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About a New Nonlinear Integro-Differential Equation of Volterra-Strum-Liouville Type with a Weakly Singular Kernel

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ABSTRACT: This present paper proposes an analytical and a numerical study of a new integro-differential Volterra-Strum-Liouville equation of the second type, having a weak singularity. We provide conditions that guarantee the existence and uniqueness of the solution to the nonlinear problem. Then, we develop a numerical technique using the production integration method. The numerical application shows the efficiency of the proposed procedure.

Key Words: Nonlinear Volterra integral equation, integro-differential equation, fixed point, product integration method.

Contents

1. Introduction

Several contemporary problems in mathematics, engineering, physics, and natural sciences, have been solved using integral equations, see for example [1]. In this paper, we propose an analytical and numerical study of a new nonlinear integro-differential equation of the Volterra-Strum-Liouville type with a weakly singular kernel. We establish sufficient conditions to ensure the existence and uniqueness of the solution, and then we implement a discretization method based on the product integration principle. The main problem is defined by :

$$\forall t \in [a,b], u(t) = f(t) + \int_a^t p(t-s)K(t,s,u(s),S(u(s)))ds, \tag{1.1}$$

where f, K and p are given functions, the unknown is u and, S is a Sturm-Liouville operator defined by

$$S : C^2([a,b]) \to \mathbb{R}, \quad u \mapsto (Q_1 u')' + Q_2 u,$$

such that $Q_1 \in C^1([a,b])$ and $Q_2 \in C^0([a,b])$ two given functions.

This work is the continuation of many results obtained for a nonlinear integral equation of the Volterra type with, weakly singular kernel (see for example [2,3,4,5]). The case for which the Sturm-Louiville operator appears nonlinearly inside the integral operator is little studied despite its great importance. Our study focuses on the type, and originality of this equation, which embodies the rapid development and the interest of scientists in this field(See, [6,7,8,9]). Moreover, the numerical part presents several challenges due to the single character of the kernel, which makes the application of traditional numerical integration methods impractical. Finally, for a study on Volterra integral equations of the second type, see [10].

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One of the most commonly used methods for approximating singular kernel integrals is the product integration method. The foundations of this approximation approach were introduced historically by Young [11] and then improved by De Hoog and Weiss Latter [12]. Further work has been realized in this direction by Atkinson [13].

The equation (1.1) will be regarded in the case where, "p" has a weak singularity defined by:

$$\begin{cases} p(0)=0, \quad p'(0)=0, \quad p''(0)\to+\infty, \\ \text{with} \quad \displaystyle\int_a^b |p''(s)|ds<\infty. \end{cases} \tag{1.2}$$

So, this leads us to assume:

$$(H1)\left\| \begin{array}{l} 1)\, p\in W^{2,1}(a-b,b-a), \\ 2)\, p(0)=p'(0)=0. \end{array}\right.$$

where,

$$W^{2,1}(a-b,b-a)=\{x\in L^1(a-b,b-a): x',x''\in L^1(a-b,b-a)\},$$

x', x'' are the first and second derivatives of x in the weak sense. $W^{2,1}(a-b,b-a)$ is a Banach space with the norm:

$$\|x\|_{W^{2,1}(a-b,b-a)}=\|x\|_{L^1(a-b,b-a)}+\|x'\|_{L^1(a-b,b-a)}+\|x''\|_{L^1(a-b,b-a)}=\int_{a-b}^{b-a}(|x(s)|+|x'(s)|+|x''(s)|)ds.$$

We can prove that $\forall t,s\in[a,b]$: $p(t-s)<\infty$ and $\frac{\partial p}{\partial t}(t-s)<\infty$. Then, the singularity comes from the term $\frac{\partial^2 p}{\partial t^2}(t-s)$, when $t\to s$ i.e. $\frac{\partial^2 p}{\partial t^2}(0)=\infty$. Indeed;

$$\left|\int_a^t \frac{\partial^2 p}{\partial t^2}(t-s)ds\right| \leq \int_a^t \left|\frac{\partial^2 p}{\partial t^2}(t-s)\right| ds \leq \int_0^{t-a}\left|\frac{\partial^2 p}{\partial \tau^2}(z)\right|dz \leq \int_{a-b}^{b-a}\left|\frac{\partial^2 p}{\partial \tau^2}(z)\right|dz \leq \|p\|_{W^{2,1}(a-b,b-a)}.$$

The paper is organized as follows: In section 2, we present the main theoretical results in which, under appropriate hypotheses, we prove the existence and uniqueness to the solution of the main problem (1.1), using ideas based on the successive Piccard method. In section 3, we illustrate these results by a numerical study where, we will show the consistency and convergence of our scheme, and apply this scheme on examples showing the accuracy and efficiency of our algorithm.

