1+u-v)_n(1+u-w)_n(1+u-d)_n(1+u-v-w-d)_n},$$

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+2)\theta}(sin\theta)^{u-2v+1}(cos\theta)^{v-1}$$
$$\times{}_7F_6\left[\begin{matrix} u, & \frac{1}{2}(u+3), & w, & d, & 2u-v-w-d+n+3, & 1-n, & 1 \\ \frac{1}{2}(u+1), & 1+u-w, & 1+u-d, & v+w+d-u-n-1, & u+n+1, & 2 \end{matrix}; e^{i\theta}cos\theta\right]d\theta$$
$$= e^{i\frac{\pi}{2}(u-2v+2)}\frac{(v-u-1)(w-u-1)(d-u-1)(n+2+u-v-w-d)(u+n)}{n(1+u)(1-v)(1-w)(1-d)(v+w+d-2u-2-n)}\frac{\Gamma(v)\Gamma(u-2v+1)}{\Gamma(u-v+1)}$$
$$\times\left\{1-\frac{(u)_n(u-v-w+2)_n(u-v-d+2)_n(u-w-d+2)_n}{(1+u-v)_n(1+u-w)_n(1+u-d)_n(3+u-v-w-d)_n}\right\}$$

and

$$\int_0^{\frac{\pi}{2}} e^{i(u-v+3)\theta}(sin\theta)^{u-2v+2}(cos\theta)^{v-1}$$

$$\times {}_7F_6\left[\begin{matrix} u, & \frac{1}{2}(u+4), & w, & d, & 2u-v-w-d+n+5, & 2-n, & 1 \\ \frac{1}{2}(u+2), & 3+u-w, & 3+u-d, & v+w+d-u-n-2, & u+n+1, & 3 & \end{matrix};e^{i\theta}cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(u-2v+3)}\frac{(u-2)(v-u-2)_2(w-u-2)_2(d-u-2)_2(-u-n)_n(3+n+u-v-w-d)_2}{(u+2)(1-u)_2(1-v)_2(1-w)_2(1-d)_2(n-1)_2(v+w+d-2u-4-n)_2}$$

$$\frac{\Gamma(v)\Gamma(u-2v+3)}{\Gamma(u-v+2)}\times\left\{\frac{(u-1)_n(u-v-w+3)_n(u-v-d+3)_n(u-w-d+3)_n}{(u-v+1)_n(u-w+1)_n(u-d+1)_n(u-v-w-d+5)_n}\right.$$

$$\left.+\frac{nu(v-2)(w-2)(d-2)(2u-v-w+n+3)}{(u-v+1)(u-w+1)(u-d+1)(v+w+d-u-n-4)(n+u-1)}-1\right\}.$$

Remark 2.1 *For several finite integrals, see papers [4, 5, 6].*

Conclusion Remark

In this paper, eleven new class of integrals due to MacRobert involving generalized hypergeometric functions have been evaluated in terms of gamma function using Masjed-Jamei and Koepf's recently derived summation theorems. As special cases of our primary conclusions, we have provided a number of novel, intriguing, and elementary integrals.

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