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The Cauchy Problem for Matrix Factorization of the Helmholtz Equation in a Multidimensional Unbounded Domain

Davron Aslonqulovich Juraev

ABSTRACT: In the present work, the Carleman matrix is constructed. Using the construction of the Carleman matrix, a regularized solution of the Cauchy problem in a multidimensional unbounded domain $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$ is found in explicit form.

Key Words: Ill-Posed Cauchy Problems, regularized solution, approximate solution, matrix factorization, elliptical system.

Contents

1. Introduction

The most actively developing modern area of scientific knowledge is the theory of correctly and incorrectly posed problems, most of which have practical value and require decision making in uncertain or contradictory conditions. The development and justification of methods for solving such a complex class of problems as ill-posed ones is an urgent problem of the present time. The theory of ill-posed problems is an apparatus of scientific research for many scientific areas, such as differentiation of approximately given functions, solving inverse boundary value problems, solving problems of linear programming and control systems, solving degenerate or ill-conditioned systems of linear equations, etc. The concept of a well-posed problem is due to J. Hadamard [24], who took the point of view that every mathematical problem corresponding to some physical or technological problem must be well-posed. In fact, what physical interpretation can a solution have if an arbitrary small change in the data can lead to large changes in the solution? Moreover, it would be difficult to apply approximation methods to such problems. This put the expediency of studying ill-posed problems in doubt. For ill-posed problems of the question arises: What is meant by an approximate solution? Clearly, it should be so defined that it is stable under small changes of the original information. A second question is: What algorithms are there for the construction of such solutions? Answers to these basic questions were given by A.N. Tikhonov (see [2]).

The concept of conditional correctness first appeared in the work of Tikhonov [2], and then in the studies of Lavrent'ev [28]- [29]. In a theoretical study of the conditional correctness (correctness according to Tikhonov) of an ill-posed problem of the existence of a solution and its belonging to the correctness set, it is postulated in the very formulation of the problem. The study of uniqueness issues in a conditionally well-posed formulation does not essentially differ from the study in a classically well-posed formulation, and the stability of the solution from the data of the problem is required only from those variations of the data that do not deduce solutions from the well-posedness set. After establishing the uniqueness and stability theorems in the study of the conditional correctness of ill-posed problems, the question arises of constructing effective solution methods, i.e. construction of regularizing operators. It is known that the Cauchy problem for elliptic equations and for systems of elliptic equations belongs to the class of ill-posed problems (see, for example, [2], [21], [24], [26], [28]- [29], [36]- [37], [39]- [40]). Boundary value

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problems, as well as numerical solutions of some problems, are considered in [3]- [7], [22]- [23], [25], [27], [34]- [35], [38], [41].

Formulas that allow one to find a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman-type formulas. In [39], Carleman established a formula giving a solution to the Cauchy–Riemann equations in a domain of a special form. Developing their idea, G.M. Goluzin and V.I. Krylov [21] derived a formula for determining the values of analytic functions from data known only on a portion of the boundary, already for arbitrary domains. A formula of the Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance [28]- [29]). Using this method, Sh. Yarmukhamedov (see, for instance [36]- [37]) constructed the Carleman functions for the Laplace and Helmholtz operators. Carleman-type formulas for various elliptic equations and systems were also obtained in [20]- [21], [26], [28]- [29], [30]- [32], [36]- [37] and [8]- [19]. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in [26]. In [32], an integral formula was proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. In [20], the Cauchy problem was considered for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary.

Based on the results of previous works [36]- [37], we have constructed the Carleman matrix and based on it the approximate solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In this article, we find an explicit formula for an approximate solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain of an even-dimensional space $\mathbb{R}^m$. The case of an odd-dimensional space will be considered in other scientific studies of the author. Our approximate solution formula also includes the construction of a family of fundamental solutions of the Helmholtz operator in space. This family is parametrized by some entire function $K(z)$, the choice of which depends on the dimension of the space. In this work, relying on the results of previous works [8]- [19], we similarly obtain better results with approximate estimates due to a special selection of the function $K(z)$. This helped to get good results when finding an approximate solution based on the Carleman matrix. In many well-posed problems it is not possible to compute the values of the function on the whole boundary.

