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Corrigendum and Addendum to "h-open sets in Topological spaces"

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ABSTRACT: The aim of this paper is to rectify an important characterization of h-open sets in topological spaces given by F. Abbas in 2021. F. Abbas offered examples of h-open sets in finite topological spaces. We'll start with a key finding that gives us instances of h-open sets in infinite topological spaces. Another significant purpose of this paper is to introduce new notions related to these h-open sets and study their fundamental properties.

Key Words: h-open sets, h-compactness, h-connectedness.

Contents

1. Introduction

F. Abbas [1] presented new types of open sets called h-open sets in 2021. He looked into some of the topological aspects of these open sets and proposed several directions. Though, few corrections has already been pointed out by Cakalli and Dagci in [2], still after thorough analysis of [1], it has been observed that a minor mistake erupted which requires correction that is given in section 3. The concepts of connectedness and compactness [3,4,5] are widely recognized and have great importance not only in general topology but also in other advanced disciplines of mathematics. So, this study will also present new notions like h-connectedness and h-compactness. In addition, we shall also examine some separation axioms via these h-open sets and characterize their fundamental properties.

2. Preliminaries

Throughout this paper, X and Y will represent two topological spaces with topologies τ and σ respectively.

Definition 2.1 *[1] A subset A of a topological space X is said to be h-open if $A \subseteq Int(A \cup U)$ for every $U \in \tau$, $U \neq \phi, X$. Complement of a h-open set is called a h-closed set.*

Clearly, every open set is h-open but converse need not be true. τ^h shall be the notation used for the family of all h-open sets of a topological space (X, τ).

Example 2.1 Let X be any non-empty set and $x \in X$. Let $\tau = \{\phi, X \setminus \{x\}, X\}$ be a topology endowed on X. Then $\tau^h = \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X.

Let us recall some basic definitions that will be used in the subsequent sections.

Definition 2.2 *[1] The h-interior of a subset A of a topological space X is defined as the union of all h-open sets in X contained in A, denoted by $Int_h(A)$.*

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It is clear that $Int_h(A)$ is h-open set, for any subset A of X.

Definition 2.3 *[1] The h-closure of a subset A of a topological space X is defined as the intersection of all h-closed sets in X containing A, denoted by $Cl_h(A)$.*

It is clear that $Cl_h(A)$ is h-closed set for any subset A of X.

Definition 2.4 *[1] A function $g : X \to Y$ is said to be h-continuous if inverse image of every open set in Y is h-open in X.*

Definition 2.5 *[1] A function $g : X \to Y$ is said to be h-irresolute if inverse image of every h-open set in Y is h-open in X.*

3. Corrigendum

To prove invalidity of Lemma 2.23 in [1], we shall consider the Example 2.2 in [1].

Lemma 3.1 *[1] A subset A of a topological space X is h-open if and only if there exists an open set U in X such that $A \subseteq U \subseteq Cl(A)$.*

Example 3.1 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and
$\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$
Here $\{b\}$ is h-open. But there does not exist any open set U such that $\{b\} \subseteq U \subseteq Cl(\{b\})$
Also, $\{a, d\} \subseteq X \subseteq Cl(\{a, d\}) = X$ for some open set $U = X$. But $\{a, d\}$ is not h-open.

Further, it was observed that in [1], Lemma 2.24 has been proved by using Lemma 2.23, but the proof could have been possible even without Lemma 2.23 by using Theorem 2.5 and Theorem 2.8.

4. Addendum

This section will begin with an intriguing result that generates some examples of h-open sets in infinite topological spaces.

Theorem 4.1 *Let X be any non-empty set endowed with cofinite topology. Then $\{x\}$ is h-open $\forall\, x \in X$.*

Proof: If X is finite, then the proof follows trivially.
Suppose X is infinite and let $x \in X$. As the set has been endowed with cofinite topology, any open set will be in the form of $X \setminus \{finite\ set\}$. Then $Int\ (\{x\} \cup (X \setminus \{finite\ set\}))$ can be X, $X \setminus \{finite\ set\}$ or $\{x\} \cup (X \setminus \{finite\ set\})$. In every possibility, we have $\{x\} \subseteq Int(\{x\} \cup U)\ \forall\, U \in \tau,\ U \neq \phi, X$. This implies that $\{x\}$ is h-open $\forall\, x \in X$. □

Corollary 4.1 *Let X be any non-empty set endowed with cofinite topology. Then all subsets of X are h-open.*

Theorem 4.2 *Let X be any non-empty set endowed with co-countable topology. Then all subsets of X are h-open.*

Proof: The proof proceeds as the proof of previous theorem. □

Example 4.1 Consider $\mathbb{R}$, the set of real numbers endowed with the topology $\tau = \{A \subseteq \mathbb{R} : A = \phi \text{ or } X \setminus A \text{ is bounded}\}$. Then all subsets of $\mathbb{R}$ are h-open.

