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Mappings of $*$-derivation-type on sum of products $a \circ b - ba^*$ on $*$-algebras

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ABSTRACT: Let $\mathcal{A}$ be a prime complex $*$-algebra equipped with a new product $\{a, b\}_* = a \circ b - ba^*$, for all $a, b \in \mathcal{A}$, where $\circ$ is the special Jordan product. In this paper, we proved that every mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(\{a, b\}_*) = \{\delta(a), b\}_* + \{a, \delta(b)\}_*$, for all $a, b \in \mathcal{A}$, is an additive $*$-derivation.

Key Words: Additive $*$-derivations, prime algebras, $*$-algebras.

Contents

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called *additive $*$-derivation* if it is an additive derivation and satisfies $\delta(a^*) = \delta(a)^*$, for all $a \in \mathcal{A}$. The study of mappings between $*$-algebras characterized by the new product $[a, b]_* = ab - ba^*$, for all $a, b \in \mathcal{A}$ (called *$*$-Lie product*), introduced by Brešar and Fošner [3], has received fair amount of attentions (for example, see the works [1], [2], [4], [6], [7] and [8]). In particular, Cui and Li [4] proved that every bijective map preserving $*$-Lie product on factor von Neumann algebras is a $*$-ring isomorphism and Yu and Zhang [8] proved that every $*$-Lie derivation on factor von Neumann algebras is an additive $*$-derivation.

Inspired by [3], we introduce a new product on $\mathcal{A}$, defined by

$$\{a, b\}_* = a \circ b - ba^*,$$

for all $a, b \in \mathcal{A}$, where $a \circ b = \frac{1}{2}(ab + ba)$, for all $a, b \in \mathcal{A}$ (called *special Jordan product*).

A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called *$*$-derivation-type with respect to the product* $\{a, b\}_*$ or *$*$-derivation-type on sum of products* $a \circ b - ba^*$ if

$$\delta(\{a, b\}_*) = \{\delta(a), b\}_* + \{a, \delta(b)\}_*,$$

for all $a, b \in \mathcal{A}$.

The following lemma is technical and straightforward.

Lemma 1.1 *Let $\mathcal{A}$ be a complex $*$-algebra. Then:*

(i) $\{a+b, c\}_* = \{a, c\}_* + \{b, c\}_*$, $\{a, b+c\}_* = \{a, b\}_* + \{a, c\}_*$ *and* $\{a, \lambda b\}_* = \lambda\{a, b\}_*$, *for all* $a, b, c \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

(ii) A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is a $*$-derivation-type with respect to the product $\{a, b\}_*$ if only if

$$\delta(a \circ b - ba^*) = \delta(a) \circ b + a \circ \delta(b) - \delta(b)a^* - b\delta(a)^*,$$

for all $a, b \in \mathcal{A}$.

Ferreira and Marietto [5] proved that every mapping preserving product $\{a, b\}_*$ on factor von Neumann algebras is a $*$-ring isomorphism. Based on this last result and on the results in [4] and [8], the aim of this paper is to prove that a mapping of $*$-derivation-type with respect to the product $\{a, b\}_*$ on a prime complex $*$-algebra is an additive $*$-derivation.

Our main result reads as follows.

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Theorem 1.1 (Main Theorem) *Let $\mathcal{A}$ be a prime complex $*$-algebra with $1_{\mathcal{A}}$ its identity and such that $\mathcal{A}$ has a nontrivial projection. Then every $*$-derivation-type with respect to the product $\{a,b\}_*$ is an additive $*$-derivation.*

2. The proof of the main theorem

The proof of the Main Theorem is made by proving several lemmas.