When solving correct problems, sometimes it is not possible to find the value of the vector function on the entire boundary. Finding the value of a vector function on the entire boundary for systems of elliptic type with constant coefficients (see, for example, [8]- [19]) is one of the topical problems in the theory of differential equations.

Let $\mathbb{R}^m$, $(m = 2k,\, k \geq 1)$ be the $m-$ dimensional real Euclidean space,

$$x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \quad y = (y_1, \ldots, y_m) \in \mathbb{R}^m,$$

$$x' = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}, \quad y' = (y_1, \ldots, y_{m-1}) \in \mathbb{R}^{m-1}.$$

$\Omega \subset \mathbb{R}^m$ is an unbounded simply-connected domain with piecewise smooth boundary consisting of the plane D: $y_m = 0$ and some smooth surface Σ lying in the half-space $y_m > 0$, i.e., $\partial\Omega = \Sigma \bigcup D$.

We introduce the following notation:

$$r = |y - x|, \quad \alpha = |y' - x'|, \quad z = i\sqrt{a^2 + \alpha^2} + y_m, \quad a \geq 0,$$

$$\partial_x = (\partial_{x_1}, \ldots, \partial_{x_m})^T, \quad \partial_x \to \xi^T, \quad \xi^T = \begin{pmatrix} \xi_1 \\ \ldots \\ \xi_m \end{pmatrix} \text{ - transposed vector } \xi,$$

$$V(x) = (V_1(x), \ldots, V_n(x))^T, \quad v^0 = (1, \ldots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m = 2k, \quad k \geq 1,$$

$$E(u) = \begin{Vmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & u_n \end{Vmatrix} - \text{diagonal matrix, } u = (u_1, \ldots, u_n) \in \mathbb{R}^n.$$

$P(\xi^T)$ is an $(n \times n)-$dimensional matrix satisfying:

$$P^*(\xi^T)P(\xi^T) = E((|\xi|^2 + \lambda^2)v^0),$$

where $P^*(\xi^T)$ is the Hermitian conjugate matrix of $P(\xi^T)$, $\lambda \in \mathbb{R}$, the elements of the matrix $P(\xi^T)$ consist of a set of linear functions with constant coefficients from the complex plane $\mathbb{C}$.

Let us consider the following first order systems of linear partial differential equations with constant coefficients

$$P\left(\partial_x\right) V(x) = 0, \tag{1.1}$$

in the region Ω where $P\left(\partial_x\right)$ is the matrix of first-order differential operators.

Also consider the set

$$S\left(\Omega\right) = \left\{V : \overline{\Omega} \longrightarrow \mathbb{R}^n\right\},$$

here V is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ and V satisfies the system (1.1).

2. Statement of the Cauchy problem

The Cauchy problem for system (1.1) is formulated as follows:

Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a continuous given function on Σ.

Problem 1. *Suppose $V(y) \in S(\Omega)$ and*

$$V(y)|_{\Sigma} = f(y), \quad y \in \Sigma. \tag{2.1}$$

Our purpose is to determine the function $V(y)$ in the domain Ω when its values are known on Σ.

If $V(y) \in S(\Omega)$, then the following Cauchy type integral formula

$$V(x) = \int\limits_{\partial G} L(y, x; \lambda)V(y)ds_y, \quad x \in \Omega, \tag{2.2}$$

is valid and

$$L(y, x; \lambda) = \left(E\left(\varphi_m(\lambda r)v^0\right) P^*\left(\partial_x\right)\right) P(t^T).$$

Here $t = (t_1, \dots, t_m)-$is the unit exterior normal, drawn at a point y, the surface $\partial\Omega$, $\varphi_m(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$, where $\varphi_m(\lambda r)$ defined by the following formula:

$$\varphi_m(\lambda r) = B_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$B_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k, \quad k \geq 1. \tag{2.3}$$

Here $H^{(1)}_{(m-2)/2}(\lambda r)-$ is the Hankel function of the first kind of $(m-2)/2-$ th order (see for instance [33]).

