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Regularizing Effect of Absorption Term in Singular and Degenerate Elliptic Problems

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ABSTRACT: In this paper we study the existence and regularity of solutions to the following singular problem

$$\begin{cases} -\text{div}\Big(a(x,u)|\nabla u|^{p-2}\nabla u\Big) + |u|^{s-1}u = h(u)f & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

proving that the lower order term $u|u|^{s-1}$ has some regularizing effects on the solutions in the case of an elliptic operator with degenerate coercivity.

Key Words: Degenerate coercivity, singular non-linearity, regularity, distributional solutions, entropy solutions, Sobolev spaces.

Contents

1. Introduction

Let us consider the following problem

$$\begin{cases} -\text{div}\Big(a(x,u)|\nabla u|^{p-2}\nabla u\Big) + |u|^{s-1}u = h(u)f & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < N$, Ω is a bounded set in $I\!R^N$ and $a : \Omega \times I\!R \longrightarrow I\!R$ is a carathéodory function such that for a.e. $x \in \Omega$ and for every $\ell \in I\!R$, we have

$$a(x,\ell) \geq \frac{\alpha}{(1+|\ell|)^{\theta}}, \tag{1.2}$$

$$a(x,\ell) \leq \beta, \tag{1.3}$$

for some real positive constants α, β and $0 \leq \theta \leq 1$. Moreover, f is a non negative $L^m(\Omega)$ function, with $m \geq 1$ and the term $h : [0,\infty) \longrightarrow [0,\infty)$ is continuous, bounded outside the origin with $h(0) \neq 0$ and such that the following properties hold true

$$\exists c,\, \gamma > 0 \text{ such that } h(\ell) \leq \frac{c}{\ell^{\gamma}} \quad \forall \ell \in (0,+\infty), \tag{1.4}$$

for some real number γ such that $0 < \gamma \leq 1$. Singular problems of this type have been largely studied in the past also for their connection with the theory of non-Newtonian fluids, boundary layer phenomena for viscous fluids and chemical heterogeneous (see for instance [18,21]).

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Let us briefly recall the mathematical framework concerning problem (1.1) we start with the case $a(x,u):=a(x), \theta=0$ and f lies just in $L^1(\Omega)$ has been studied in [3]. Problem (1.1) in the non-singular case $h(u)=1$, the author of [9] studied the existence and regularity of weak solution to the elliptic problem with degenerate coercivity

$$\begin{cases} -\operatorname{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big)+|u|^{r-1}u=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in the case where $f\in L^m(\Omega)$ with $m\geq 1$ and $\theta>0$.

If $p=2$, the problem (1.5) have been treated in [11], i.e, in the case of the following problem

$$\begin{cases} -\operatorname{div}\big(a(x,u)\nabla u\big)+|u|^{r-1}u=f & \text{in } \Omega \\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{1.6}$$

the authors studied the lower order term $|u|^{r-1}u$ in (1.5) and (1.6) that has the regularizing effects of the solutions in the case where $f\in L^m(\Omega)$, with $m\geq 1$ and $\theta\geq 0$. When p = 2 and the lower-order term does not appear in (1.5), the existence and regularity of solution to problem (1.5) are proved in [4]. The extension of this work to general case is investigated in [1].

Now we turn our attention recalling some results when the authors had added the singular sourcing term. Problems of p-Laplacien type (i.e $\theta=0$), have been well studied in both the existence and regularity aspects with f having different summbility (see [12]). This frame work has been extended to the problems with a lower order, considering

$$\begin{cases} -\Delta u+u^s=\frac{f}{u^\gamma} & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $f\in L^m(\Omega)$, $m\geq 1$, $0\leq\gamma<1$. Existence and regularity was established in [13]. Recently Olivia [22] have proved the existence and regularity of solution to the problem

$$\begin{cases} -\Delta_p u+g(u)=h(u)f & \text{in } \Omega \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

f is a nonnegative and it belongs to $f\in L^m(\Omega)$, $m\geq 1$, for some $0<\gamma\leq 1$. While $g(s)$ is continuous, $g(0)=0$ and, as $s\to\infty$, could act as s^q with $q\geq -1$, the p-Laplacian operator is $\Delta_p u:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and h is continuous, it possibly blows up at the origin and it is bounded at infinity.

In [25], the authors studied the following degenerate elliptic problem with a singular nonlinearity:

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u))=fh(u) & \text{in } \Omega \\ u\geq 0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

f is a nonnegative and it belongs to $f\in L^m(\Omega)$, $m\geq 1$, and h satisfied the condition in (1.4). Following this way in this paper, we are interested again in the regularity results. By adding the singular term to the right of (1.5), we investigate the regularity of solutions of problems of kind (1.1) in light of the influence of some lower order terms. Let us observe that we refer to [15,16,19] for more details on singular problems.

In the study of problem (1.1), there are one to two difficulties, the first one is the fact that, due to hypothesis (1.2) the differential operator $A(u)=\operatorname{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big)$ is not coercive on $W_0^{1,p}(\Omega)$, when u is large (see [24]). Due to the lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (see [20]) cannot be applied. The second difficulty comes from the right-hand side is singular in the variable u. We overcome these difficulties by replacing operator A by another one defined by means of truncations, and approximating the singular term by non singular one. We will prove in section 3 that these problems admit a bounded $W_0^{1,p}(\Omega)$ solution u_n, $n\in\mathbb{N}$ by using Schauder's fixed point theorem. In section 4 we will get some a priori estimates and convergence results on the sequence of approximating solutions. In the end, we pass to the limit in the approximate problems.

Notations:

In the entire paper Ω is an open and bounded subset of $I\!R^N$, with $N>1$, we denote by ∂A the boundary and by $|A|$ the Lebesgue measure of a subset A of $I\!R^N$.