Lemma 2.1 *Let $\mathcal{A}$ be a $*$-algebra and a mapping $\delta : \mathcal{A} \to \mathcal{A}$ of $*$-derivation-type with respect to the product $\{a,b\}_*$. Then $\delta(0) = 0$.*

Proof: Indeed, $\delta(0) = \delta(\{0,0\}_*) = \{\delta(0), 0\}_* + \{0, \delta(0)\}_* = 0$. □

Lemma 2.2 *Let $\mathcal{A}$ be a prime complex $*$-algebra with $1_{\mathcal{A}}$ its identity and such that $\mathcal{A}$ has a nontrivial projection. Then every mapping $\delta : \mathcal{A} \to \mathcal{A}$ of $*$-derivation-type with respect to the product $\{a,b\}_*$ is additive.*

Following the techniques used by Cui and Li [4], Yu and Zhang [8] and Ferreira and Marietto [5], we shall organize the proof of Lemma 2.2 in a series of properties. We begin, though, with a well-known result that will be used throughout this paper.

Let p_1 be an arbitrary non-trivial projection of $\mathcal{A}$ and write $p_2 = 1_{\mathcal{A}} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ $(i,j = 1,2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function*.

Property 2.1 *For arbitrary elements $a_{ij} \in \mathcal{A}_{ij}$, for $i,j \in \{1,2\}$, hold: (i) $\delta(a_{11}+a_{12}) = \delta(a_{11})+\delta(a_{12})$, (ii) $\delta(a_{11}+a_{21}) = \delta(a_{11})+\delta(a_{21})$, (iii) $\delta(a_{12}+a_{22}) = \delta(a_{12})+\delta(a_{22})$ and (iv) $\delta(a_{21}+a_{22}) = \delta(a_{21})+\delta(a_{22})$.*

Proof: First, note that $\{ip_2, a_{11}\}_* = 0$ which implies that $\{ip_2, a_{11} + a_{12}\}_* = \{ip_2, a_{12}\}_*$, by Lemma 1.1(i). Hence, we compute

$$\begin{aligned}
&\{\delta(ip_2), a_{11} + a_{12}\}_* + \{ip_2, \delta(a_{11} + a_{12})\}_* \\
=&\delta(\{ip_2, a_{11} + a_{12}\}_*) \\
=&\delta(\{ip_2, a_{11}\}_*) + \delta(\{ip_2, a_{12}\}_*) \\
=&\{\delta(ip_2), a_{11}\}_* + \{ip_2, \delta(a_{11})\}_* + \{\delta(ip_2), a_{12}\}_* + \{ip_2, \delta(a_{12})\}_* \\
=&\{\delta(ip_2), a_{11} + a_{12}\}_* + \{ip_2, \delta(a_{11}) + \delta(a_{12})\}_*
\end{aligned}$$

which results that

$$\{ip_2, \delta(a_{11} + a_{12}) - \delta(a_{11}) - \delta(a_{12})\}_* = 0.$$

Set $t = \delta(a_{11} + a_{12}) - \delta(a_{11}) - \delta(a_{12}) = t_{11} + t_{12} + t_{21} + t_{22}$. So, this leads to

$$\{ip_2, t_{11} + t_{12} + t_{21} + t_{22}\}_* = (ip_2) \circ (t_{11} + t_{12} + t_{21} + t_{22}) - (t_{11} + t_{12} + t_{21} + t_{22})(ip_2)^* = 0$$

which yields that $\frac{3}{2}t_{12} + \frac{1}{2}t_{21} + 2t_{22} = 0$. It follows that $t_{12} = t_{21} = t_{22} = 0$. Next, note that $\{ip_1 - \frac{1}{3}ip_2, a_{12}\}_* = 0$ which shows that $\{ip_1 - \frac{1}{3}ip_2, a_{11} + a_{12}\}_* = \{ip_1 - \frac{1}{3}ip_2, a_{11}\}_*$. Hence, we compute

$$\begin{aligned}
&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{11} + a_{12}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{11} + a_{12})\}_* \\
=&\delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{11} + a_{12}\}_*) \\
=&\delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{11}\}_*) + \delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{12}\}_*) \\
=&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{11}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{11})\}_* + \{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{12}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{12})\}_* \\
=&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{11} + a_{12}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{11}) + \delta(a_{12})\}_*
\end{aligned}$$

which implies that

$$\{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{11} + a_{12}) - \delta(a_{11}) - \delta(a_{12})\}_* = 0.$$

We then obtain

$$\{ip_1 - \tfrac{1}{3}ip_2, t_{11}\}_* = (ip_1 - \tfrac{1}{3}ip_2) \circ t_{11} - t_{11}(ip_1 - \tfrac{1}{3}ip_2)^* = 0$$

which leads to $t_{11} = 0$. As a consequence, we have $\delta(a_{11} + a_{12}) = \delta(a_{11}) + \delta(a_{12})$.