Let $K(z)$ be an entire function taking real values for real z, $(z = a + ib,\ a, b \in \mathbb{R})$ such that

$$K(a) \neq 0, \quad \sup_{b \geq 1} \left|b^p K^{(p)}(z)\right| = N(a, p) < \infty,$$
$$-\infty < a < \infty, \quad p = 0, 1, \dots, m. \tag{2.4}$$

We define the function $\Psi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Psi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \operatorname{Im}\left[\frac{K(z)}{z - x_m}\right] \frac{a\, I_0(\lambda a)}{\sqrt{a^2 + \alpha^2}} da,$$
$$z = i\sqrt{a^2 + \alpha^2} + y_m, \quad m = 2k,\ k \geq 1. \tag{2.5}$$

Where $c_m = (-1)^{k-1}(k-1)!(m-2)\omega_m$; $I_0(\lambda a) = J_0(i\lambda a)-$is the Bessel function of the first kind of zero order [1], ω_m- area of a unit sphere in space $\mathbb{R}^m$.

Formula (2.2) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Psi(y,x;\lambda) = \varphi_m(\lambda r) + g(y,x;\lambda), \tag{2.6}$$

where $g(y,x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then formula (2.2) has the following form

$$V(x) = \int\limits_{\partial G} L(y,x;\lambda)V(y)ds_y, \quad x \in \Omega, \tag{2.7}$$

where

$$L(y,x;\lambda) = \left(E\left(\Psi(y,x;\lambda)v^0\right)P^*\left(\partial_x\right)\right)P(t^T).$$

Formula (2.7) can be generalized for the case when Ω is the unbounded domain.

Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain, with a piecewise smooth boundary $\partial\Omega$ ($\partial\Omega-$extends to infinity).

We denote by Ω_R the part Ω lying inside the circle of radius R with center at zero:

$$\Omega_R = \{y : y \in \Omega, \quad |y| < R\}, \quad \Omega_R^\infty = \Omega \backslash \Omega_R, \quad R > 0.$$

Theorem 2.1. *Let $V(y) \in S(\Omega)$, Ω be a finitely connected unbounded domain in $\mathbb{R}^m$, with piecewise-smooth boundary $\partial\Omega$. If for each fixed $x \in \Omega$ we have the equality*

$$\lim_{R\to\infty} \int\limits_{\Omega_R^\infty} L(y,x;\lambda)V(y)ds_y = 0, \tag{2.8}$$

then the formula (2.7) is true.

Proof. Indeed, for a fixed $x \in \Omega$ ($|x| < R$) and taking (2.7) into account, we have

$$\int\limits_{\partial\Omega} L(y,x;\lambda)V(y)ds_y = \int\limits_{\partial\Omega_R} L(y,x;\lambda)V(y)ds_y$$

$$+\int\limits_{\partial\Omega_R^\infty} L(y,x;\lambda)V(y)ds_y = V(x) + \int\limits_{\partial\Omega_R^\infty} L(y,x;\lambda)V(y)ds_y, \qquad x \in \Omega_R.$$

Taking into account condition (2.8), for $R \to \infty$, we obtain (2.7).

Suppose that an unbounded domain Ω lies inside the layer of smallest width defined by the inequality

$$0 < y_m < h, \quad h = \frac{\pi}{\rho}, \quad \rho > 0,$$

and $\partial\Omega$ extends to infinity.

Suppose that for some $d_0 > 0$ the area $\partial\Omega$ satisfies the growth condition

$$\int\limits_{\partial\Omega} \exp\left[-d_0\rho_0\,|y'|\right] ds_y < \infty, \quad 0 < \rho_0 < \rho. \tag{2.9}$$

Suppose $V(y) \in S(\Omega)$ that it satisfies the boundary growth condition

$$|V(y)| \le \exp\left[\exp\rho_2\,|y'|\right], \quad \rho_2 < \rho, \quad y \in \Omega. \tag{2.10}$$

In (2.5) we put

$$K(z) = \exp\left[-d\, i\rho_1 \left(z - \frac{h}{2}\right) - d_1\, i\rho_0 \left(d - \frac{h}{2}\right)\right],$$

$$K(x_m) = \exp\left[d \cos\rho_1 \left(x_m - \frac{h}{2}\right) + d_1 \cos i\rho_0 \left(x_m - \frac{h}{2}\right)\right], \tag{2.11}$$

$$0 < \rho_1 < \rho, \quad 0 < x_m < h,$$

where

$$d = 2c\exp\left(\rho_1 |x'|\right), \quad d_1 > \frac{d_0}{\cos\left(\rho_0 \frac{h}{2}\right)}, \quad c \geq 0, \quad d > 0.$$

Then the integral representation (2.7) is true.