For any $q>1$, $q'=\frac{q}{q-1}$ is the Hölder conjugate exponent of q, while for any $1<p<N$, $p^*=\frac{Np}{N-p}$ is the Sobolev conjugate exponent of p. For fixed $k>0$ we will use of the truncation T_k defined as $T_k(t)=max\big(-k,min(k,t)\big)$, we will also use the following functions

$$V_{\delta,k}(t)=\begin{cases}1 & t\leq k\\ \frac{k+\delta-t}{\delta} & k<t<k+\delta,\\ 0 & t\geq k+\delta,\end{cases} \tag{1.10}$$

and

$$S_{\delta,k}(t):=1-V_{\delta,k}(t). \tag{1.11}$$

For the sake of implicity we will often use the simplified notation

$$\int_\Omega f:=\int_\Omega f(x)dx,$$

when referring to integrals when no ambiguity on the variable of integration is possible. If no otherwise specified, we will denote by c serval constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c can depend on Ω, γ, N, k, ...) but the will never depend on the indexes of the sequences we will often introduce.

2. Statement of definitions and the main results

2.1. Statement of definitions

In this context we deal with some class of solutions

Definition 2.1. *A nonnegative measurable function u is a weak solution to problem (1.1) if $u\in W_0^{1,1}(\Omega)$ and if*

$$a(x,u)|\nabla u|^{p-2}\nabla u\in(L^1(\Omega))^N,\ \ h(u)f\in L^1(\Omega),\ |u|^{s-1}u\in L^1(\Omega),$$

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u.\nabla\varphi+\int_\Omega|u|^{s-1}u\varphi=\int_\Omega fh(u)\varphi\ \ \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{2.1}$$

Definition 2.2. *A nonnegative measurable function u is an entropy solution to problem (1.1) if $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$ and*

$$a(x,T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u)\in(L^1(\Omega))^N,|u|^s\in L^1(\Omega),\ h(u)f\in L^1(\Omega),$$

and if

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u.\nabla T_k(u-\varphi)dx+\int_\Omega|u|^{s-1}uT_k(u-\varphi)dx\leq\int_\Omega fh(u)T_k(u-\varphi)dx, \tag{2.2}$$

for every $k>0$ and for any $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Definition 2.3. *The Marcinkwicz space $M^\jmath(\Omega)$, consists of all measurable functions $v:\Omega\longrightarrow I\!R$ that satisfy the following condition: there exists $c>0$ such that*

$$meas\{|v|\geq k\}\leq\frac{c}{k^\jmath}\ \ for\ all\ \ k>0. \tag{2.3}$$

and, then, we define

$$\|u\|_{M^\jmath(\Omega)}:=(\inf\{c>0:(2.3)\ holds\ \})^{\frac{1}{\jmath}}.$$

Moreover, $L^1(\Omega)\subset M^1(\Omega)$ and $L^\jmath(\Omega)\subset M^\jmath(\Omega)\subset L^{\jmath-\varepsilon}(\Omega)$ for every $\jmath>1$ and $0<\varepsilon\leq\jmath-1$.

Let

$$p_0:=1+\frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma}. \tag{2.4}$$

2.2. Statement of the main results

The main results of this paper are stated as follows:

Theorem 2.4. *Let a satisfy (1.2) and (1.3). Let h satisfy (1.4) with $0 < \gamma \leq 1$ and let f be a positive function in $L^m(\Omega)$, $m > 1$, $1 < p < N$.*

i) If $s \geq \frac{1+\theta-\gamma}{m-1}$, then there exists a weak solution u to problem (1.1) such that

$$u \in W_0^{1,p}(\Omega) \cap L^{ms+\gamma}(\Omega).$$

ii) If $\frac{1+\theta-\gamma}{pm-1} < s < \frac{1+\theta-\gamma}{m-1}$, then there exists a weak solution u to problem (1.1) such that

$$u^{ms+\gamma} \in L^1(\Omega) \text{ and } u \in W_0^{1,\sigma}(\Omega) \quad, 1 < \sigma = \frac{pms}{1+\theta+s-\gamma}.$$

iii) If $0 < s \leq \frac{1+\theta-\gamma}{pm-1}$, then there exists an entropy solution u to problem (1.1) such that

$$u^{ms+\gamma} \in L^1(\Omega) \quad \text{and} \quad |\nabla u| \in M^{\frac{pms}{1+\theta+s-\gamma}}(\Omega).$$

Remark 2.5. *If $p = 2$ and $\gamma = 0$; the result of Theorem 2.4 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [11], Theorem 1.5).*

Theorem 2.6. *Under the assumptions (1.2)-(1.3) and h satisfy (1.4), with $0 < \gamma \leq 1$ and let $f \in L^m(\Omega)$ be non negative function, with, $m > 1$, $p_0 < p < N$.*

i) If $0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, then there exists a weak solution u to problem (1.1) such that $u^{ms+\gamma} \in L^1(\Omega)$ and $u \in W_0^{1,\sigma}(\Omega)$, where $1 < \sigma = \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}$.

ii) If $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, then item(ii) of Theorem 2.4 holds.

Remark 2.7. *If $\gamma = 0$; the result of Theorem 2.4 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [9], Theorem 3) and Theorem 2.6 coincides with ([9], Theorem 4).*

Theorem 2.8. *Let a satisfy (1.2) and (1.3). Let h satisfy (1.4) with $0 < \gamma \leq 1$ and let f be a positive function in $L^1(\Omega)$, $1 < p < N$.*

a) If $s > \frac{1+\theta-\gamma}{p-1}$, then there exists a weak solution u to problem (1.1) such that $u^{s+\gamma} \in L^1(\Omega)$ and $u \in W_0^{1,r}(\Omega)$, where $1 < r < \frac{ps}{s+1+\theta-\gamma}$.

b) If $0 < s \leq \frac{1+\theta-\gamma}{p-1}$, then there exists an entropy solution u to problem (1.1) such that

$$u^{s+\gamma} \in L^1(\Omega) \quad \text{and} \quad |\nabla u| \in M^{\frac{ps}{s+1+\theta-\gamma}}(\Omega).$$

Remark 2.9. *If $p = 2$ and $\gamma = 0$; the result of Theorem 2.8 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [11], Theorem 1.4).*