Using a similar reasoning as above, we prove the cases (ii), (iii) and (iv). □

Property 2.2 *For arbitrary elements $a_{ij} \in \mathcal{A}_{ij}$, for $i, j\ (i \neq j) \in \{1, 2\}$, holds $\delta(a_{12} + a_{21}) = \delta(a_{12}) + \delta(a_{21})$.*

Proof: Note that $\{ip_1 - \frac{1}{3}ip_2, a_{12} + a_{21}\}_* = \{ip_1 - \frac{1}{3}ip_2, a_{21}\}_*$. Hence, we compute

$$\begin{aligned}
&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{12} + a_{21}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{12} + a_{21})\}_* \\
=&\delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{12} + a_{21}\}_*) \\
=&\delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{12}\}_*) + \delta(\{ip_1 - \tfrac{1}{3}ip_2, a_{21}\}_*) \\
=&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{12}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{12})\}_* + \{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{21}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{21})\}_* \\
=&\{\delta(ip_1 - \tfrac{1}{3}ip_2), a_{12} + a_{21}\}_* + \{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{12}) + \delta(a_{21})\}_*
\end{aligned}$$

which yields that

$$\{ip_1 - \tfrac{1}{3}ip_2, \delta(a_{12} + a_{21}) - \delta(a_{12}) - \delta(a_{21})\}_* = 0.$$

Set $t = \delta(a_{12} + a_{21}) - \delta(a_{12}) - \delta(a_{21}) = t_{11} + t_{12} + t_{21} + t_{22}$. Then

$$\begin{aligned}
\{ip_1 - \tfrac{1}{3}ip_2, t_{11} + t_{12} + t_{21} + t_{22}\}_* = &(ip_1 - \tfrac{1}{3}ip_2) \circ (t_{11} + t_{12} + t_{21} + t_{22}) \\
&- (t_{11} + t_{12} + t_{21} + t_{22})(ip_1 - \tfrac{1}{3}ip_2)^* = 0
\end{aligned}$$

which implies that $2t_{11} + \frac{4}{3}t_{21} - \frac{2}{3}t_{22} = 0$, by which we obtain that $t_{11} = t_{21} = t_{22} = 0$. Next, since $\{ip_1 - 3ip_2, a_{21}\}_* = 0$, then $\{ip_1 - 3ip_2, a_{12} + a_{21}\}_* = \{ip_1 - 3ip_2, a_{12}\}_*$. It follows that

$$\begin{aligned}
&\{\delta(ip_1 - 3ip_2), a_{12} + a_{21}\}_* + \{ip_1 - 3ip_2, \delta(a_{12} + a_{21})\}_* \\
=&\delta(\{ip_1 - 3ip_2, a_{12} + a_{21}\}_*) \\
=&\delta(\{ip_1 - 3ip_2, a_{12}\}_* + \delta(\{ip_1 - 3ip_2, a_{21}\}_*) \\
=&\{\delta(ip_1 - 3ip_2), a_{12}\}_* + \{ip_1 - 3ip_2, \delta(a_{12})\}_* + \{\delta(ip_1 - 3ip_2), a_{21}\}_* + \{ip_1 - 3ip_2, \delta(a_{21})\}_* \\
=&\{\delta(ip_1 - 3ip_2), a_{12} + a_{21}\}_* + \{ip_1 - 3ip_2, \delta(a_{12}) + \delta(a_{21})\}_*
\end{aligned}$$

which results that

$$\{ip_1 - 3ip_2, \delta(a_{12} + a_{21}) - \delta(a_{12}) - \delta(a_{21})\}_* = 0.$$