For a fixed $x \in \Omega$ and $y \to \infty$, we estimate the function $\Psi(y, x; \lambda)$ and its derivatives $\frac{\partial\Psi(y, x; \lambda)}{\partial y_j}$, $(j = 1, \ldots, m-1)$, $\frac{\partial\Psi(y, x; \lambda)}{\partial y_m}$. For the estimation $\frac{\partial\Psi(y, x; \lambda)}{\partial y_j}$ we use the following equality

$$\frac{\partial\Psi(y, x; \lambda)}{\partial y_j} = \frac{\partial\Psi(y, x; \lambda)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j - x_j)\frac{\partial\Psi(y, x; \lambda)}{\partial s},$$

$$s = \alpha^2, \quad j = 1, \ldots, m-1. \tag{2.12}$$

Really,

$$\left|\exp\left[-d\, i\rho_1 \left(z - \frac{h}{2}\right) - d_1 i\rho_0 \left(z - \frac{h}{2}\right)\right]\right|$$

$$= \exp \mathrm{Re}\left[-d i\rho_1 \left(z - \frac{h}{2}\right) - d_1 i\rho_0 \left(z - \frac{h}{2}\right)\right]$$

$$= \exp\left[-d\rho_1 \sqrt{a^2 + \alpha^2} \cos\rho_1 \left(y_m - \frac{h}{2}\right) - d_1\rho_0 \sqrt{a^2 + \alpha^2} \cos\rho_0 \left(y_m - \frac{h}{2}\right)\right].$$

As

$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \leq \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2},$$

$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \leq \rho_0 \left(y_m - \frac{h}{2}\right) \leq \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2}.$$

Consequently,

$$\cos\rho \left(y_m - \frac{h}{2}\right) > 0, \quad \cos\rho_0 \left(y_m - \frac{h}{2}\right) \geq \cos\frac{h\rho_0}{2} > \delta_0 > 0.$$

It does not vanish in the region Ω and

$$|\Psi(y, x; \lambda)| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1 |y'|\right)\right], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial\Omega,$$

$$\left|\frac{\partial\Psi(y, x; \lambda)}{\partial y_j}\right| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1 |y'|\right)\right], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial\Omega,\ j = 1, \ldots, m-1,$$

$$\left|\frac{\partial\Psi(y, x; \lambda)}{\partial y_m}\right| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1 |y'|\right)\right], \quad \varepsilon > 0, \quad y \to \infty, \quad y \in \Omega \bigcup \partial\Omega.$$

We now choose ρ_1 with the condition $\rho_2 < \rho_1 < \rho$. Then condition (2.9) is fulfilled and the integral formula (2.7) is true. **Theorem 2.1 is proved.** □

Condition (2.11) can be weakened.

We denote by $S_\rho(\Omega)$ is the class of vector-valued functions from $S(\Omega)$, satisfying the following growth condition:

$$S_\rho(\Omega) = \{V(y) \in S(\Omega), \quad |V(y)| \leq \exp\left[\mathrm{o}\left(\exp\rho\,|y'|\right)\right], \quad y \to \infty, \quad y \in \Omega\}. \tag{2.13}$$

The following is valid

Theorem 2.2. *Suppose $V(y) \in S_\rho(\Omega)$ that it satisfies the growth condition*

$$|V(y)| \leq C\exp\left[c\cos\rho_1\left(y_m - \frac{h}{2}\right)\exp\left(\rho_1\,|y'|\right)\right], \qquad c \geq 0, \quad 0 < \rho_1 < \rho, \quad y \in \partial\Omega, \tag{2.14}$$

where $C-$is some constant. Then the formula (2.7) is valid.

