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A Fixed Point Theorem for Weak (ψ - ϕ)-Jaggi Type Contraction

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ABSTRACT: In this paper, we introduce the weak (ψ-ϕ)-Jaggi type contraction. The existence and uniqueness of fixed point for such contraction is investigated. It is very helpful in extending the existing results of corresponding literature. In addition, we also provide an example in support of our theorem.

Key Words: Fixed point, Jaggi type hybrid contraction, weak (ψ-ϕ)-Jaggi type contraction.

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1. Introduction

Fixed point theory simply deals with solution of the equation $f(x) = x$ where f is a self-map on a non-empty set X. The fixed point problem first appeared in the solution of an initial value problem. Then Liouville [9] in 1837 and Picard [11] in 1890 solved the problem using successive approximation method and which also provided the solution of the fixed point equation. Before 1922, there was no any direct method to evaluate the fixed point of a map. Then in 1922, Banach [3] was the first who introduced the contraction principle to evaluate the fixed point. Metric fixed point theory has been investigated by many researchers as it is the natural and strong connection of the theoretical results in non-linear functional analysis with applied sciences. Later a lot of generalization of Banach contraction principle was done by weakening its hypothesis. In 1969, Boyd and Wong [4] have attempted to generalize Banach contraction principle by replacing the Lipschitz constant by some real valued function whose value is less than 1. Later in 1969, Meir and Keeler [10] have generalized Banach contraction principle for the case of weakly uniformly strict contraction. In 1977, Jaggi [7] first time introduced the rational expression for contraction and proved the fixed point theorem for such contraction. In 2001, Rhoades ([12]-[13]) has shown that the result of Alber and Guerre [2] for Hilbert spaces is also valid in complete metric spaces. Then in 2006, Abbas and Beg [1] proved the fixed point theorems for mappings which satisfy a generalized weak contractive conditions. Then in 2008, Choudhary and Dutta ([5]-[6]) motivated from Rhoades, Abbas and Beg showed the existence of unique fixed point in complete metric space for weakly contractive mappings. In 2009, Zhang and Song [14] proved fixed point theorems for weak ϕ-contraction. In 2019, Karapinar and Fulga [8] introduced the new notion of Jaggi type hybrid contraction as follows: A self-mapping T on (X, d) is called a Jaggi type hybrid contraction if

$$(d(Tx, Ty)) \leq \psi(\mathfrak{I}_s^T(x, y)),$$

for all distinct $x, y \in X$ where $s \geq 0$ and $\alpha_i \geq 0$ for $i = 1, 2$ such that $\alpha_1 + \alpha_2 = 1$ and

$$\mathfrak{I}_s^T(x,y) = \begin{cases} [\alpha_1(\frac{d(x,Tx)d(y,Ty)}{d(x,y)})^s + \alpha_2(d(x,y))^s]^{\frac{1}{s}}, & \text{if } s > 0 \\ (d(x,Tx))^{\alpha_1}(d(y,Ty))^{\alpha_2}, & \text{if} s = 0. \end{cases}$$

If x and y are different elements in X and $x, y \notin F_T(X)$ where

$$F_T(X) = \{x \in X : Tx = x\}.$$

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2. Main Results

In this paper, after getting motivation from Karapinar and Fulga [8], we will introduce weak (ψ-ϕ)-Jaggi type contraction and show the existence of unique fixed point for a self-map T on a complete metric space (X, d). We also provide an example to show the validity of our main result.

Definition 2.1. A self-mapping T on (X, d) is called a weak (ψ-ϕ)-Jaggi type hybrid contraction if

$$\psi(d(Tx, Ty)) \leq \psi(\mathfrak{J}_s^T(x, y)) - \phi(\mathfrak{J}_s^T(x, y)), \tag{2.1}$$

for all distinct $x, y \in X$ where $s \geq 0$ and $\alpha_i \geq 0$ for $i = 1, 2$ such that $\alpha_1 + \alpha_2 = 1$ and

$$\mathfrak{J}_s^T(x, y) = \begin{cases} [\alpha_1(\frac{d(x,Tx)d(y,Ty)}{d(x,y)})^s + \alpha_2(d(x, y))^s]^{\frac{1}{s}}, & \text{if } s > 0 \\ (d(x, Tx))^{\alpha_1}(d(y, Ty))^{\alpha_2}, & \text{if} s = 0. \end{cases}$$

If x and y are different elements in X and $x, y \notin F_T(X)$ where

$$F_T(X) = \{x \in X : Tx = x\}.$$

and $\psi, \phi : [0, \infty) \to [0, \infty)$ both are continuous and non-decreasing functions with $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$.

Theorem 2.2. Let (X, d) be a complete metric space and T be a continuous self map on X satisfying equation (2.1). Then T has a unique fixed point.

Proof: Let x_0 be an arbitrary element of X. Consider an iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ and $Tx_0 = x_1$.