Theorem 2.10. *Assume that (1.2) and (1.3) hold true. Let h satisfy (1.4) with $0 < \gamma \leq 1$. Let $p_0 = 1 + \frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma} < p < N$ and let f be positive function in $L^1(\Omega)$. Then there exists a weak solution u to problem (1.1) such that if $0 < s \leq \frac{N(1-\gamma)+\gamma}{N-1}$*

$$\text{then } u \text{ belong to } W_0^{1,r}(\Omega), \text{ with } 1 < r < \frac{N[p-\theta-1+\gamma]}{N-\theta+\gamma-1}.$$

Remark 2.11. *If $\gamma = 0$; the result of Theorem 2.8 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [9], Theorem 1) and Theorem 2.10 coincides with ([9], Theorem 2).*

3. A priori estimates and Preliminary facts

Let us introduce the following scheme of approximation

$$\begin{cases} -\mathrm{div}\big(a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\big)+|u_n|^{s-1}u_n=h_n(u_n)f_n & \text{in } \Omega\\ u_n=0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f_n=T_n(f)$. Moreover, we set

$$h_n(t)=\begin{cases} T_n(h(t)) \text{ for } & t>0,\\ \min(n,h(0)) & \text{otherwise.}\end{cases} \tag{3.2}$$

The right hand side of (3.1) is nonnegative, that u_n is non negative. The existence of weak solution $u_n\in W_0^{1,p}(\Omega)$ is guaranteed by the following lemma.

Lemma 3.1. *Problem (3.1) has a non negative solution u_n in $W_0^{1,p}(\Omega)$, such that*

$$\int_\Omega |u_n|^{ms+\gamma}dx\le c\int_\Omega |f|^m dx \tag{3.3}$$

and the solution u_n satisfies

$$\int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n.\nabla\varphi+\int_\Omega |u_n|^{s-1}u_n\varphi=\int_\Omega f_nh_n(u_n)\varphi, \tag{3.4}$$

where $0<\gamma\le 1$ and φ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Proof. This proof derived from Schauder's fixed point argument in [23]. For fixed $n\in I\!N$ let us define a map

$$G: L^p(\Omega)\to L^p(\Omega),$$

such that, for any v be a function in $L^p(\Omega)$ gives the weak solution w to the following problem

$$-div(a(x,T_n(w))|\nabla w|^{p-2}\nabla w)+|w|^{s-1}w=f_nh_n(v). \tag{3.5}$$

The existence of a unique $w\in W_0^{1,p}(\Omega)$ corresponding to a $v\in L^p(\Omega)$ follows from the classical result of [1,20]. Moreover, since the datum $f_nh_n(v)$ is bounded, we have that $w\in L^\infty(\Omega)$ and there exists a positive constant d_1, independents of v and w (but possibly depending in n), such that $\|w\|_{L^\infty(\Omega)}\le d_1$. Again, thanks to the regularity of the datum $f_nh_n(v)$, we have can choose w as test function in the weak formulation (3.4), we have

$$\int_\Omega a(x,T_n(w))|\nabla w|^{p-2}\nabla w.\nabla w+\int_\Omega |w|^{s-1}w.w=\int_\Omega f_nh_n(v).w, \tag{3.6}$$

then, it follows from (1.2)

$$\alpha\int_\Omega \frac{|\nabla w|^p}{(1+n)^\theta}dx\le n^2\int_\Omega |w|dx,$$

using the Poincaré inequality we have

$$\int_\Omega \frac{|\nabla w|^p}{(1+n)^\theta}dx\le \frac{c_1}{\alpha}n^2\int_\Omega |\nabla w|dx,$$

then

$$\int_\Omega |\nabla w|^p dx\le \frac{c_1}{\alpha}(1+n)^{\theta+2}\int_\Omega |\nabla w|dx\le c(n,\alpha)|\Omega|^{\frac{1}{p'}}\left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}, \tag{3.7}$$

we obtain

$$\int_\Omega |\nabla w|^p dx\le c^{p'}(n,\alpha)|\Omega|,$$

using the Poincaré inequality on the left hand side

$$\|w\|_{L^p(\Omega)} \leq c^{\frac{p'}{p}}|\Omega|^{\frac{1}{p}} = c(n,\alpha,|\Omega|), \tag{3.8}$$

where $c(n,\alpha,|\Omega|)$ is a positive constant independent of v, thus, we have that the ball S of radius $c(n,\alpha,|\Omega|)$ is invariant for G.

Now, we are going to prove that the map G is continuous in S. Consider a sequence (v_k) that converges to v in $L^p(\Omega)$. We recall that $w_k = f_n h_n(v_k)$ are bounded, we have that $w_k \in L^\infty(\Omega)$ and there exists a positive constant d, independent of v_k and w_k, such that $\|w_k\|_{L^\infty(\Omega)} \leq d$. Then by dominated convergence theorem

$$\|f_n h_n(v_k) - f_n h_n(v)\|_{L^p(\Omega)} \longrightarrow 0.$$

Hence, by the uniqueness of the weak solution, we can say that $w_k = G(v_k)$ converges to $w = G(v)$ in $L^p(\Omega)$. Thus G is continuous over $L^p(\Omega)$.
What finally needs to be checked is that $G(S)$ is relatively compact in $L^p(\Omega)$. Let v_k be a bounded sequence, and let $w_k = G(v_k)$. It is possible to reason in the same way as to get (3.8), we have

$$\int_\Omega |\nabla w_k|^p dx = \int_\Omega |\nabla G(v_k)|^p dx \leq c(n,\alpha,\gamma),$$

where c is clearly independent from v_k , so that, $G(L^p(\Omega))$ is relatively compact in $L^p(\Omega)$. Now, applying the Schauder's fixed point theorem that G has a fixed point $u_n \in S$ that is solution to (3.1) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

To show (3.3), we will consider the cases $m > 1$ and $m = 1$.
Case $m > 1$, choosing $\varphi = |u_n|^{s(m-1)+\gamma}$ in (3.4), we have

$$\int_\Omega |u_n|^{sm+\gamma} dx \leq \int_\Omega |f||u_n|^{s(m-1)} dx,$$

therefore

$$\int_\Omega |u_n|^{sm+\gamma} \leq c\left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega |u_n|^{sm+\gamma}\right)^{1-\frac{1}{m}},$$

wich implies (3.3).