As a consequence, we have

$$\{ip_1 - 3ip_2, t_{12}\}_* = (ip_1 - 3ip_2) \circ t_{12} - t_{12}(ip_1 - 3ip_2)^* = 0$$

which leads to $t_{12} = 0$. Therefore, $\delta(a_{12} + a_{21}) = \delta(a_{12}) + \delta(a_{21})$. □

Property 2.3 *For arbitrary elements $a_{ij} \in \mathcal{A}_{ij}$, for $i, j \in \{1, 2\}$, hold: (i) $\delta(a_{11} + a_{12} + a_{21}) = \delta(a_{11}) + \delta(a_{12}) + \delta(a_{21})$ and (ii) $\delta(a_{12} + a_{21} + a_{22}) = \delta(a_{12}) + \delta(a_{21}) + \delta(a_{22})$.*

Proof: By Property 2.2, we have

$$\begin{aligned}
&\{\delta(ip_2), a_{11}+a_{12}+a_{21}\}_* + \{ip_2, \delta(a_{11}+a_{12}+a_{21})\}_* \\
=&\delta(\{ip_2, a_{11}+a_{12}+a_{21}\}_*) \\
=&\delta(\{ip_2, a_{11}\}_*) + \delta(\{ip_2, a_{12}\}_*) + \delta(\{ip_2, a_{21}\}_*) \\
=&\{\delta(ip_2), a_{11}\}_* + \{ip_2, \delta(a_{11})\}_* + \{\delta(ip_2), a_{12}\}_* + \{ip_2, \delta(a_{12})\}_* \\
&+ \{\delta(ip_2), a_{21}\}_* + \{ip_2, \delta(a_{21})\}_* \\
=&\{\delta(ip_2), a_{11}+a_{12}+a_{21}\}_* + \{ip_2, \delta(a_{11})+\delta(a_{12})+\delta(a_{21})\}_*
\end{aligned}$$

which implies that

$$\{ip_2, \delta(a_{11}+a_{12}+a_{21}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21})\}_* = 0.$$

Set $t = \delta(a_{11}+a_{12}+a_{21}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) = t_{11}+t_{12}+t_{21}+t_{22}$. This produces

$$\{ip_2, t_{11}+t_{12}+t_{21}+t_{22}\}_* = (ip_2)\circ(t_{11}+t_{12}+t_{21}+t_{22}) - (t_{11}+t_{12}+t_{21}+t_{22})(ip_2)^* = 0$$

which yields that $\frac{3}{2}t_{12} + \frac{1}{2}t_{21} + 2t_{22} = 0$. Thus we conclude that $t_{12} = t_{21} = t_{22} = 0$. Next, by Property 2.1(i), we compute

$$\begin{aligned}
&\{\delta(ip_1-3ip_2), a_{11}+a_{12}+a_{21}\}_* + \{ip_1-3ip_2, \delta(a_{11}+a_{12}+a_{21})\}_* \\
=&\delta(\{ip_1-3ip_2, a_{11}+a_{12}+a_{21}\}_*) \\
=&\delta(\{ip_1-3ip_2, a_{11}\}_*) + \delta(\{ip_1-3ip_2, a_{12}\}_*) + \delta(\{ip_1-3ip_2, a_{21}\}_*) \\
=&\{\delta(ip_1-3ip_2), a_{11}\}_* + \{ip_1-3ip_2, \delta(a_{11})\}_* + \{\delta(ip_1-3ip_2), a_{12}\}_* + \{ip_1-3ip_2, \delta(a_{12})\}_* \\
&+ \{\delta(ip_1-3ip_2), a_{21}\}_* + \{ip_1-3ip_2, \delta(a_{21})\}_* \\
=&\{\delta(ip_1-3ip_2, a_{11}+a_{12}+a_{21}\}_* + \{ip_1-3ip_2, \delta(a_{11})+\delta(a_{12})+\delta(a_{21})\}_*
\end{aligned}$$

which implies that

$$\{ip_1-3ip_2, \delta(a_{11}+a_{12}+a_{21}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21})\}_* = 0.$$