2. Existence and Uniqueness

In this section, we establish the existence and uniqueness of the equation (1.1) where, we propose some assumptions to achieve this goal. These assumptions are similar to the conditions established in [2,3,4,5]. Let K be a given function, defined by

$$K:[a,b]^2\times\mathbb{R}^2\to\mathbb{R}, \quad (t,s,u,v)\mapsto K(t,s,u,v).$$

We assume that the functions f, K, Q_1 and Q_2 satisfy

$$(H2)\left\| \begin{array}{l} 1)\, f\in C^2([a,b]), \frac{\partial K}{\partial t}\in C^1([a,b]^2\times\mathbb{R}^2), \\ 2)\, \exists M\in\mathbb{R}_+, \forall t,s\in[a,b], \forall u,v,\overline{u},\overline{v}\in\mathbb{R}: \\ \max\left(|K(t,s,u,v)|, \left|\frac{\partial K}{\partial t}(t,s,u,v)\right|, \left|\frac{\partial^2 K}{\partial t^2}(t,s,u,v)\right|\right)\leq M, \\ 3)\, \exists\overline{M}\in\mathbb{R}_+, \forall s\in[a,b]: \max(|Q_1(s)|,|Q_1'(s)|,|Q_2(s)|)\leq\overline{M}, \\ 4)\, \exists A,B,\overline{A},\overline{B},\overline{\overline{A}},\overline{\overline{B}}, \forall t,s\in[a,b], \forall u,v,\overline{u},\overline{v}\in\mathbb{R}: \\ |K(t,s,u,v)-K(t,s,\overline{u},\overline{v})|\leq A|u-\overline{u}|+B|v-\overline{v}|, \\ \\ |\dfrac{\partial K}{\partial t}(t,s,u,v)-\dfrac{\partial K}{\partial t}(t,s,\overline{u},\overline{v})|\leq\overline{A}|u-\overline{u}|+\overline{B}|v-\overline{v}|, \\ \\ |\dfrac{\partial^2 K}{\partial t^2}(t,s,u,v)-\dfrac{\partial^2 K}{\partial t^2}(t,s,\overline{u},\overline{v})|\leq\overline{\overline{A}}|u-\overline{u}|+\overline{\overline{B}}|v-\overline{v}|. \end{array}\right.$$

Now, we define the operator G_f as

$$\forall \xi \in C^2([a,b]), \forall t \in [a,b] : G_f(\xi)(t) = f(t) + \int_a^t p(t-s)K(t,s,\xi(s),S(\xi(s)))ds.$$

Proposition 2.1. *For all $f \in C^2([a,b])$, G_f is continuous from $C^2([a,b])$ into itself.*

Proof. It is enough to use the assumptions (H2) to establish this result. □

Theorem 2.1. *Under the hypotheses $(H1)$ and $(H2)$ and knowing that there exist points $a = T_0, T_1, \cdots,$ $T_n = b$ such that for $0 \le i \le n$ and $\forall t \in [T_i, T_{i+1}]$:*

$$\max\left(A, B\overline{M}, \overline{A}, \overline{BM}, \overline{\overline{A}}, \overline{\overline{BM}}\right) \int_{T_i}^{\min(T_i,t)} |p(t-s)|ds \le \varrho < \frac{1}{7},$$

$$\max\left(A, B\overline{M}, \overline{A}, \overline{BM}\right) \int_{T_i}^{\min(T_i,t)} |\frac{\partial p}{\partial t}(t-s)|ds \le \varrho < \frac{1}{7},$$

$$\max\left(A, B\overline{M}\right) \int_{T_i}^{\min(T_i,t)} |\frac{\partial^2 p}{\partial t^2}(t-s)|ds \le \varrho < \frac{1}{7},$$

where, ϱ is a constant independent of t and s. Then (1.1) admits a unique solution in $C^2([a,b])$.

Proof. We construct the two sequences $\{U_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ which are defined by

$$\begin{cases} U_0(t) = f(t), \\ U_n(t) = f(t) + \displaystyle\int_a^t p(t-s)K(t,s,U_{n-1},S(U_{n-1}(s)))ds, \quad \forall n \in \mathbb{N}^*, \end{cases}$$

and

$$\begin{cases} \varphi_0(t) = f(t), \\ \varphi_n(t) = U_n(t) - U_{n-1}(t), \quad \forall n \in \mathbb{N}^*. \end{cases} \tag{2.1}$$

By derivation, we get

$$\begin{cases} U_0'(t) = & f'(t), \\ U_n'(t) = & f'(t) + \displaystyle\int_a^t p(t-s)\frac{\partial K}{\partial t}(t,s,U_{n-1},S(U_{n-1}(s)))ds \\ & + \displaystyle\int_a^t \frac{\partial p}{\partial t}(t-s)K(t,s,U_{n-1},S(U_{n-1}(s)))ds, \quad \forall n \in \mathbb{N}^*, \end{cases}$$

and

$$\begin{cases} U_0''(t) = & f''(t), \\ U_n''(t) = & f''(t) + \displaystyle\int_a^t p(t-s)\frac{\partial^2 K}{\partial t^2}(t,s,U_{n-1},S(U_{n-1}(s)))ds \\ & +2\displaystyle\int_a^t \frac{\partial p}{\partial t}(t-s)\frac{\partial K}{\partial t}(t,s,U_{n-1},S(U_{n-1}(s)))ds \\ & + \displaystyle\int_a^t \frac{\partial^2 p}{\partial t^2}(t-s)K(t,s,U_{n-1},S(U_{n-1}(s)))ds, \quad \forall n \in \mathbb{N}^*. \end{cases}$$

In addition,

$$\begin{cases} \varphi_0'(t) = f'(t), \\ \varphi_n'(t) = U_n'(t) - U_{n-1}'(t), \quad \forall n \in \mathbb{N}^*, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \varphi_0''(t) = f''(t), \\ \varphi_n''(t) = U_n''(t) - U_{n-1}''(t), \quad \forall n \in \mathbb{N}^*, \end{cases} \tag{2.3}$$