Case $m = 1$. Choosing $\varphi = u_n^\gamma$, then

$$\int_\Omega |u_n|^{s-1} u_n u_n^\gamma dx \leq \int_\Omega \frac{f}{u_n^\gamma} u_n^\gamma dx \leq f dx,$$

which is the estimate (3.3), as desired.

Lemma 3.2. *[4] Let u be a measurable function in $M^r(\Omega)$, $r > 0$, and suppose that there exists a positive constant $\rho > 0$ such that*

$$\int_\Omega |\nabla T_k(u)|^p dx \leq Ck^\rho \quad \forall k > 0.$$

Then $|\nabla u| \in M^{\frac{pr}{\rho+r}}(\Omega)$.

Lemma 3.3. *Let u_n be a sequence of measurable functions such that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists a measurable function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ and, moreover,*

$$T_k(u_n) \longrightarrow T_k(u) \ \ \textit{weakly in } W_0^{1,p}(\Omega) \ \ \textit{and} \ \ u_n \longrightarrow u \ \ \textit{a.e.in } \Omega.$$

Proof. Let us prove that $u_n \longrightarrow u$ locally in measure. To begin with, we observe that, for $t, \varepsilon > 0$, we have

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}.$$

Therefore,

$$meas\{|u_n - u_m| > t\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > t\}.$$

Choosing k large enough, we obtain

$$meas\{|u_n| > k\} < \varepsilon \ \text{ and } \ meas\{|u_m| > k\} < \varepsilon.$$

We can assume that $\{T_k(u_n)\}$ is a Cauchy sequence in $L^q(\Omega)$ for every $q < p^* = \frac{Np}{N-p}$. Then $\forall n, m \geq n_0(k,t)$:

$$meas\{|T_k(u_n) - T_k(u_m)| > t\} \leq t^{-q} \int_\Omega |T_k(u_n) - T_k(u_m)|^q dx \leq \varepsilon$$

This proves that $\{u_n\}$ is a Cauchy sequence in measure in Ω. Therefore, there exists a measurable function u such that $u_n \longrightarrow u$ in measure. Hence that $u_n \longrightarrow u$ a.e.in Ω, and so

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{weakly in} \quad W_0^{1,p}(\Omega).$$

4. Proof of the results

This section is devoted to proving theorems cited above. We start with

Proof of Theorem 2.4. We separate our proof in three parts, according to the values of s

Part I. Let $s \geq \frac{1+\theta-\gamma}{m-1}$ and we take $\varphi = (1+u_n)^{1+\theta} - 1$ as a test function in (3.4), using (1.2), we obtain

$$\alpha \int_\Omega |\nabla u_n|^p dx + \int_\Omega u_n^s[(1+u_n)^{1+\theta} - 1] \leq c \int_\Omega f u_n^{1+\theta-\gamma} dx.$$

In the following, by dropping the second positive term, we get

$$\alpha \int_\Omega |\nabla u_n|^p dx \leq c \int_\Omega f u_n^{1+\theta-\gamma} dx.$$

Applying Hölder's inequality in the right-hand side of above estimate, we obtain

$$\int_\Omega f u_n^{1+\theta-\gamma} dx \leq c \left[\int_\Omega u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx \right]^{1-\frac{1}{m}}.$$

Then, we get

$$\int_\Omega |\nabla u_n|^p dx \leq c \left[\int_\Omega u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx \right]^{1-\frac{1}{m}}. \tag{4.1}$$

The condition $\frac{m(1+\theta-\gamma)}{m-1} \leq ms$, ensure that $s \geq \frac{1+\theta-\gamma}{m-1}$. Then by (3.3) the right-hand side of (4.1) is uniformly bounded, so we can get

$$\int_\Omega |\nabla u_n|^p dx \leq c. \tag{4.2}$$

In order to prove that the limit function u is a solution of (1.1) in the sense of Definition 2.1, we need to show that we can pass to the limit in the weak formulation of the approximating problems (3.1).

Now we focus on the left hand side of (3.4), by (4.2) we conclude that there exists a subsequence, still indexed by $\{u_n\}$, and a measurable function u in $W_0^{1,p}(\Omega)$, such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \longrightarrow u$ a.e in Ω. Fatou's lemma implies $u \in L^{ms+\gamma}(\Omega)$. Now, adapting the approach of [6, Lemma 5] (see also [14,25]) then there exists a subsequence (still denoted $\{u_n\}$) such that

$$\nabla u_n \longrightarrow \nabla u \ \text{ a.e } \ \text{in } \ \Omega. \tag{4.3}$$

Next, we pass to the limit in (3.4). By (4.3), we can easily obtain

$$|\nabla u_n|^{p-2}.|\nabla u_n| \rightharpoonup |\nabla u|^{p-2}.|\nabla u| \ \text{ weakly in } L^{p'}(\Omega).$$

Moreover,

$$a(x,T_n(u_n)).\nabla\varphi \longrightarrow a(x,u).\nabla\varphi \text{ in } L^p(\Omega).$$

Consequently, we have

$$\int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n.\nabla\varphi dx \longrightarrow \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u.\nabla\varphi dx.$$

Therefore, we can pass to the limit in the first term of the left-hand side of (3.4). We will show that

$$|u_n|^{s-1}u_n \to |u|^{s-1}u \text{ in } L^1(\Omega). \tag{4.4}$$

We take $S_{\eta,k}(u_n)$ as a test function in the weak formulation (3.1), where $S_{\eta,k}$ is the function given in (1.11), we deduce

$$\int_\Omega a(x,T_n(u_n))|\nabla u_n|^p S'_{\eta,k}(u_n)dx + \int_\Omega |u_n|^{s-1}u_n S_{\eta,k}(u_n)dx \leq \sup_{t\in[k,\infty)}[h(t)]\int_\Omega f_n S_{\eta,k}(u_n),$$

which, observing that the first term on the left hand side is non negative and taking the limit with respect to $\eta\to 0$, implies

$$\int_{\{u_n\geq k\}}|u_n|^{s-1}u_n dx \leq \sup_{t\in[k,\infty)}[h(t)]\int_{\{u_n\geq k\}} f_n dx,$$

which, since f_n converges to f in $L^m(\Omega)$, easily implies that $|u_n|^{s-1}u_n$ is equi-integrable and so it converges to $|u|^{s-1}u$ in $L^1(\Omega)$, this concludes (4.4).