Proof. Divide the area Ω by a line $y_m = \dfrac{h}{2}$ into two areas

$$\Omega_1 = \left\{y:\, 0 < y_m < \frac{h}{2}\right\} \text{ and } \Omega_2 = \left\{y:\, \frac{h}{2} < y_m < h\right\}.$$

Consider the domain Ω_1. In the formula (2.5) together $K(z)$ we put $K_1(z)$

$$K_1(z) = K(z)\exp\left[-\delta i\tau\left(z - \frac{h}{2}\right) - \delta_1 i\rho\left(z - \frac{h}{2}\right)\right], \qquad \rho < \tau < 2\rho, \quad \delta > 0, \quad \delta_1 > 0. \tag{2.15}$$

Here $K(z)$ it is determined from (2.11). With this notation, (2.9) is true.

Really,

$$\left|\exp\left[-i\tau\left(z - \frac{h}{4}\right) - \delta_1 i\rho\left(z - \frac{h}{4}\right)\right]\right|$$

$$= \exp\left[-\delta\tau\sqrt{a^2+\alpha^2}\cos\tau\left(y_m - \frac{h}{4}\right)\right]$$

$$= \exp\left[-\delta\tau\sqrt{a^2+\alpha^2}\right] \leq \exp\left[-\delta\exp\tau\,|y'|\right],$$

as

$$-\frac{\pi}{2} \leq -\tau\frac{\pi}{4} \leq \tau\left(y_m - \frac{h}{4}\right) \leq \tau\frac{\pi}{2} < \frac{h}{2} \text{ and } \cos\tau\left(y_m - \frac{h}{4}\right) \geq \cos\tau\frac{h}{4} \geq \delta_0 > 0.$$

We denote the corresponding $\Psi(y, x; \lambda)$ by $\Psi^+(y, x; \lambda)$. As

$$\cos\tau\left(y_m - \frac{h}{4}\right) \geq \delta_0, \quad y \in \Omega_1 \bigcup \partial\Omega_1,$$

then for fixed $x \in \Omega_1$, $y \in \Omega_1 \bigcup \partial\Omega_1$, for $\Psi^+(y, x; \lambda)$ and its derivatives are true asymptotic estimates

$$|\Psi^+(y, x; \lambda)| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\,|y'|\right)\right], \quad y \to \infty, \quad \rho < \tau < 2\rho,$$

$$\left|\frac{\partial\Psi^+(y, x; \lambda)}{\partial y_j}\right| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\,|y'|\right)\right], \quad y \to \infty, \quad \rho < \tau < 2\rho, \quad j = 1, \ldots, m-1,$$

$$\left|\frac{\partial\Psi^+(y, x; \lambda)}{\partial y_m}\right| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\,|y'|\right)\right], \quad y \to \infty, \quad \rho < \tau < 2\rho.$$

Suppose $V(y) \in S_\rho(\Omega_1)$ that in a domain Ω_1 satisfies the growth condition

$$|V(y)| \leq C \exp\left[\exp\left(2\rho - \varepsilon\right)|y'|\right], \quad \varepsilon > 0. \tag{2.16}$$

We choose τ the inequality $2\rho - \varepsilon < \tau < 2\rho$ in (2.15).

Then the condition (2.15) is satisfied for the region Ω_1, therefore, the following integral formula holds

$$V(x) = \int\limits_{\partial\Omega_1} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega_1, \tag{2.17}$$

where

$$L(y, x; \lambda) = \left(E\left(\Psi^+(y, x; \lambda) v^0\right) P^*\left(\partial_x\right)\right) P(t^T).$$

If $V(y) \in S_\rho(\Omega_2)$ satisfies the growth condition (2.14) in Ω_2, then for $2\rho - \varepsilon < \tau < 2\rho$, similarly we obtain the following integral formula

$$V(x) = \int\limits_{\partial\Omega_2} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega_2, \tag{2.18}$$

where

$$L(y, x; \lambda) = \left(E\left(\Psi^-(y, x; \lambda) v^0\right) P^*\left(\partial_x\right)\right) P(t^T).$$