Thus, from (2.1), (2.2) and (2.3), we find that

$$\sum_{i=0}^{n}\varphi_i(t)=U_n(t),\quad \sum_{i=0}^{n}\varphi_i'(t)=U_n'(t),\quad \sum_{i=0}^{n}\varphi_i''(t)=U_n''(t).$$

Let now $t\in[T_0,T_1]$, it is obvious that the sequences $\{U_i(t)\}_{i=1,\cdots,n}$ and $\{\varphi_i(t)\}_{i=1,\cdots,n}$ are included in $C^2([a,b])$. Moreover, we have

$$\begin{aligned}|\varphi_n(t)| &\le A\int_a^t|p(t-s)||\varphi_{n-1}(t)|ds+B\int_a^t|p(t-s)||S(\varphi_{n-1}(s))|ds\\ &\le \max(A,B\overline{M})\int_a^t|p(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}\\ &\le \varrho\|\varphi_{n-1}\|_{C^2([T_0,T_1])}.\end{aligned}$$

and

$$\begin{aligned}|\varphi_n'(t)| &\le \overline{A}\int_a^t|p(t-s)||\varphi_{n-1}(t)|ds+\overline{B}\int_a^t|p(t-s)||S(\varphi_{n-1}(s))|ds\\ &+A\int_a^t|\frac{\partial p}{\partial t}(t-s)||\varphi_{n-1}(t)|ds+B\int_a^t|\frac{\partial p}{\partial t}(t-s)||S(\varphi_{n-1}(s))|ds\\ &\le \max(\overline{A},\overline{BM})\int_a^t|p(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}\\ &+\max(A,B\overline{M})\int_a^t|\frac{\partial p}{\partial t}(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}\\ &\le 2\varrho\|\varphi_{n-1}\|_{C^2([T_0,T_1])}.\end{aligned}$$

Also,

$$\begin{aligned}|\varphi_n''(t)| &\le \overline{\overline{A}}\int_a^t|p(t-s)||\varphi_{n-1}(t)|ds+\overline{\overline{B}}\int_a^t|p(t-s)||S(\varphi_{n-1}(s))|ds\\ &+2\overline{A}\int_a^t|\frac{\partial p}{\partial t}(t-s)||\varphi_{n-1}(t)|ds+2\overline{B}\int_a^t|\frac{\partial p}{\partial t}(t-s)||S(\varphi_{n-1}(s))|ds\\ &+A\int_a^t|\frac{\partial^2 p}{\partial t^2}(t-s)||\varphi_{n-1}(t)|ds+B\int_a^t|\frac{\partial^2 p}{\partial t^2}(t-s)||S(\varphi_{n-1}(s))|ds\\ \le \max(\overline{\overline{A}},\overline{\overline{B}M})&\int_a^t|p(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}+2\max(\overline{A},\overline{BM})\int_a^t|\frac{\partial p}{\partial t}(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}+.\\ &+\max(A,B\overline{M})\int_a^t|\frac{\partial^2 p}{\partial t^2}(t-s)|ds\|\varphi_{n-1}\|_{C^2([T_0,T_1])}\\ &\le 4\varrho\|\varphi_{n-1}\|_{C^2([T_0,T_1])}.\end{aligned}$$

So,

$$\|\varphi_n\|_{C^2([T_0,T_1])}\le 7\varrho\|\varphi_{n-1}\|_{C^2([T_0,T_1])},$$

this implies that

$$\|\varphi_n\|_{C^2([T_0,T_1])}\le \frac{1}{1-7\varrho}\|\varphi_0\|_{C^2([T_0,T_1])}.$$

Let us show that $\{U_n(t)\}_{n\in\mathbb{N}}$ converges uniformly to $U(t)$; Indeed

$$\lim_{n\to+\infty}U_n=\sum_{n=0}^{\infty}\|\varphi_n\|_{C^2(T_0,T_1)}=U(t)\in C^2([T_0,T_1]).$$

Let us now prove that $U(t)$ is a solution of (1.1), we set

$$U(t) = U_n(t) + \Delta_n(t),$$

which gives on the one hand

$$U'(t) = U_n'(t) + \Delta_n'(t),$$
$$U''(t) = U_n''(t) + \Delta_n''(t),$$

on the other hand,

$$|U(t) - f(t) - \int_a^t p(t-s)K(t,s,u(s),S(u(s)))ds|$$
$$\leq |U_n(t) + \Delta_n(t) - f(t) - \int_a^t p(t-s)K(t,s,u(s),S(u(s)))ds|$$
$$\leq |\Delta_n(t)| + \left|\int_a^t p(t-s)(K(t,s,U_{n-1}(s),S(U_{n-1}(s))) - K(t,s,U(s),S(U(s))))ds\right|$$
$$\leq |\Delta_n(t)| + A\int_a^t |p(t-s)||\Delta_n(s)|ds + B\int_a^t |p(t-s)||S(\Delta_n(s))|ds$$
$$\leq |\Delta_n(t)| + \max(A, B\overline{M})\int_a^t p(t-s)ds\|\Delta_n\|_{C^2(T_0,T_1)}.$$
$$\leq \|\Delta_n\|_{C^2([T_0,T_1])} + \varrho\|\Delta_{n-1}\|_{C^2([T_0,T_1])},$$