It follows that

$$\{ip_1-3ip_2, t_{11}\}_* = (ip_1-3ip_2)\circ t_{11} - t_{11}(ip_1-3ip_2)^* = 0$$

which shows that $t_{11} = 0$. Therefore, we can conclude that $\delta(a_{11}+a_{12}+a_{21}) = \delta(a_{11}) + \delta(a_{12}) + \delta(a_{21})$.
Using a similar reasoning as above, we prove the case (ii). □

Property 2.4 *For arbitrary elements $a_{ij} \in \mathcal{A}_{ij}$, for $i, j \in \{1, 2\}$, holds $\delta(a_{11}+a_{12}+a_{21}+a_{22}) = \delta(a_{11}) + \delta(a_{12}) + \delta(a_{21}) + \delta(a_{22})$.*

Proof: By Property 2.3(i), we compute

$$\begin{aligned}
&\{\delta(ip_1), a_{11}+a_{12}+a_{21}+a_{22}\}_* + \{ip_1, \delta(a_{11}+a_{12}+a_{21}+a_{22})\}_* \\
=&\delta(\{ip_1, a_{11}+a_{12}+a_{21}+a_{22}\}_*) \\
=&\delta(\{ip_1, a_{11}\}_*) + \delta(\{ip_1, a_{12}\}_*) + \delta(\{ip_1, a_{21}\}_*) + \delta(\{ip_1, a_{22}\}_*) \\
=&\{\delta(ip_1), a_{11}\}_* + \{ip_1, \delta(a_{11})\}_* + \{\delta(ip_1), a_{12}\}_* + \{ip_1, \delta(a_{12})\}_* \\
&+ \{\delta(ip_1), a_{21}\}_* + \{ip_1, \delta(a_{21})\}_* + \{\delta(ip_1), a_{22}\}_* + \{ip_1, \delta(a_{22})\}_* \\
=&\{\delta(ip_1), a_{11}+a_{12}+a_{21}+a_{22}\}_* + \{ip_1, \delta(a_{11})+\delta(b_{21})+\delta(a_{21})+\delta(a_{22})\}_*
\end{aligned}$$

which implies that

$$\{ip_1, \delta(a_{11}+a_{12}+a_{21}+a_{22}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) - \delta(a_{22})\}_* = 0.$$

Set $t = \delta(a_{11}+a_{12}+a_{21}+a_{22}) - \delta(a_{11}) - \delta(a_{12}) - \delta(a_{21}) - \delta(a_{22}) = t_{11}+t_{12}+t_{21}+t_{22}$. It follows that

$$\{ip_1, t_{11}+t_{12}+t_{21}+t_{22}\}_* = (ip_1)\circ(t_{11}+t_{12}+t_{21}+t_{22}) - (t_{11}+t_{12}+t_{21}+t_{22})(ip_1)^* = 0$$

which results that $2t_{11} + \frac{1}{2}t_{12} + \frac{3}{2}t_{21} = 0$. Thus we conclude that $t_{11} = t_{12} = t_{21} = 0$.

Using a similar reasoning as above and the Property 2.3(ii), we prove that $t_{22} = 0$. As a consequence we have, $\delta(a_{11}+a_{12}+a_{21}+a_{22}) = \delta(a_{11}) + \delta(a_{12}) + \delta(a_{21}) + \delta(a_{22})$. □

Property 2.5 *For arbitrary elements $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$, for $i,j\,(i \neq j) \in \{1,2\}$, hold: (i) $\delta(a_{12}+b_{12}) = \delta(a_{12}) + \delta(b_{12})$ and (ii) $\delta(a_{21}+b_{21}) = \delta(a_{21}) + \delta(b_{21})$.*

Proof: First, we note that the following identity holds

$$\begin{aligned}\{p_1+a_{12}, p_2+b_{12}\}_* =&(p_1+a_{12})\circ(p_2+b_{12}) - (p_2+b_{12})(p_1+a_{12})^* \\ =&\tfrac{1}{2}a_{12} + \tfrac{1}{2}b_{12} - a_{12}^* - b_{12}a_{12}^*.\end{aligned}$$