Some results regarding the notions of h-continuous and h-irresolute functions can be seen in [1]. We would like to provide another important result as follows:

Theorem 4.3 *Every open and continuous function is h-irresolute.*

Proof: Straightforward. □

Definition 4.1 *A topological space X is said to be h-disconnected if and only if it can be expressed as a union of two disjoint non-empty h-open subsets of X.*
A topological space that is not h-disconnected is called as a h-connected space.

Clearly, every disconnected space is h-disconnected and every h-connected space is connected. However, converse of both of the results need not be true. Sierpinski space can serve as a beautiful example.

- An indiscrete topological space is an example of h-connected space.
- Discrete space having more than one element is always a h-disconnected space.
- Sorgenfrey line is always h-disconnected.

Theorem 4.4 *Let (X, τ) be a topological space. Then the following are equivalent:*

1. *X is h-connected space.*
2. *X and ϕ are only h-clopen subset of X.*
3. *Every h-continuous function from X onto discrete space with atleast two points is a constant function.*

Proof: (1) $\Rightarrow$ (2): Suppose X is h-connected and there exists some h-clopen subset of X say U other than X and ϕ. Then $X = U \cup U^c$, a contradiction as X is h-connected. Hence, (2) follows.
(2) $\Rightarrow$ (3): Suppose X is h-connected and let $g : X \to \{a, b\}$ be a h-continuous function onto a discrete space. We claim that g is a constant function. On contrary, let us suppose that g is not a constant function. By hypothesis, $g^{-1}\{a\}$ and $g^{-1}\{b\}$ are h-open in X. Also, $X = g^{-1}\{a\} \cup g^{-1}\{b\}$. This contradicts the fact that X is h-connected and hence the proof follows.
(3) $\Rightarrow$ (1): Suppose X is h-disconnected. Then $X = U_1 \cup U_2$, where U_1 and U_2 are h-open subsets of X. Define $g : X \to \{a, b\}$ by $g(x) = a \ \forall \ x \in \ U_1$ and $g(x) = b \ \forall \ x \in \ U_2$. Clearly, g is a h-continuous function. But g is not constant, a contradiction. Hence X is h-disconnected space. □

Theorem 4.5 *Let X and Y be two topological spaces endowed with topologies τ and σ respectively. Suppose X is h-connected, then the following holds:*

1. *h-continuous image of X is connected.*
2. *h-irresolute image of X is h-connected.*

Proof: (1) Suppose $g : X \to Y$ is h-continuous, surjective and X is h-connected. We have to show that Y is connected. On contrary, let us suppose that Y is disconnected. Then $Y = U \cup V$, where U and V are non-empty disjoint open sets of Y. Further, $g^{-1}(U)$ and $g^{-1}(V)$ are h-open in X as g is h-continuous. Also, $g^{-1}(U) \cup g^{-1}(V) = X$. This implies that X is h-disconnected, a contradiction. Hence, Y is connected.
(2) Suppose $g : X \to Y$ is h-irresolute, surjective and X is h-connected. We have to show that Y is h-connected. On contrary, let us suppose that Y is h-disconnected. Then $Y = U \cup V$, where U and V are non-empty disjoint h-open sets of Y. Further, $g^{-1}(U)$ and $g^{-1}(V)$ are h-open in X as g is h-irresolute. Also, $g^{-1}(U) \cup g^{-1}(V) = X$. This implies that X is h-disconnected, a contradiction. Hence, Y is h-connected. □

Corollary 4.2 *Let $g : X \to Y$ be open, continuous and surjective map. Suppose X is h-connected, then image of X is h-connected.*

Remark 4.1 *It should be noted that h-closure of a h-connected set in a topological space need not be h-connected, which clearly follows from the example given below:*

Example 4.2 Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, X\}$. Then $\tau^h = \{\phi, \{a\}, \{b, c, d\}, X\}$. But here $\{a, b, c\}$ is h-connected but $Cl_h(\{a, b, c\}) = X$ is h-disconnected.

Definition 4.2 *A collection $\{H_i : i \in I\}$ of h-open sets is said to be a h-open cover for X if $X \subseteq \bigcup_{i \in I} H_i$.*

It is evident from the above definition that every open cover is h-open cover. But converse need not be true, which follows from the following example:

Example 4.3 Consider the set $\mathbb{R}$ of real numbers with the cofinite topology. Then $\{\{x\} : x \in \mathbb{R}\}$ is h-open cover but not open cover.

Definition 4.3 *A topological space X is said to be h-compact if every h-open cover has a finite sub-cover.*

Theorem 4.6 *Every h-compact space is compact.*

Proof: Let X be a h-compact space and $\{H_i : i \in I\}$ be an open cover for X. Since every open set is h-open set, it follows that $\{H_i : i \in I\}$ is h-open cover. Further, X is h-compact implies $\{H_i : i \in I\}$ has a finite sub-cover. Therefore, we have X compact. □

But converse of the above theorem need not be true. Consider the following example:

Example 4.4 The set $\mathbb{R}$ of real numbers with the co-finite topology is compact but not h-compact as the h-open cover $\{\{x\} : x \in \mathbb{R}\}$ has no finite sub-cover.