given that

$$\lim_{n\to+\infty} \|\Delta_n\|_{C^2([T_0,T_1])} = 0,$$

we conclude that U is a solution to (1.1). To complete the proof, we demonstrate the uniqueness of the solution, let u and $\tilde{u}$ be two solutions of (1.1), for $t \in [T_0, T_1]$, we find that

$$|u(t) - \tilde{u}(t)| \leq \varrho\|u - \tilde{u}\|_{C^2([T_0,T_1])}$$
$$|u'(t) - \tilde{u}'(t)| \leq 2\varrho\|u - \tilde{u}\|_{C^2([T_0,T_1])}$$

and

$$|u''(t) - \tilde{u}''(t)| \leq 4\varrho\|u - \tilde{u}\|_{C^2([T_0,T_1])}$$

which allows to obtain

$$|u(t) - \tilde{u}(t)| \leq 7\varrho\|u - \tilde{u}\|_{C^2([T_0,T_1])}$$

and as $1 - 7\varrho \neq 0$ (because $\varrho < \frac{1}{7}$), we find that

$$u = \tilde{u}.$$

Let us now construct the solution on $[T_0, T_2]$. For $t \in [T_1, T_2]$, the equation (1.1) is equivalent to

$$u(t) = F(t) + \int_{T_1}^t p(t-s)K(t,s,u,S(u))ds \tag{2.4}$$

with

$$F(t) = \int_a^{T_1} p(t-s)K(t,s,u,S(u))ds + f(t) \tag{2.5}$$

where, in (2.5), u is the solution obtained in step 1; the equation (2.4) is the same equation where we translate "a" to T_1, we apply the same steps as before, we prove that (2.4) admits a unique solution denoted v, on $[T_1, T_2]$, we show that:

$$\lim_{t\to T_1^+} v(t) = F(T_1) = \lim_{t\to T_1^-} u(t), \tag{2.6}$$

$$\lim_{t\to T_1^+} v'(t) = F'(T_1) = \lim_{t\to T_1^-} u'(t), \tag{2.7}$$

$$\lim_{t\to T_1^+} v''(t) = F''(T_1) = \lim_{t\to T_1^-} u''(t), \tag{2.8}$$

which allows us to obtain a unique solution on $[T_0, T_2]$, denoted w, defined

$$w(t) = \begin{cases} u(t), & t \in [T_0, T_1] \\ v(t), & t \in [T_1, T_2] \end{cases}$$

Thus, from (2.6), (2.7) and (2.8), we have $w \in C^2([T_0, T_2])$, repeating this process on $[T_2, T_3], \cdots, [T_{n-1}, T_n]$ and since we have a finite number of intervals then, we construct then a unique solution in $[a, b]$ of the equation (1.1).

□

3. Numerical study

In this section, we numerically study the equation (1.1). We implement the Nyström method on the regular terms, however, the part that contains a weak singularity, we use the product integration method to eliminate the singularity on the kernel, this approach requires the approximation of the regular part K by the following formula:

$$P_{n1}[K](t, s, u(s), S(u(s))) = \frac{s - t_j}{h} K(t, t_{j+1}, u(t_{j+1}), S(u(t_{j+1}))) + \frac{t_{j+1} - s}{h} K(t, t_j, u(t_j), S(u(t_j))),$$

$t_j \le s \le t_{j+1}$, where, for $n \in \mathbb{N}^*$, we define the subdivision L_n of the interval $[a, b]$ as the set

$$L_n = \Big\{ t_i = a + ih : h = \frac{b-a}{n}, i = 0, 1, \cdots, n \Big\}.$$

Thus, we define our approximate system by:

$$U_0 = f(t),$$

$$U(t_n) = f(t_n) + h \sum_{i=0}^{n-1} p(t_n - t_i)\omega_i K(t_n, t_i, U(t_i), S(U(t_i))),$$

$$V_0 = f'(t),$$

$$V(t_n) = f'(t_n) + h \sum_{i=0}^{n-1} p(t_n - t_i)\omega_i \frac{\partial K}{\partial t}(t_n, t_i, U(t_i), S(U(t_i))) +$$

$$+ h \sum_{i=0}^{n-1} \frac{\partial p}{\partial t}(t_n - t_i)\omega_i K(t_n, t_i, U(t_i), S(U(t_i))),$$

$$W_0 = f''(t),$$

$$W(t_n) = f''(t_n) + h \sum_{i=0}^{n-1} p(t_n - t_i)\omega_i \frac{\partial^2 K}{\partial t^2}(t_n, t_i, U(t_i), S(U(t_i))) +$$

$$+ 2h \sum_{i=0}^{n-1} \frac{\partial p}{\partial t}(t_n - t_i)\omega_i \frac{\partial K}{\partial t}(t_n, t_i, U(t_i), S(U(t_i))) + \alpha_{n,1} K(t_n, t_0, U(t_0), S(U(t_0))),$$

$$+ \sum_{i=1}^{n-1} (\alpha_{n,i+1} + \beta_{n,1}) K(t_n, t_i, U(t_i), S(U(t_i))) + \beta_{n,n} K(t_n, t_n, U(t_n), S(U(t_n))),$$

where, for $0 \leq j \leq n-1$

$$\alpha_{n,i+1} = \frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{\partial^2 p}{\partial t^2}(t_n - s)(t_{i+1} - s)ds,$$

$$\beta_{n,i+1} = \frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{\partial^2 p}{\partial t^2}(t_n - s)(s - t_i)ds.$$