Now, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Because if for some $k \in \mathbb{N}$, $d(x_k, x_{k+1}) = 0$ then $x_k = x_{k+1} = Tx_k$, this implies x_k is the fixed point of T.

We will prove the theorem by examining two different cases: $s > 0$ and $s = 0$.

Case (i). When $s > 0$, putting $x = x_{n-1}$ and $y = Tx_{n-1}$ in the equation (2.1), we get

$$\psi(d(Tx_{n-1}, Tx_n)) \leq \psi(\mathfrak{J}_s^T(x_{n-1}, x_n)) - \phi(\mathfrak{J}_s^T(x_{n-1}, x_n)), \tag{2.2}$$

where

$$\begin{aligned} \mathfrak{J}_s^T(x_{n-1}, x_n) &= [\alpha_1(\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)})^s + \alpha_2(d(x_{n-1}, x_n))^s]^{\frac{1}{s}}, \\ &= [\alpha_1(d(x_n, x_{n+1})^s + \alpha_2(d(x_{n-1}, x_n))^s]^{\frac{1}{s}}. \end{aligned} \tag{2.3}$$

Therefore equation (2.2) becomes

$$\begin{aligned} \psi(d(Tx_{n-1}, Tx_n)) &\leq \psi([\alpha_1(d(x_n, x_{n+1}))^s + \alpha_2(d(x_{n-1}, x_n))^s]^{\frac{1}{s}}) \\ &- \phi([\alpha_1(d(x_n, x_{n+1})^s + \alpha_2(d(x_{n-1}, x_n))^s]^{\frac{1}{s}}). \end{aligned} \tag{2.4}$$

Let, if possible suppose that $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$, then equation (2.4) becomes

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})), \\ &< \psi(d(x_n, x_{n+1})). \end{aligned}$$

a contradiction.

So,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{2.5}$$

It follows that sequence $d(x_n, x_{n+1})$ is monotonically decreasing and is bounded below by 0. So, the sequence $d(x_n, x_{n+1})$ converges to some $r \geq 0$, that is

$$d(x_n, x_{n+1}) \to r \, as \, n \to \infty. \tag{2.6}$$

Making $n \to \infty$ in equation (2.4), we get $\psi(r) \leq \psi(r) - \phi(r)$, which holds only when $r = 0$ as ψ is a non decreasing function.

$$d(x_n, x_{n+1}) \to 0 \; as \; n \to \infty. \tag{2.7}$$

Next, we shall show that $\{x_n\}$ is a Cauchy sequence.

If possible, suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$, such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.8}$$

Further for $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.8). Then

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.9}$$

Then using triangle inequality, equations (2.8) and (2.9), we have

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\leq \epsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{2.10}$$

Making $k \to \infty$ and using equation (2.7) in (2.10), we get

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.11}$$

Again

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}). \tag{2.12}$$

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}). \tag{2.13}$$

Taking $k \to \infty$ and using equations (2.7), (2.11) in equations (2.12) and (2.13), we get

$$\lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.14}$$

Putting $x = x_{m(k)-1}, y = x_{n(k)-1}$, in the equation (2.1), we get

$$\psi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \leq \psi(\mathfrak{J}_s^T(x_{m(k)-1}, x_{n(k)-1})) - \phi(\mathfrak{J}_s^T(x_{m(k)-1}, x_{n(k)-1})), \tag{2.15}$$

where

$$\mathfrak{J}_s^T(x_{m(k)-1}, x_{n(k)-1}) = [\alpha_1(\frac{d(x_{m(k)-1}, Tx_{m(k)-1})d(x_{n(k)-1}, Tx_{n(k)-1})}{d(x_{m(k)-1}, x_{n(k)-1})})^s + \alpha_2(d(x_{m(k)-1}, x_{n(k)-1}))^s]^{\frac{1}{s}} \tag{2.16}$$

Taking $k \to \infty$ and using (2.7), (2.11) and (2.16) in (2.15), we get

$$\psi(\epsilon) \leq \psi(\alpha_2^{\frac{1}{s}} \epsilon) - \phi(\alpha_2^{\frac{1}{s}} \epsilon). \tag{2.17}$$

Subcase (i): When $\alpha_2 = 0$.

From equation (2.17), we have $\psi(\epsilon) \leq 0$, but ψ is a non-negative function, so $\psi(\epsilon) = 0$ and this holds only when $\epsilon = 0$.

Subcase (ii): When $\alpha_2 = 1$.

From equation (2.17), we have $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$, but ψ is non-decreasing function, so this holds only when $\epsilon = 0$.

Subcase (iii): When $0 < \alpha_2 < 1$.

Clearly, $\epsilon > \alpha_2^{\frac{1}{s}} \epsilon$, so equation (2.17) implies that $\psi(\epsilon) \leq \psi(\alpha_2^{\frac{1}{s}} \epsilon)$ but as ψ is non-decreasing function, so this holds only when $\epsilon = 0$.