The next step we want to pass to the limit in the right hand side of (3.4). Let us take $0\leq\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in the weak formulation of (3.1), by using the young inequality and the hypotheses (1.2) and (1.3), we have

$$\begin{aligned}
\int_\Omega h_n(u_n)f_n\varphi &= \int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}.\nabla u_n\nabla\varphi dx + \int_\Omega u_n^{s-1}u_n\varphi dx\\
&\leq C||\varphi||_{L^\infty(\Omega)} + \beta\int_\Omega |\nabla u_n|^{p-1}\nabla\varphi dx + \frac{1}{s}\int_\Omega u_n^s dx + \frac{1}{s'}\int_\Omega \varphi^{s'}dx\\
&\leq C||\varphi||_{L^\infty(\Omega)} + \beta\frac{p-1}{p}\int_\Omega|\nabla u_n|^p dx + \beta\frac{1}{p}\int_\Omega|\nabla\varphi|^p dx + \frac{1}{s}\int_\Omega u_n^s dx + \frac{1}{s'}\int_\Omega\varphi^{s'}dx\\
&\leq C||\varphi||_{L^\infty(\Omega)} + C\left[\int_\Omega|\nabla\varphi|^p dx + \int_\Omega|\nabla u_n|^p dx\right] + \frac{1}{s}\|u_n\|_{L^s(\Omega)} + \frac{1}{s'}\|\varphi\|_{L^{s'}(\Omega)},
\end{aligned}$$

then

$$\int_\Omega h_n(u_n)f_n\varphi \leq C||\varphi||_{L^\infty(\Omega)} + C[||\varphi||_{W_0^{1,p}(\Omega)} + ||u_n||_{W_0^{1,p}(\Omega)}] + \frac{1}{s}\|u_n\|_{L^s(\Omega)} + \frac{1}{s'}\|\varphi\|_{L^{s'}(\Omega)}. \tag{4.5}$$

From now on, we assume that $h(t)$ is unbounded as t tends to 0. An application of the Fatou Lemma in (4.5) with respect to n gives

$$\int_\Omega h(u)f\varphi \leq c, \tag{4.6}$$

where c does not depend on n.
Hence $fh(u)\varphi\in L^1(\Omega)$ for any nonnegative $\varphi\in W_0^{1,p}(\Omega)$. As a consequence, if $h(t)$ is unbounded as t tends to 0, we deduce that

$$\{u=0\}\subset\{f=0\} \tag{4.7}$$

up to a set of zero Lebesgue measure.
Now, for $\delta>0$, we split the right hand side of (3.4) as

$$\int_\Omega h_n(u_n)f_n\varphi dx = \int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx + \int_{\{u_n>\delta\}} h_n(u_n)f_n\varphi dx, \tag{4.8}$$

and we pass to limit as $n \to +\infty$ and then $\delta \to 0$, we remark that we need to choose $\delta \neq \{\eta; |u = \eta| > 0\}$, which is at most a countable set, for the second term (4.8) we have

$$0 \le h_n(u_n) f_n \varphi \chi_{\{u_n > \delta\}} \le \sup_{t \in]\delta, \infty)} [h(t)] f \varphi \in L^1(\Omega), \tag{4.9}$$

which precis to apply the Lebesgue Theorem with respect n. Hence on has

$$\lim_{n \to +\infty} \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx = \int_{\{u > \delta\}} h(u) f \varphi dx.$$

Moreover it follows by (4.6) that

$$\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx = \int_{\{u > 0\}} h(u) f \varphi dx. \tag{4.10}$$

Now in order to get rid of the first term of the right hand side of (4.8), we take $V_\delta(u_n)\varphi$ is a test function in the weak formulation of (3.1), where $V_\delta(u_n) := V_{\delta,\delta}(u_n)$ is defined in (1.11) and by Lemma 1.1 contained in [9], we have $V_\delta(u_n)$ belongs to $W_0^{1,p}(\Omega)$, then (recall $V_\delta'(u_n) \le 0 \; for \; u_n \ge 0$)

$$\int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \int_\Omega h_n(u_n) f_n V_\delta(u_n) \varphi dx = \int_\Omega a(x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi V_\delta(u_n) dx$$
$$- \frac{1}{\delta} \int_{\{\delta < u_n < 2\delta\}} a(x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \varphi \nabla u_n dx + \int_\Omega |u_n|^{s-1} u_n V_\delta(u_n) \varphi dx,$$

by using (1.2) and (1.3), we have

$$\int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \beta \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi V_\delta(u_n) dx + \int_\Omega |u_n|^{s-1} u_n V_\delta(u_n) \varphi dx,$$

using that V_δ is bounded we deduce that $|\nabla u_n|^{p-2} \nabla u_n V_\delta(u_n)$ converges to $|\nabla u|^{p-2} \nabla u V_\delta(u)$ weakly in $(L^{p'}(\Omega))^N$ as n tends to infinity. This implies that

$$\lim_{n \to +\infty} \int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \beta \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi V_\delta(u) dx + \int_\Omega |u|^{s-1} u V_\delta(u) \varphi dx. \tag{4.11}$$