Hence, by Property 2.4, we compute

$$\begin{aligned}&\delta(\tfrac{1}{2}a_{12}+\tfrac{1}{2}b_{12}) + \delta(-a_{12}^*) + \delta(-b_{12}a_{12}^*) \\ =&\delta(\tfrac{1}{2}a_{12}+\tfrac{1}{2}b_{12} - a_{12}^* - b_{12}a_{12}^*) \\ =&\delta(\{p_1+a_{12}, p_2+b_{12}\}_*) \\ =&\{\delta(p_1+a_{12}), p_2+b_{12}\}_* + \{p_1+a_{12}, \delta(p_2+b_{12})\}_* \\ =&\{\delta(p_1)+\delta(a_{12}), p_2+b_{12}\}_* + \{p_1+a_{12}, \delta(p_2)+\delta(b_{12})\}_* \\ =&\{\delta(p_1), p_2\}_* + \{p_1, \delta(p_2)\}_* + \{\delta(p_1), b_{12}\}_* + \{p_1, \delta(b_{12}\}_* \\ &+ \{\delta(a_{12}), p_2\}_* + \{a_{12}, \delta(p_2)\}_* + \{\delta(a_{12}), b_{12}\}_* + \{a_{12}, \delta(b_{12})\}_* \\ =&\delta(\{p_1,p_2\}_*) + \delta(\{p_1,b_{12}\}_*) + \delta(\{a_{12},p_2\}_*) + \delta(\{a_{12},b_{12}\}_*) \\ =&\delta(\tfrac{1}{2}b_{12}) + \delta(\tfrac{1}{2}a_{12} - a_{12}^*) + \delta(-b_{12}a_{12}^*) \\ =&\delta(\tfrac{1}{2}a_{12}) + \delta(\tfrac{1}{2}b_{12}) + \delta(-a_{12}^*) + \delta(-b_{12}a_{12}^*)\end{aligned}$$

which implies that $\delta(\frac{1}{2}a_{12}+\frac{1}{2}b_{12}) = \delta(\frac{1}{2}a_{12}) + \delta(\frac{1}{2}b_{12})$. It therefore follows that $\delta(a_{12}+b_{12}) = \delta(a_{12}) + \delta(b_{12})$.

Similarly, we prove the case (ii) using the identity

$$\begin{aligned}\{p_2+a_{21}, p_1+b_{21}\}_* =&(p_2+a_{21})\circ(p_1+b_{21}) - (p_1+b_{21})(p_2+a_{21})^* \\ =&\tfrac{1}{2}a_{21} + \tfrac{1}{2}b_{21} - a_{21}^* - b_{21}a_{21}^*.\end{aligned}$$

□

Property 2.6 *For arbitrary elements $a_{ii}, b_{ii} \in \mathcal{A}_{ii}$, for $i \in \{1,2\}$, hold: (i) $\delta(a_{11}+b_{11}) = \delta(a_{11}) + \delta(b_{11})$ and (ii) $\delta(a_{22}+b_{22}) = \delta(a_{22}) + \delta(b_{22})$.*

Proof: By Lemma 2.1, we compute

$$\begin{aligned}&\{\delta(ip_2), a_{11}+b_{11}\}_* + \{ip_2, \delta(a_{11}+b_{11})\}_* \\ =&\delta(\{ip_2, a_{11}+b_{11}\}_*) \\ =&\delta(\{ip_2, a_{11}\}_*) + \delta(\{ip_2, b_{11}\}_*) \\ =&\{\delta(ip_2), a_{11}\}_* + \{ip_2, \delta(a_{11})\}_* + \{\delta(ip_2), b_{11}\}_* + \{ip_2, \delta(b_{11})\}_* \\ =&\{\delta(ip_2), a_{11}+b_{11}\}_* + \{ip_2, \delta(a_{11})+\delta(b_{11})\}_*\end{aligned}$$

which implies that

$$\{ip_2, \delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11})\}_* = 0.$$