Here $\Psi^-(y, x; \lambda)$ it is defined by the formula (6), in which $K(z)$ it is replaced by the function $K_2(z):$

$$K_2(z) = K(z) \exp\left[-\delta i \tau\left(z - h_1\right) - \delta_1 i \rho\left(z - \frac{h}{2}\right)\right], \tag{2.19}$$

where

$$h_1 = \frac{h}{2} + \frac{h}{4}, \quad \frac{h}{2} < y_m < h, \quad \frac{h}{2} < x_m < h_1, \quad \delta > 0, \quad \delta_1 > 0.$$

In the formulas obtained with this formula, the integrals (according to (2.10)) converge uniformly for $\delta \geq 0$, when $V(y) \in S_\rho(\Omega)$. In these formulas we put $\delta = 0$ and, combining the formulas obtained, we find

$$V(x) = \int\limits_{\partial\Omega} L(y, x; \lambda) V(y) ds_y, \quad x \in \Omega, \quad x_m \neq \frac{h}{2}, \tag{2.20}$$

where

$$L(y, x; \lambda) = \left(E\left(\tilde{\Psi}(y, x; \lambda) v^0\right) P^*\left(\partial_x\right)\right) P(t^T).$$

(integrals over the cross section $y_m = \dfrac{h}{2}$ are mutually destroyed)

$$\tilde{\Psi}(y, x; \lambda) = (\Psi^+(y, x; \lambda))_{\delta=0} = (\Psi^-(y, x; \lambda))_{\delta=0}.$$

Here, $\tilde{\Psi}(y, x; \lambda)$ is determined by the formula (2.5), in which $K(z)$ is determined from (2.15), where $\delta = 0$ is laid. According to the continuation principle, formula (2.20) is true for $\forall x \in \Omega$. Under condition (2.16), formula (2.20) is true for $\forall \delta_1 \geq 0$. Assuming $\delta_1 = 0$, we obtain the proof of the theorem. **Theorem 2.2 is proved.** □

In the formula (2.5), choosing

$$K(z) = \frac{1}{(z - x_m + 2h)^k} \exp(\sigma z), \quad k \geq 2,$$
$$K(x_m) = \frac{1}{(2h)^k} \exp(\sigma x_m), \quad 0 < x_m < h, \quad h = \frac{\pi}{\rho}, \tag{2.21}$$

we get

$$\Psi_\sigma(y,x) = -\frac{e^{-\sigma x_m}}{c_m(2h)^{-k}}\frac{\partial^{k-1}}{\partial s^{k-1}}\int_0^\infty \mathrm{Im}\frac{\exp(\sigma z)}{(z-x_m+2h)^k(z-x_m)}\frac{a I_0(\lambda a)}{\sqrt{a^2+\alpha^2}}da. \tag{2.22}$$

Then the integral formula (2.7) has the following form:

$$V(x) = \int_{\partial\Omega} L_\sigma(y,x;\lambda)V(y)ds_y, \quad x\in\Omega, \tag{2.23}$$

where

$$L_\sigma(y,x;\lambda) = \left(E\left(\Psi_\sigma(y,x;\lambda)v^0\right)P^*\left(\partial_x\right)\right)P(t^T).$$

3. Regularized solution of problem (1.1)–(2.1)

Theorem 3.1. *Let $V(y)\in S_\rho(\Omega)$ satisfy in the following inequality*

$$|V(y)|\le M, \quad y\in D. \tag{3.1}$$

If

$$V_\sigma(x) = \int_{\Sigma} L_\sigma(y,x;\lambda)V(y)ds_y, \quad x\in\Omega, \tag{3.2}$$

then the following estimations are correct:

$$|V(x)-V_\sigma(x)|\le K_\rho(\lambda,x)\sigma^k M e^{-\sigma x_m}, \quad x\in\Omega, \tag{3.3}$$

$$\left|\frac{\partial V(x)}{\partial x_j}-\frac{\partial V_\sigma(x)}{\partial x_j}\right|\le K_\rho(\lambda,x)\sigma^k M e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega, \quad j=1,\dots,m, \tag{3.4}$$

where $K_\rho(\lambda,x)$ shows the bounded functions on compact subsets of the domain Ω.