On the other hand, we rewrite this system as

$$U(t_n) = T_1, \tag{3.1}$$

$$V(t_n) = T_2, \tag{3.2}$$

$$W(t_n) = T_3 + \beta_{n,n}K(t_n, t_n, U(t_n), S(U(t_n))), \tag{3.3}$$

such that

$$T_1 = f(t_n) + h\sum_{i=0}^{n-1} p(t_n - t_i)\omega_i K(t_n, t_i, U(t_i), S(U(t_i))),$$

$$T_2 = f'(t_n) + h\sum_{i=0}^{n-1} p(t_n - t_i)\omega_i \frac{\partial K}{\partial t}(t_n, t_i, U(t_i), S(U(t_i)))+$$

$$+h\sum_{i=0}^{n-1} \frac{\partial p}{\partial t}(t_n - t_i)\omega_i K(t_n, t_i, U(t_i), S(U(t_i))),$$

and

$$T_3 = f''(t_n) + h\sum_{i=0}^{n-1} p(t_n - t_i)\omega_i \frac{\partial^2 K}{\partial t^2}(t_n, t_i, U(t_i), S(U(t_i)))+$$

$$+2h\sum_{i=0}^{n-1} \frac{\partial p}{\partial t}(t_n - t_i)\omega_i \frac{\partial K}{\partial t}(t_n, t_i, U(t_i), S(U(t_i))) + \alpha_{n,1}K(t_n, t_0, U(t_0), S(U(t_0))),$$

$$+\sum_{i=1}^{n-1}(\alpha_{n,i+1} + \beta_{n,1})K(t_n, t_i, U(t_i), S(U(t_i))).$$

Theorem 3.1. *Let h be small enough, then the system of equations (3.1), (3.2) and (3.3) admits a unique solution in $\mathbb{R}^{3n}$.*

Proof. We can rewrite this system as follows

$$\chi\begin{pmatrix} U(t_n) \\ V(t_n) \\ W(t_n) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 + \beta_{n,n}K(t_n, t_n, U(t_n), S(U(t_n))) \end{pmatrix},$$

then we have

$$\left|\chi\begin{pmatrix} U(t_n) \\ V(t_n) \\ W(t_n) \end{pmatrix} - \chi\begin{pmatrix} \tilde{U}(t_n) \\ \tilde{V}(t_n) \\ \tilde{W}(t_n) \end{pmatrix}\right| \leq |\beta_{n,n}||K(t_n, t_n, U(t_n), S(U(t_n))) - K(t_n, t_n, \tilde{U}(t_n), S(\tilde{U}(t_n)))|$$

$$\leq |\beta_{n,n}|(A|U(t_n) - \tilde{U}(t_n)| + B|S(U(t_n)) - S(\tilde{U}(t_n))|)$$

$$\leq |\beta_{n,n}| \max(A, B\overline{M})|U(t_n) - \tilde{U}(t_n)|,$$

as $\beta_{n,n} \to 0, n \to \infty$ so the operator χ verifies the fixed point theorem hence the existence and uniqueness of the system of equations (3.1), (3.2) and (3.3). □

3.1. Error study

Before proceeding to the study of the error, we must first show that this method is consistent. To do this, we define the local consistency error of the equation (1.1) by the following formula

$$\Delta(h,t_n)=\delta^1(h,t_n)+\delta_1^2(h,t_n)+\delta_2^2(h,t_n)+\delta_1^3(h,t_n)+\delta_2^3(h,t_n)+\delta_3^3(h,t_n),$$

where,

$$\delta^1(h,t_n)=\int_{t_0}^{t_n}p(t_n-s)K(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n-1}\omega_j p(t_n-t_j)K(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.4)$$

$$\delta_1^2(h,t_n)=\int_{t_0}^{t_n}p(t_n-s)\frac{\partial K}{\partial t}(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n-1}\omega_j p(t_n-t_j)\frac{\partial K}{\partial t}(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.5)$$

$$\delta_2^2(h,t_n)=\int_{t_0}^{t_n}\frac{\partial p}{\partial t}(t_n-s)K(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n-1}\omega_j \frac{\partial p}{\partial t}(t_n-t_j)K(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.6)$$

$$\delta_1^3(h,t_n)=\int_{t_0}^{t_n}p(t_n-s)\frac{\partial^2 K}{\partial t^2}(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n-1}\omega_j p(t_n-t_j)\frac{\partial^2 K}{\partial t^2}(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.7)$$

$$\delta_2^3(h,t_n)=\int_{t_0}^{t_n}\frac{\partial p}{\partial t}(t_n-s)\frac{\partial K}{\partial t}(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n-1}\omega_j \frac{\partial p}{\partial t}(t_n-t_j)\frac{\partial K}{\partial t}(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.8)$$

$$\delta_3^3(h,t_n)=\int_{t_0}^{t_n}\frac{\partial^2 p}{\partial t^2}(t_n-s)K(t_n,s,u(s),S(u(s)))ds-\sum_{j=0}^{n}\delta_{i,j}K(t_n,t_j,u(t_j),S(u(t_j))), \quad (3.9)$$

with,

$$\delta_{i,j}=\begin{cases}\alpha_{n,1}, & j=0,\\ \alpha_{n,j+1}+\beta_{n,j}, & 1\le j\le n-1,\\ \beta_{n,n}, & j=n.\end{cases}$$