Set $t=\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11})=t_{11}+t_{12}+t_{21}+t_{22}$. Then

$$\{ip_2,t_{11}+t_{12}+t_{21}+t_{22}\}_*=(ip_2)\circ(t_{11}+t_{12}+t_{21}+t_{22})-(t_{11}+t_{12}+t_{21}+t_{22})(ip_2)^*=0$$

which results that $\frac{3}{2}t_{12}+\frac{1}{2}t_{21}+2t_{22}=0$. As a consequence, we obtain $t_{12}=t_{21}=t_{22}=0$. Next, by Property 2.5(i), for an arbitrary element $c_{12}\in\mathcal{A}_{12}$, we compute

$$\begin{aligned}
&\{\delta(a_{11}+b_{11}),c_{12}\}_*+\{a_{11}+b_{11},\delta(c_{12})\}_*\\
=&\delta(\{a_{11}+b_{11},c_{12}\}_*)\\
=&\delta(\{a_{11},c_{12}\}_*+\{b_{11},c_{12}\}_*)\\
=&\delta(\{a_{11},c_{12}\}_*)+\delta(\{b_{11},c_{12}\}_*)\\
=&\{\delta(a_{11}),c_{12}\}_*+\{a_{11},\delta(c_{12})\}_*+\{\delta(b_{11}),c_{12}\}_*+\{b_{11},\delta(c_{12})\}_*\\
=&\{\delta(a_{11})+\delta(b_{11}),c_{12}\}_*+\{a_{11}+b_{11},\delta(c_{12})\}_*.
\end{aligned}$$

As a consequence, we obtain

$$\{\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11}),c_{12}\}_*=0$$

which results that

$$\{t_{11},c_{12}\}_*=t_{11}\circ c_{12}-c_{12}t_{11}^*=0.$$

It therefore follows that $t_{11}c_{12}=0$ which yields $t_{11}=0$, in view of primeness of $\mathcal{A}$. As a consequence, we obtain $\delta(a_{11}+b_{11})=\delta(a_{11})+\delta(b_{11})$.

Similarly, we prove the case (ii). □

Property 2.7 *δ is an additive mapping.*

Proof: The result is a direct consequence of Properties 2.4, 2.5 and 2.6. □

In the rest of this paper we prove that δ is a $*$-derivation. We assume that all lemmas satisfy the conditions of the Main Theorem.

Lemma 2.3 *The following assertions hold: (i) $\delta(1_{\mathcal{A}})=0$ and $\delta(i1_{\mathcal{A}})=0$, (ii) $\delta(a^*)=\delta(a)^*$, for all $a\in\mathcal{A}$, and (iii) $\delta(ia)=i\delta(a)$, for all $a\in\mathcal{A}$.*

Proof: First, by Lemma 1.1(ii) we have

$$\delta(1_{\mathcal{A}})\circ 1_{\mathcal{A}}+1_{\mathcal{A}}\circ\delta(1_{\mathcal{A}})-\delta(1_{\mathcal{A}})1_{\mathcal{A}}^*-1_{\mathcal{A}}\delta(1_{\mathcal{A}})^*=\delta(1_{\mathcal{A}}\circ 1_{\mathcal{A}}-1_{\mathcal{A}}1_{\mathcal{A}}^*)=0$$

and

$$\delta(i1_{\mathcal{A}})\circ 1_{\mathcal{A}}+(i1_{\mathcal{A}})\circ\delta(1_{\mathcal{A}})-\delta(1_{\mathcal{A}})(i1_{\mathcal{A}})^*-1_{\mathcal{A}}\delta(i1_{\mathcal{A}})^*=\delta((i1_{\mathcal{A}})\circ 1_{\mathcal{A}}-1_{\mathcal{A}}(i1_{\mathcal{A}})^*)=2\delta(i1_{\mathcal{A}})$$

which implies that $\delta(1_{\mathcal{A}})-\delta(1_{\mathcal{A}})^*=0$ and $2i\delta(1_{\mathcal{A}})=\delta(i1_{\mathcal{A}})+\delta(i1_{\mathcal{A}})^*$, respectively. Note that the element $2i\delta(1_{\mathcal{A}})$ is self-adjoint and as a consequence we get $\delta(1_{\mathcal{A}})+\delta(1_{\mathcal{A}})^*=0$. It follows that $\delta(1_{\mathcal{A}})=0$. Next, by Lemma 1.1(ii) again, we have