From all the above discussed three subcases it is clear that $\epsilon = 0$, a contradiction to our assumption. So, $\{x_n\}$ is a Cauchy sequence. Now as (X, d) is a complete metric space so $\{x_n\}$ is a convergent sequence in X. Let $\{x_n\}$ converges to x. Since T is a continuous map, it follows that

$$d(x, Tx) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

which shows that x is a fixed point of T.

Uniqueness:

If possible, let us suppose that x_1 and x_2 are two distinct fixed points of T, that is $Tx_1 = x_1$ and $Tx_2 = x_2$.

Now, putting $x = x_1$ and $x = x_2$ in the equation (2.1), we have

$$\psi(d(Tx_1, Tx_2)) \leq \psi(\mathfrak{J}_s^T(x_1, x_2)) - \phi(\mathfrak{J}_s^T(x_1, x_2)) \tag{2.18}$$

where

$$\begin{aligned} \mathfrak{J}_s^T(x_1, x_2) &= [\alpha_1(\frac{d(x_1, Tx_1)d(x_2, Tx_2}{d(x_1, x_2)})^s + \alpha_2(d(x_1, x_2))^s]^{\frac{1}{s}}, \\ &= (\alpha_2)^{\frac{1}{s}} d(x_1, x_2). \end{aligned}$$

From equation (2.18), we obtain that

$$\psi(d(x_1, x_2)) \leq \psi((\alpha_2)^{\frac{1}{s}} d(x_1, x_2)) - \phi((\alpha_2)^{\frac{1}{s}} d(x_1, x_2)),$$

Now using the same argument as above after equation (2.14), we get $d(x_1, x_2) = 0 \Rightarrow x_1 = x_2$, a contradiction. So T has a unique fixed point.

Case (ii). When $s = 0$. Putting $x = x_{n-1}$ and $y = Tx_{n-1}$ in the equation (2.1), we get

$$\begin{aligned} \psi(d(Tx_{n-1}, Tx_n)) &\leq \psi(\mathfrak{J}_s^T(x_{n-1}, x_n)) - \phi(\mathfrak{J}_s^T(x_{n-1}, x_n)), \\ &< \psi(\mathfrak{J}_s^T(x_{n-1}, x_n)). \end{aligned} \tag{2.19}$$

where

$$\mathfrak{J}_s^T(x_{n-1}, x_n) = (d(x_{n-1}, x_n))^{\alpha_1}(d(x_n, x_{n+1}))^{\alpha_2}.$$

Using this in equation (2.19), we get

$$\psi(d(x_n, x_{n+1})) < \psi((d(x_{n-1}, x_n))^{\alpha_1}(d(x_n, x_{n+1}))^{\alpha_2}),$$

Using non-decreasing property of ψ, this implies that

$$d(x_n, x_{n+1}) < (d(x_{n-1}, x_n))^{\alpha_1}(d(x_n, x_{n+1}))^{\alpha_2},$$

which becomes

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n),$$

as $\alpha_1 + \alpha_2 = 1$.

So, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence. Now repeating the same process as after equation (2.5) we can easily prove that sequence $\{x_n\}$ is a Cauchy sequence and hence convergent. Let us consider the sequence $\{x_n\}$ converges to x. Now using the continuity of T we can show that x is the unique fixed point of T.

Example 2.3. Let $X = [0, 1]$ and the distance function is $d(x, y) = |x - y|$. Clearly (X, d) is a complete metric space. Define $T : X \to X$ as $Tx = \frac{x}{8}$. Taking $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{1}{4}$ and $s = 2$. Define $\psi(t) = 2t$ and $\phi(t) = t$. Now evaluating left hand side of equation (2.1)

$$\psi(d(Tx, Ty)) = \frac{|x - y|}{4}. \tag{2.20}$$

Now for right hand side of equation (2.1), we evaluate value of $\mathfrak{J}_s^T(x, y)$

$$\mathfrak{J}_s^T(x, y) = [\frac{(\frac{7}{8})^4 (xy)^2 + 3(x-y)^4}{4(x-y)^2}]^{\frac{1}{2}} \tag{2.21}$$

Using equations (2.20), (2.21) and value of ψ and ϕ in equation (2.1), we obtain that

$$\begin{aligned} 2(\frac{|x-y|}{4}) &\leq [\frac{(\frac{7}{8})^4 (xy)^2 + 3(x-y)^4}{4(x-y)^2}]^{\frac{1}{2}}, \\ \frac{(x-y)^2}{4} &\leq (\frac{(\frac{7}{8})^4 (xy)^2 + 3(x-y)^4}{4(x-y)^2}, \\ (x-y)^4 &\leq (\frac{7}{8})^4 (xy)^2 + 3(x-y)^4. \end{aligned}$$

which is true. So, all the conditions of Theorem 2.2 are satisfied. Clearly, 0 is the unique fixed point T. Hence Theorem 2.2 is verified.

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