Lemma 3.1. *We have*

$$|\delta^1(h,t_n)|\le(b-a)\varpi(h,H_0),\quad |\delta_1^2(h,t_n)|\le(b-a)\varpi(h,H_1),$$

$$|\delta_2^2(h,t_n)|\le(b-a)\varpi(h,H_2),\quad |\delta_1^3(h,t_n)|\le(b-a)\varpi(h,H_3),$$

$$|\delta_2^3(h,t_n)|\le(b-a)\varpi(h,H_4),$$

where,

$$\varpi(h,H_i)=\max_{|\iota-\theta|<h}|H_i(t_n,\iota)-H_i(t_n,\theta)|,\quad i=0,1,2,3,4,$$

$$H_0(t,s)=p(t-s)k(t,s,u(s),S(u(s))),\quad H_1(t,s)=p(t-s)\frac{\partial k}{\partial t}(t,s,u(s),S(u(s))),$$

$$H_2(t,s)=\frac{\partial p}{\partial t}(t-s)k(t,s,u(s),S(u(s))),\quad H_3(t,s)=p(t-s)\frac{\partial^2 k}{\partial t^2}(t,s,u(s),S(u(s))),$$

$$H_4(t,s)=\frac{\partial p}{\partial t}(t-s)\frac{\partial k}{\partial t}(t,s,u(s),S(u(s))),$$

Proof. For all $g \in C^0([a,b])$, then

$$\left|\int_a^b g(t)dt - \frac{h}{2}g(t_0) - h\sum_{i=1}^{n-1} g(t_i) - \frac{h}{2}g(t_n)\right| \leq (b-a)\varpi(h,g), \quad \varpi(h,g)\to 0 \quad h\to 0.$$

Thus we find that

$$\begin{aligned}|\delta^1(h,t_n)| &\leq (b-a)\max_{|\iota-\theta|<h}(p(t_n-\iota)k(t_n,s\iota,u(\iota),S(u(\iota))) - p(t_n-\theta)k(t_n,s\theta,u(\theta),S(u(\theta))))\\ &\leq (b-a)\max_{|\iota-\theta|<h}|H_0(t_n,\iota)-H_0(t_n,\theta)|\\ &\leq (b-a)\varpi(h,H_0).\end{aligned}$$

Analogously, we obtain the other estimates. □

Lemma 3.2. *We have*

$$|\delta_3^3(h,t_n)| \leq \left(\varrho[\varpi(h,f)+\varpi(h,\tilde{H})] + \max_{t\in[a,b],x\in\mathbb{R}}\varpi(h,k(t,\cdot,x,S(x)))\right)\|p\|_{W^{1,2}(a-b,b-a)},$$

where,

$$\tilde{H} = \int_a^t p(t-s)k(t,s,u(s),S(u(s)))ds.$$

Proof. Firstly, we get

$$\begin{aligned}|\delta_3^3(h,t_n)| &\leq \int_{t_0}^{t_n}\left|\frac{\partial^2 p}{\partial t^2}(t_n-s)\right| |k(t_n,s,u(s),S(u(s))) - P_{n1}[k](t_n,s,u(s),S(u(s)))|\,ds\\ &\leq \max_{|\iota-\theta|<h}(|k(t_n,\iota,u(\iota),S(u(\iota))) - k(t_n,\theta,u(\theta),S(u(\theta)))|)\int_{t_0}^{t_n}\left|\frac{\partial^2 p}{\partial t^2}(t_n-s)\right|\end{aligned}$$

$$\leq \left(\max_{|\iota-\theta|<h}\max(A,B\overline{M})|u(\iota)-u(\theta)| + \max_{|\iota-\theta|<h}\max_{x\in\mathbb{R}}|k(t_n,\iota,x,S(x)) - k(t_n,\theta,x,S(x))|\right)\|p\|_{W^{1,2}(a-b,b-a)}.$$

So,

$$|u(\iota)-u(\theta)| \leq |f(\iota)-f(\theta)| + \left|\int_a^t p(t-\iota)k(t_n,\iota,u(\iota),S(u(\iota)))ds - \int_a^t p(t-\theta)k(t_n,\theta,u(\theta),S(u(\theta)))ds\right|.$$

Thus,

$$|\delta_3^3(h,t_n)| \leq \left(\varrho[\varpi(h,f)+\varpi(h,\tilde{H})] + \max_{t\in[a,b],x\in\mathbb{R}}\varpi(h,k(t,\cdot,x,S(x)))\right)\|p\|_{W^{1,2}(a-b,b-a)}.$$

□

From Lemma 3.1-3.2, we easily conclude the next proposition.