$$\begin{aligned}
0=&\delta((i1_{\mathcal{A}})\circ(i1_{\mathcal{A}})-(i1_{\mathcal{A}})(i1_{\mathcal{A}})^*)\\
=&\delta(i1_{\mathcal{A}})\circ(i1_{\mathcal{A}})+(i1_{\mathcal{A}})\circ\delta(i1_{\mathcal{A}})-\delta(i1_{\mathcal{A}})(i1_{\mathcal{A}})^*-(i1_{\mathcal{A}})\delta(i1_{\mathcal{A}})^*\\
=&i(3\delta(i1_{\mathcal{A}})-\delta(i1_{\mathcal{A}})^*).
\end{aligned}$$

This results that $3\delta(i1_{\mathcal{A}})-\delta(i1_{\mathcal{A}})^*=0$ which leads to $\delta(i1_{\mathcal{A}})=0$. Thus, for an arbitrary element $a\in\mathcal{A}$, we have

$$\delta(a)-\delta(a)^*=\delta(a)\circ 1_{\mathcal{A}}+a\circ\delta(1_{\mathcal{A}})-\delta(1_{\mathcal{A}})a^*-1_{\mathcal{A}}\delta(a)^*=\delta(a\circ 1_{\mathcal{A}}-1_{\mathcal{A}}a^*)=\delta(a)-\delta(a^*)$$

and

$$\begin{aligned}2\delta(ia) =&\delta((i1_{\mathcal{A}})\circ a - a(i1_{\mathcal{A}})^*)\\ =&\delta(i1_{\mathcal{A}})\circ a + (i1_{\mathcal{A}})\circ\delta(a) - \delta(a)(i1_{\mathcal{A}})^* - a\delta(i1_{\mathcal{A}})^*\\ =&2i\delta(a).\end{aligned}$$

Therefore, $\delta(a^*) = \delta(a)^*$ and $\delta(ia) = i\delta(a)$, for all $a \in \mathcal{A}$. □

Lemma 2.4 *The mapping δ is a derivation.*

Proof: For arbitrary elements $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\delta(b\circ a) - \delta(ab^*) =&\delta(b\circ a - ab^*)\\ =&\delta(b)\circ a + b\circ\delta(a) - \delta(a)b^* - a\delta(b)^*,\end{aligned} \tag{2.1}$$

by Property 2.7. Replacing b by ib in (2.1), we get

$$\begin{aligned}\delta((ib)\circ a) - \delta(a(ib)^*) =&\delta((ib)\circ a - a(ib)^*)\\ =&\delta(ib)\circ a + (ib)\circ\delta(a) - \delta(a)(ib)^* - a\delta(ib)^*\end{aligned}$$

which implies that

$$\delta(b\circ a) + \delta(ab^*) = \delta(b)\circ a + b\circ\delta(a) + \delta(a)b^* + a\delta(b)^*, \tag{2.2}$$

by Lemma 2.3(iii). Hence, subtracting (2.1) from (2.2), we get

$$\delta(ab^*) = \delta(a)b^* + a\delta(b)^*. \tag{2.3}$$

Therefore, from the Lemma 2.3(ii), we can conclude that

$$\delta(ab) = \delta(a)b + a\delta(b),$$

for all $a, b \in \mathcal{A}$. Consequently, the mapping δ is a derivation. □

Finalizing the proof of the Main Theorem, we conclude that δ is a $*$-derivation, by Property 2.7 and the Lemmas 2.3(ii) and 2.4.

As a consequence of the Main Theorem, we have the following result.

Corollary 2.1 *Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$ and $\delta : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a mapping of $*$-derivation-type with respect to the product $\{a, b\}_*$. Then there exists an element $t \in \mathscr{B}(\mathscr{H})$ satisfying $t + t^* = 0$ and such that $\delta(a) = at - ta$, for all $a \in \mathscr{B}(\mathscr{H})$.*

Proof: The prove is the same as in [8, Corollary 2.1.]. □

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