Theorem 4.7 *Let X and Y be two topological spaces endowed with topologies τ and σ respectively. Suppose X is h-compact, then the following holds:*

1. *h-continuous image of X is compact.*
2. *h-irresolute image of X is h-compact.*

Proof: (1) Suppose $g : X \to Y$ be a h-continuous, surjective map and X be a h-compact space. Consider an open cover $\{H_i : i \in I\}$ for Y. By given hypothesis, $\{g^{-1}(H_i) : i \in I\}$ is h-open cover for X having a finite subcover. Let $\{g^{-1}(H_i) : i = 1, 2, 3....n\}$ be that finite subcover. Also, $Y = \bigcup_{i=1}^{n} H_i$ as g is onto, which completes the proof.

(2) Suppose $g : X \to Y$ be a h-irresolute, surjective map and X be a h-compact space. Consider a h-open cover $\{H_i : i \in I\}$ for Y. By given hypothesis, $\{g^{-1}(H_i) : i \in I\}$ is h-open cover for X having a finite subcover. Let $\{g^{-1}(H_i) : i = 1, 2, 3....n\}$ be that finite subcover. Also, $Y = \bigcup_{i=1}^{n} H_i$ as g is onto, which completes the proof. □

Corollary 4.3 *Let $g : X \to Y$ be open, continuous and surjective map. Suppose X is h-compact, then image of X is h-compact.*

Definition 4.4 *A topological space (X, τ) is said to be*
(1) hT_0 if for every two $p \neq q$ in X, there exists a h-open set H_1 containing p but not q.
(2) hT_1 if for every two $p \neq q$ in X, there exists a h-open set H_1 containing p but not q and an h-open set H_2 containing q but not p.
(3) hT_2 if for every two $p \neq q$ in X, there exists h-open sets H_1 and H_2 containing p and q respectively such that $H_1 \cap H_2 = \phi$.

Example 4.5 Consider $\mathbb{R}$, the set of real numbers with the usual topology, then $\mathbb{R}$ is hT_0, hT_1 as well as hT_2.

It is clear that $hT_2 \Rightarrow hT_1 \Rightarrow hT_0$.

Note : $hT_0 \nRightarrow hT_1$ follows from the following example, but $hT_1 \Rightarrow hT_2$ is open to question.

Example 4.6 $X = \{a, b, c\},\ \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau^h = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is hT_0 but not hT_1.

Theorem 4.8 *Let (X, τ) be a topological space. Then the following are equivalent:*

1. *X is hT_0 space.*
2. *For every two $p \neq q$ of X, $Cl_h(\{p\}) \neq Cl_h(\{q\})$.*

Proof: (1) $\Rightarrow$ (2) : Let $p \neq q$ be any two points of X. Since X is hT_0 space, there exists a h-open set H containing p but not q. Then H_c is a h-closed set containing q but not p. We know that $Cl_h(\{q\})$ is the smallest h-closed set containing q. Thus, we have $Cl_h(\{q\}) \subset H^c$ which implies that $p \notin Cl_h(\{q\})$. Hence the proof.
(2) $\Rightarrow$ (1): Let us suppose $r \in X$ such that $r \in Cl_h(\{p\})$ but $r \notin Cl_h(\{q\})$. Now we shall show that $Cl_h(\{q\})$ does not contain p. On contrary, suppose $Cl_h(\{q\})$ contains p. This implies that $Cl_h(\{p\})$ is subset of $Cl_h(\{q\})$, a contradiction as $r \notin Cl_h(\{q\})$. As a result, $(Cl_h(\{q\}))^c$ is a h-open set that contains p and does not contain q. □

Theorem 4.9 *Let (X, τ) be a topological space. Then the following are equivalent:*

1. *X is hT_1 space.*
2. *$\{x\}$ is h-closed $\forall\ x \in X$.*

Proof: (1) $\Rightarrow$ (2) : Let $x \in X$ arbitrary and y be a point of X that does not belong to $\{x\}$. Clearly, x and y are distinct points and by given hypothesis, there exists a h-open set H_y containing y but not x. Thus, y belongs to H_y, subset of $\{x\}^c$. This implies that $\{x\}^c$ is h-open.
(2) $\Rightarrow$ (1) : Let x and y be any two distinct points of X. It is clear that y does not belong to $\{x\}$. By given hypothesis, $\{x\}^c$ is h-open that contains y but not x, which completes the proof. □

5. Conclusion

A brief study of h-connectedness, h-compactness and separation axioms via h-open sets is presented in this paper. Examples are also provided wherever required. Some well known properties in topological spaces are explored with respect to the aforementioned concepts.

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