Proposition 3.3. *Let H_i and $\varpi(h,H_i), i=1,2,3,4,$ be defined in Lemma 3.1 and $\tilde{H}$ defined in Lemma 3.2. Then*

$$\begin{aligned}\max_{1\leq j\leq n}|\Delta(h,t_j)| &\leq (b-a)(\varpi(h,H_1)+\varpi(h,H_2)+\varpi(h,H_3)+\varpi(h,H_4))+\\ &+\left(\varrho[\varpi(h,f)+\varpi(h,\tilde{H})] + \max_{t\in[a,b],x\in\mathbb{R}}\varpi(h,k(t,\cdot,x,S(x)))\right)\|p\|_{W^{1,2}(a-b,b-a)}.\end{aligned}$$

Now, we define the error E_i by

$$E_i = \varepsilon_i^1 + \varepsilon_i^2 + \varepsilon_i^3$$

such that

$$\varepsilon_i^1 = U_i - u(t_i),$$
$$\varepsilon_i^2 = V_i - u'(t_i),$$
$$\varepsilon_i^3 = W_i - u''(t_i).$$

So, we find

$$\varepsilon_i^1 = h\sum_{j=0}^{i-1} p(t_i - t_j)\omega_j \left[K(t_i, t_j, U(t_j), S(U(t_j))) - K(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta^1(h, t_i), \tag{3.10}$$

$$\varepsilon_i^2 = h\sum_{j=0}^{i-1} p(t_i - t_j)\omega_j \left[\frac{\partial K}{\partial t}(t_i, t_j, U(t_j), S(U(t_j))) - \frac{\partial K}{\partial t}(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta_1^2(h, t_i) + \tag{3.11}$$
$$+h\sum_{j=0}^{i-1} \frac{\partial p}{\partial t}(t_i - t_j)\omega_j \left[K(t_i, t_j, U(t_j), S(U(t_j))) - K(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta_2^2(h, t_i),$$

$$\varepsilon_i^3 = h\sum_{j=0}^{i-1} p(t_i - t_j)\omega_j \left[\frac{\partial^2 K}{\partial t^2}(t_i, t_j, U(t_j), S(U(t_j))) - \frac{\partial^2 K}{\partial t^2}K(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta_1^3(h, t_i) + \tag{3.12}$$
$$+2h\sum_{j=0}^{i-1} \frac{\partial p}{\partial t}(t_i - t_j)\omega_j \left[\frac{\partial K}{\partial t}(t_i, t_j, U(t_j), S(U(t_j))) - \frac{\partial K}{\partial t}(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta_2^3(h, t_i) +$$
$$+\sum_{j=0}^{i} \delta_{ij} \left[K(t_i, t_j, U(t_j), S(U(t_j))) - K(t_i, t_j, u(t_j), S(u(t_j)))\right] - \delta_3^3.$$

According to the definitions (3.10)-(3.12) and with the hypotheses $(H1)$, $(H2)$ we can find the following estimates

$$|\varepsilon_i^1| \leq h\|p\|_{C^0} W \max(A, B\overline{M}) \sum_{j=0}^{i-1} E_j + |\delta^1(h, t_i)|, \tag{3.13}$$

$$|\varepsilon_i^2| \leq h\|p\|_{C^0} W \max(\overline{A}, \overline{BM}) \sum_{j=0}^{i-1} E_j + |\delta_1^2(h, t_i)| + h\|p\|_{C^1} W \max(A, B\overline{M}) \sum_{j=0}^{i-1} E_j + |\delta_2^2(h, t_i)|, \tag{3.14}$$

$$|\varepsilon_i^3| \leq h\|p\|_{C^0} W \max(\overline{\overline{A}}, \overline{\overline{B}M}) \sum_{j=0}^{i-1} E_j + |\delta_1^3(h, t_i)| + 2h\|p\|_{C^1} W \max(\overline{A}, \overline{BM}) \sum_{j=0}^{i-1} E_j + |\delta_2^3(h, t_i)| + \tag{3.15}$$
$$+\sum_{j=0}^{i} \delta_{ij} \max(A, B\overline{M}) E_j + |\delta_3^3(h, t_i)|.$$

The next lemma is intermediate result using to prove the convergence of our scheme.

Lemma 3.4. *Let $a_0, a_1, \ldots$ satisfy*

$$|a_n| \leq A\sum_{i=0}^{n-1} |a_i| + B,$$

where $A > 0$, $B > 0$. Then

$$|a_n| \leq (1 + A)^{n-1}(B + A|a_0|).$$

Proof. The result is easily establish with an inductive argument. □

The following theorem shows that the method is convergent.

Theorem 3.2. *Under hypotheses $(H1)-(H2)$ and assuming that the interval $[a,b]$ can be divided into a finite number of subintervals*

$$[a=a_0,a_1],[a_1,a_2],\cdots,[a_{m-1},a_m=b],$$

such that if j_k denotes the largest integer less than or equal to a_k/h and $\delta_{n,j}=0$ for $j>n$, the poind $\delta_{n,j}$ satisfies the condition

$$\sum_{j=j_k}^{j_{k+1}-1}\frac{h\|p\|_{C^1(a,b)}W\left(3\max(\overline{A},\overline{BM})+(2+\delta_{ij})\max(A,B\overline{M})+\max(\overline{\overline{A}},\overline{\overline{B}M})\right)E_j}{1-\delta_{ii}\max(A,B\overline{M})}\leq 7\varrho<1.$$

This subdivision must be independent of h, so

$$\max_{1\leq j\leq n}|E_j|\leq\left(\frac{1}{1-7\varrho}\right)^{m+2}\max_{1\leq j\leq n}|\Delta(h,t_j)|\to 0,\quad h\to 0.$$