Since $V_\delta(u)$ converges to $\chi_{\{u=0\}}$ a.e in Ω as δ tends to 0 and since $u \in W_0^{1,p}(\Omega)$, then $|\nabla u|^{p-2} \nabla u \nabla \varphi V_\delta(u)$ converges to 0 a.e. in Ω as δ tends to 0. Applying the Lebesgue Theorem on the right hand side of (4.11) we obtain that

$$\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \beta \int_{\{u=0\}} |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \int_{\{u=0\}} |u|^{s-1} u \varphi dx = 0, \tag{4.12}$$

by (4.10) and (4.12), we deduce that

$$\lim_{n \to +\infty} \int_\Omega h_n(u_n) f_n \varphi dx = \int_\Omega h(u) f \varphi dx \quad 0 \le \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{4.13}$$

Moreover, decomposing any $\varphi = \varphi^+ - \varphi^-$, and using that (4.13) is linear in φ, we deduce that (4.13) holds for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We treated $h(t)$ unbounded as t tends to 0, as regards bounded function h the proof is easier and only difference deals with the passage to the limit in the left hand side of (4.13). We can avoid introducing δ and we can substitute (4.9) with

$$0 \le f_n h_n(u_n) \varphi \le f \|h\|_{L^\infty(\Omega)} \varphi.$$

Using the same argument above we have that

$$\lim_{n\to+\infty}\int_\Omega f_n h_n(u_n)\varphi dx=\int_\Omega f h(u)\varphi dx, \tag{4.14}$$

whence one deduces (1.5). This concludes the proof of part I.

Part II. Let

$$\frac{1+\theta-\gamma}{pm-1}<s<\frac{1+\theta-\gamma}{m-1}.$$

Taking $\varphi=(1+u_n)^{s(m-1)+\gamma}-1$ as a test function in (3.4). Using assumption (1.2) and dropping the nonnegative term, we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}}dx\le c\int_\Omega |f||u_n|^{s(m-1)}dx. \tag{4.15}$$

Applying Hölder's inequality in the right-hand side of the estimate (4.15), we get

$$\int_\Omega |f||u_n|^{s(m-1)}dx\le c\left[\int_\Omega u_n^{ms+\gamma}dx\right]^{1-\frac{1}{m}}\le c.$$

Then, we obtain

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}}dx\le c. \tag{4.16}$$

Let $1\le\sigma<p$. Writing

$$\int_\Omega |\nabla u_n|^\sigma dx=\int_\Omega \frac{|\nabla u_n|^\sigma(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}{(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}dx$$

and using Hölder's inequality, we get

$$\int_\Omega \frac{|\nabla u_n|^\sigma(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}{(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}dx\le c\left[\int_\Omega (1+u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{1-\frac{\sigma}{p}}.$$

Then by (4.16), we arrive at

$$\int_\Omega |\nabla u_n|^\sigma dx\le c\left[\int_\Omega (1+u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{1-\frac{\sigma}{p}}. \tag{4.17}$$

We now choose σ in order to have

$$\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]\le ms.$$

The last inequality is equivalent to

$$\sigma\le\frac{pms}{s+1+\theta-\gamma}.$$

Thanks to Lemma 3.1, if implies that

$$\frac{1+\theta-\gamma}{pm-1}<s \ \text{ implies } \ \frac{pms}{s+1+\theta-\gamma}>1.$$

In that case, the right-hand side of (4.17) is uniformly bounded and so we have

$$\int_\Omega |\nabla u_n|^\sigma dx\le c,\ \sigma=\frac{pms}{1+\theta+s-\gamma}.$$

Up to a subsequence, there exists a function $u \in W_0^{1,\sigma}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,\sigma}(\Omega) \text{ and } u_n \longrightarrow u \text{ a.e in } \Omega.$$

By Lemma 5 (see [6]), we have $\nabla u_n \longrightarrow \nabla u \ a.e \ in \ \Omega$. Fatou's Lemma implies $u^{ms+\gamma} \in L^1(\Omega)$ we will now pass to the limit in (3.4). We can easily obtain

$$|\nabla u_n|^{p-2}\nabla u_n \longrightarrow |\nabla u|^{p-2}\nabla u \text{ weakly in } L^{\frac{\sigma}{p-1}}(\Omega) \text{ and } a(x, T_n(u_n))\varphi \longrightarrow a(x,u)\varphi, \text{ in } L^{(\frac{\sigma}{p-1})'}(\Omega).$$

Therefore,we have

$$\int_\Omega a(x, T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \longrightarrow \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u \nabla\varphi dx.$$

The remaining two parts in (3.4) are the same as part I.
PART III. Suppose that Let $0 < s \leq \frac{1+\theta-\gamma}{pm-1}$. We show that there exists an entropy solution to problem (3.1). Estimate (4.16) implies that

$$\int_{\Omega\cap\{|u_n|<k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}} dx \leq c,$$

and consequently

$$\int_\Omega |\nabla T_k(u_n)|^p dx = \int_{\Omega\cap\{|u_n|<k\}} |\nabla T_k(u_n)|^p dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}. \tag{4.18}$$

Thanks to Lemma 3.3 there exists a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, besides, passing if necessary to subsequence, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } a.e. \text{in } \Omega.$$

Then, we can pass to the limit in (4.18), to get

$$\int_\Omega |\nabla T_k(u)|^p dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}.$$

Lemma 3.2 gives us, if

$$s \leq \frac{1+\theta-\gamma}{pm-1} \leq \frac{1+\theta-\gamma}{m-1},$$

then, we obtain

$$|\nabla u| \in M^{\frac{pms}{1+\theta+s-\gamma}}(\Omega).$$

Since $|u_n|^{ms+\gamma}$ is uniformly bounded in $L^1(\Omega)$, applying Fatou Lemma implies that $|u|^{ms+\gamma} \in L^1(\Omega)$. We will show that u is an entropy solution of (1.1). Indeed, let us choose

$$T_k(u_n - \varphi), \quad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

as a test function in (3.4), then we have

$$\int_\Omega a(x, T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n.\nabla T_k(u_n - \varphi) + \int_\Omega |u_n|^{s-1}u_n T_k(u_n-\varphi) = \int_\Omega f_n h_n(u_n) T_k(u_n-\varphi). \tag{4.19}$$

Let us pass to the limit in (4.19). For the second term on the left-hand side and for the right-hand side, we can use (4.14) to obtain the limit. For the first term on the left-hand side, we will firstly show that $\nabla T_k(u_n) \to \nabla T_k(u)$ a.e. in Ω. Let $\varphi = T_k(u_n) - T_k(u)$ in (3.4), then we obtain

$$\int_\Omega a(x, T_n(T_k(u_n)))|\nabla u_n|^{p-2}\nabla u_n.[\nabla T_k(u_n) - \nabla T_k(u)]dx$$

$$+\int_\Omega |u_n|^{s-1}u_n[T_k(u_n)-T_k(u)] = \int_\Omega f_n h_n(u_n)[T_k(u_n)-T_k(u)].$$

As a consequence, we have

$$\begin{aligned}\int_\Omega a(x,T_n(T_k(u_n)))[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)] \times [\nabla T_k(u_n)-\nabla T_k(u)]dx \\ = \int_\Omega f_n h_n(u_n)[T_k(u_n)-T_k(u)]dx - \int_\Omega |u_n|^{s-1}u_n[T_k(u_n)-T_k(u)]dx \\ - \int_\Omega a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)[\nabla T_k(u_n)-\nabla T_k(u)]dx. \qquad (4.20)\end{aligned}$$

We are going to show that the three terms of the right-hand side in (4.20) all converge to zero. For the first term, we can use the (4.14) to take the limit. As the result of the proof in part one, we obtain

$$u_n^s \to u^s \text{ in } L^1(\Omega). \qquad (4.21)$$

Again, by (4.21), we deduce

$$\int_\Omega u_n^s[T_k(u_n)-T_k(u)]dx \to 0 \text{ as } n\to\infty.$$

We can easily know the fact that $a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \in L^{p'}(\Omega)$. Thus, for every measurable set $E\subset\Omega$, we can write

$$\int_E |a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-1}|^{p'}dx \to 0 \text{ as} |E|\to 0.$$

Because

$$a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \to a(x,T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \text{ a.e. in } \Omega,$$

using Vitali's Theorem, we obtain

$$a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \to a(x,T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \text{ in } L^{p'}(\Omega).$$

Hence one can apply By Lemma 3.3, to obtain that

$$\nabla T_k(u_n)-\nabla T_k(u) \rightharpoonup 0 \text{ weakly in } L^p(\Omega).$$

Therefore,

$$\int_\Omega a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)[\nabla T_k(u_n)-\nabla T_k(u)]dx \to 0 \text{ as } n\to\infty.$$

Therefore from the previous we deduce

$$\int_\Omega a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)[\nabla T_k(u_n)-\nabla T_k(u)]dx \to 0.$$

So we can apply Lemma 5 in [6] (see also [25,14]) to obtain that $\nabla T_k(u_n)\to\nabla T_k(u)$ in $L^p(\Omega)$. Therefore,

$$\nabla T_k(u_n)\to\nabla T_k(u) \text{ a.e. in } \Omega.$$

Let $m=k+|\varphi|$. The first term on the left-hand side in (4.19) can be rewritten as

$$\int_\Omega a(x,T_n(u_n))|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n-\varphi)dx.$$

Since $\nabla T_m(u_n) \to \nabla T_m(u)$ a.e.in Ω, as a result of the Fatou's Lemma, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega a(x, T_n(u_n))|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n-\varphi)dx \\
&\geq \int_\Omega a(x,u)|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n-\varphi)dx \\
&= \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\nabla T_k(u_n-\varphi)dx.
\end{aligned}$$

So we see that u is an entropy solution of (1.1).

Proof of Theorem 2.6. We separate our proof in two parts, according to the values of s

Part I. Suppose $0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$. It is obvious that $s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, which implies $ms + \gamma \leq \sigma^*$ for $\sigma \geq 1$. Thus, from (4.17) and applying Sobolev's embedding Theorem, we obtain

$$\int_\Omega |u_n|^{\sigma^*}dx \leq c\left[\int_\Omega (1+u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{\frac{(p-\sigma)\sigma^*}{p\sigma}}.$$

On other hand, if $\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma] \leq \sigma^*$ implies that

$$\sigma \leq \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}.$$

Therefore, since $m > 1$ and $p > p_0 > 1 + \frac{(N-1)[1+\theta-s(m-1)-\gamma]}{N}$, we have $\frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma} > 1$, and

$$\int_\Omega |u_n|^{\sigma^*}dx \leq c + c\left(\int_\Omega |u_n|^{\sigma^*}dx\right)^{\frac{(p-\sigma)\sigma^*}{p\sigma}}. \tag{4.22}$$

In other hand by Young's inequality and from (4.22), we can get

$$\int_\Omega |u_n|^{\sigma^*}dx \leq c.$$

We now observe that, by (4.17) and since $\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma] \leq \sigma^*$, we have

$$\int_\Omega |\nabla u_n|^\sigma dx \leq c, \text{ such that } \sigma \leq \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}.$$

The remaining proof of this part is the same as part II in Theorem 2.4, we have can show that u is a distributional solution to problem (1.1).

Part II. Let $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$. Since $p > p_0$, it follows that

$$\frac{N(1-\gamma)+\gamma}{m(N-1)} > \frac{1+\theta-\gamma}{pm-1},$$

thus, we can show that u is a distributional solution to the problem (1.1) by the same method as in Part II of Theorem 2.4.

Proof of Theorem 2.8. We separate our proof in two parts, according to the values of s.