Proof. According to the estimates (3.13)-(3.15) we obtain

$$|E_i|\leq\sum_{j=0}^{i-1}\frac{h\|p\|_{C^1(a,b)}W\left(3\max(\overline{A},\overline{BM})+(2+\delta_{ij})\max(A,B\overline{M})+\max(\overline{\overline{A}},\overline{\overline{B}M})\right)E_j}{1-\delta_{ii}\max(A,B\overline{M})}$$
$$+\frac{|\Delta(h,t_i)|}{1-\delta_{ii}\max(A,B\overline{M})}.$$

Using the Lemma 3.4, we get the desired result. □

3.2. Numerical examples:

Example 1: we consider the nonlinear problem,

$$u(t)=\int_0^t(t-s)^{1.5}\frac{t((-s^4+12s^3)^2+s^6+1)}{1+u^2(s)+[-(s^2u'(s))'+su(s)]^2}ds+f(t).$$

where,

$$f(t)=\left(1-\frac{2}{5}\sqrt{t}\right)t^3,t\in[0,1],$$

and the exact solution is $u(t)=t^3$. It is clear that the hypotheses $(H1)$-$(H2)$ are verified. The next table (1) and figure (1) show the numerical results. These results confirm the theoretical study and show the numerical efficiency of our algorithm built, where we notice that the efficiency is established from $n=50$.

Table 1: Numerical Results of Ex.1

n	E
10	1.98E-2
50	1.90E-3
100	6.84E-4
250	1.75E-4
500	6.25E-5

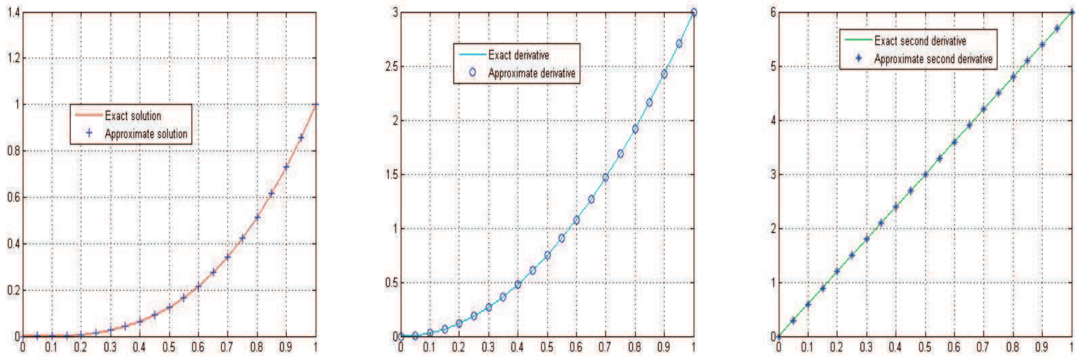

Figure 1: Results of Ex.1 according $n = 20$.

Example 2: we consider the nonlinear problem,

$$u(t) = \int_0^t \frac{(t-s)^2}{10} \ln(t-s) \sin\left(-\phi(s) + \arcsin\left(\frac{t+s}{3}\right) + u(s) + [-(e^{2s}u'(s))' + (s+1)u(s)]\right) ds + f(t).$$

where,

$$\phi(s) = \sin(2s)(s^2+2s) - 2e^{2s}(\sin(2s)(1-4s) + \cos(2s)(4+2s)), \quad f(t) = \frac{23t^4}{4320} + t\sin(2t) - \frac{6t^4}{432}\ln(t),$$

$t \in [0, 1]$ and the exact solution $u(t) = t\sin(2t)$. The next table (2) and figure (2) show the numerical results. These results confirm the theoretical study and show the numerical efficiency of our algorithm built, where we notice that the efficiency is established from $n = 50$.

Table 2: Numerical Results of Ex.2

N	E
10	1.35E-3
50	7.11E-5
100	1.96E-5
250	3.52E-6
500	9.53E-7

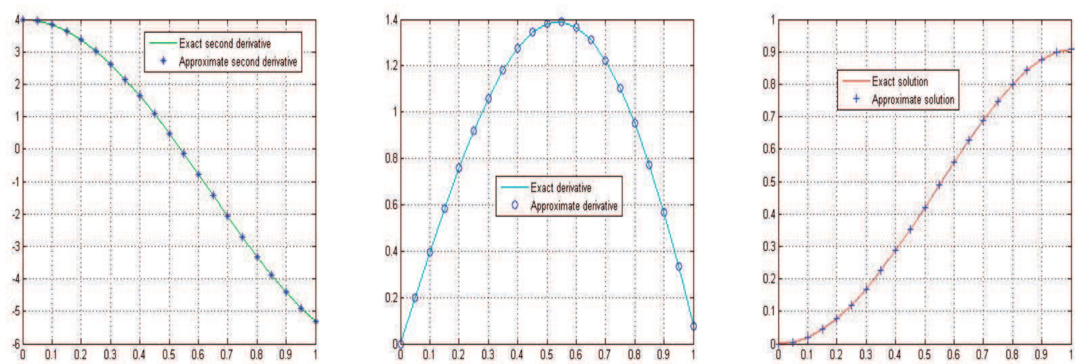

Figure 2: Results of Ex.2 according $N = 20$.

4. Conclusion

This work demonstrates the theoretical conditions of a new class of integro-differential equations with, weakly singular kernel. The Numerical results demonstrate their efficiency and accuracy. However, several challenges remain in solving nonlinear integral equations, especially those of the first type. Therefore, we will apply these solution methods to nonlinear integral equations of the first type as a starting point.

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