Part a. Let $s > \frac{1+\theta-p\gamma}{p-1}$. If we choose $\varphi = (1+u_n)^\gamma - 1$ as a test function in (3.4). Using assumption (1.2) and dropping the nonnegative term, we can get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\gamma}}dx \leq c + c\int_\Omega f dx \leq c. \tag{4.23}$$

From the other hand, let $r < p$, writing

$$\int_\Omega |\nabla u_n|^r dx = \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1+\theta-\gamma)}{p}}}(1+u_n)^{\frac{r(1+\theta-\gamma)}{p}} dx$$
$$\leq \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{(1+\theta-\gamma)}}dx\right)^{\frac{r}{p}} \left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}}dx\right)^{1-\frac{r}{p}} \leq c\left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}}dx\right)^{1-\frac{r}{p}}.$$

Thanks to Lemma 3.1, if

$$\frac{r}{p-r}(1+\theta-\gamma) \leq s \text{ , i.e. } r < \frac{ps}{1+\theta+s-\gamma}.$$

Then

$$s > \frac{1+\theta-\gamma}{p-1} \quad \text{implies} \quad \frac{ps}{1+\theta+s-\gamma} > 1.$$

In that case, the right-hand sides is uniformly bounded and so we get

$$\int_\Omega |\nabla u_n|^r dx \leq c \text{ such that } r < \frac{ps}{1+\theta+s-\gamma}.$$

As a consequence, there exists a function $u \in W_0^{1,r}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,r}(\Omega) \text{ and } u_n \longrightarrow u \text{ a.e in } \Omega.$$

Let $g_n = f_n h_n(u_n) - T_n(|u_n|^{s-1}u_n)$, because g_n is bounded in $L^1(\Omega)$, and u_n is a solution of

$$\begin{cases} -\text{div}\big(a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\big) = g_n, \\ u_n \in W_0^{1,p}(\Omega), \end{cases}$$

then use the argument of Lemma 1 (see [5]), one may get

$$\nabla u_n \longrightarrow \nabla u \;\; a.e \;\; in \;\Omega. \tag{4.24}$$

We are going to show that u is a distributional solution to problem (1.1) by passing to the limit in (3.4). We suppose that $\varphi \in C_0^\infty(\Omega)$. Since $|\nabla u_n|^{p-2}\nabla u_n \in L^{\frac{r}{p-1}}(\Omega)$ and (4.24) hold, we have

$$|\nabla u_n|^{p-2}\nabla u_n \longrightarrow |\nabla u|^{p-2}\nabla u \text{ weakly in } L^{\frac{r}{p-1}}(\Omega).$$

Using Vitali's Theorem, we obtain

$$a(x,T_n(u_n))\nabla\varphi \longrightarrow a(x,u)\nabla\varphi \text{ in } L^{(\frac{r}{p-1})'}(\Omega),$$

where $(\frac{r}{p-1})' = \frac{p-1-r}{p-1}$. Therefore, we can pass to the limit in the first term on the left-hand side of (3.4). For the second term on the left hand-side and the first term on the right-hand side in (3.4) we can namely arguing exactly as part I in Theorem 2.4. Therefore, we conclude that u is a distributional solution to problem (1.1).

Part b. Let $0 < s \leq \frac{1+\theta-\gamma}{p-1}$. Let us choose $T_k(u_n)$ as a test function in (3.4), using assumption (1.2) and removing the second term non negative, we get

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq ck^{1-\gamma}(1+k)^\theta \leq c(1+k)^{1-\gamma+\theta}. \tag{4.25}$$

Now by Lemma 3.3, there exists a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$. Moreover,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega)\; \forall k>0 \text{ and } u_n \longrightarrow u \text{ a.e in } \Omega.$$

Fatou's Lemma implies that $|u|^{s+\gamma} \in L^1(\Omega)$. We can pass to the limit in (4.25), to obtain

$$\int_\Omega |\nabla T_k(u)|^p dx \leq c(1+k)^{1+\theta-\gamma}.$$

As a result of the Lemma 3.2, we obtain

$$|\nabla u| \in M^{\frac{ps}{1+\theta+s-\gamma}}(\Omega).$$

By the same method as in part II of Theorem 2.4, we can show that u is an entropy solution of (1.1).

Proof of Theorem 2.10. Let $0 < s \leq \frac{N(1-\gamma)+\gamma}{N-1}$. Then $s \leq \frac{N(1-\gamma)+\gamma}{N-1}$ implies $s+\gamma \leq r^*$. Using (4.23), we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\gamma}} dx \leq c.$$

Let $1 \leq r < p$, let us write

$$\int_\Omega |\nabla u_n|^r dx = \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1+\theta-\gamma)}{p}}} (1+u_n)^{\frac{r(1+\theta-\gamma)}{p}} dx.$$

Then it follows from Hölder's inequality that

$$\int_\Omega |\nabla u_n|^r dx \leq c\left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}} dx\right)^{1-\frac{r}{p}}. \tag{4.26}$$

On other hand by Sobolev embedding Theorem and from (4.26), we can get

$$\left(\int_\Omega |u_n|^{r^*} dx\right)^{\frac{1}{r^*}} \leq c\left(\int_\Omega |\nabla u_n|^r dx\right)^{\frac{1}{r}} \quad \text{where } r^* = \frac{Nr}{N-r}.$$

Which implies,

$$\left(\int_\Omega |u_n|^{r^*} dx\right)^{\frac{1}{r^*}} \leq c\left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}} dx\right)^{\frac{(p-r)r^*}{pr}}.$$

being $\frac{r(1+\theta-\gamma)}{p-r} \leq r^*$, so that

$$r \leq \frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma}.$$

Then, by $m=1$ we have $p_0 = 1 + \frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma}$. By virtue of $p > p_0 > 1 + \frac{(N-1)[1+\theta-\gamma]}{N}$, ensures that

$$\frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma} > 1.$$

Then, we can obtain

$$\int_\Omega u_n^{r^*} dx \leq c\left(\int_\Omega (1+u_n)^{r^*} dx\right)^{\frac{(p-r)r^*}{p-r}} \leq c + c\left(\int_\Omega u_n^{r^*} dx\right)^{\frac{(p-r)r^*}{pr}}.$$

Using Young inequality in the above estimate gives

$$\int_\Omega |u_n|^{r^*} dx \leq c.$$

Which together with (4.26) and $\frac{r}{p-r}(\theta+1-\gamma) \leq r^*$ implies

$$\int_\Omega |\nabla u_n|^r dx \leq c \quad r < \frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma}.$$

Just as in the proof of part I in the Theorem 2.8, we can conclude that u is a distributional solution of (1.1).

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