Proof. Let us first estimate inequality (3.3). Using the integral formula (2.23) and the equality (3.2), we obtain

$$V(x) = \int_{\Sigma} L_\sigma(y,x;\lambda)U(y)ds_y + \int_D L_\sigma(y,x;\lambda)V(y)ds_y$$

$$= L_\sigma(x) + \int_D L_\sigma(y,x;\lambda)V(y)ds_y, \quad x\in\Omega.$$

Taking into account the inequality (3.1), we estimate the following

$$\begin{aligned}|V(x)-V_\sigma(x)| &\le \left|\int_D L_\sigma(y,x;\lambda)V(y)ds_y\right| \\ &\le \int_D |L_\sigma(y,x;\lambda)|\,|V(y)|\,ds_y \le M\int_D |L_\sigma(y,x;\lambda)|\,ds_y, \quad x\in\Omega.\end{aligned} \tag{3.5}$$

For this aim, we estimate the integrals $\int_D |\Psi_\sigma(y,x;\lambda)|\,ds_y$, $\int_D \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y$, $(j=1,\dots,m-1)$ and $\int_D \left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$ on the part D of the plane $y_m=0$.

Separating the imaginary part of (2.22), we obtain

$$\Psi_\sigma(y,x) = \frac{e^{\sigma(y_m-x_m)}}{c_m(2h)^{-k}}\frac{\partial^{k-1}}{\partial s^{k-1}}\left[\int_0^\infty\left(\frac{(\beta+\beta_1)\cos\sigma\alpha_1}{\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_1^2+\beta^2\right)}\right.\right.$$
$$\left.\left.+\frac{\left(-\alpha_1^2+\beta_1\beta\right)}{\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_1^2+\beta^2\right)}\frac{\sin\sigma\alpha_1}{\alpha_1}\right)a\,I_0(\lambda a)\,da\right], \tag{3.6}$$

where

$$\alpha_1^2 = a^2+\alpha^2, \quad \beta = y_m - x_m, \quad \beta_1 = y_m - x_m + 2h.$$

Given (3.6) and the inequality

$$I_0(\lambda a) \le \sqrt{\frac{2}{\lambda\pi a}}, \tag{3.7}$$

we have

$$\int_D |\Psi_\sigma(y,x;\lambda)|\,ds_y \le K_\rho(\lambda,x)\sigma^k e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega. \tag{3.8}$$

To estimate the second integral, we use the equality

$$\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j} = \frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j-x_j)\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2, \quad j=1,\dots,m-1. \tag{3.9}$$

Given equality (3.6), inequality (3.7) and equality (3.9), we obtain

$$\int_D\left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_j}\right|ds_y \le K_\rho(\lambda,x)\sigma^k e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega. \tag{3.10}$$

Now, we estimate the integral $\int_D\left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y$, we obtain

$$\int_D\left|\frac{\partial\Psi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y \le K_\rho(\lambda,x)\sigma^k e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega. \tag{3.11}$$

From inequalities (3.8), (3.10) and (3.11), bearing in mind (3.5), we obtain an estimate (3.3).

Now the inequality (3.4) can be proved. To do this, we take the derivatives from equalities (2.23) and (3.2) with respect to x_j, $(j=1,\dots,m)$ then we get:

$$\frac{\partial V(x)}{\partial x_j} = \int_\Sigma\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}V(y)ds_y + \int_D\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}V(y)ds_y,$$
$$\frac{\partial V_\sigma(x)}{\partial x_j} = \int_\Sigma\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}V(y)ds_y, \quad x\in\Omega, \quad j=1,\dots,m. \tag{3.12}$$

Taking into account the (3.12) and inequality (3.1), we estimate the following

$$\left|\frac{\partial V(x)}{\partial x_j} - \frac{\partial_\sigma V(x)}{\partial x_j}\right| \le \left|\int_D\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}V(y)ds_y\right|$$
$$\le \int_D\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right||V(y)|\,ds_y \le M\int_D\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y, \tag{3.13}$$
$$x\in\Omega, \quad j=1,\dots,m.$$

To do this, we estimate the integrals $\int\limits_{D}\left|\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_j}\right|ds_y, (j=1,\dots,m-1)$ and $\int\limits_{D}\left|\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_m}\right|ds_y$ on the part D of the plane $y_m=0$.

To estimate the first integrals, we use the equality

$$\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_j}=\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial x_j}=-2(y_j-x_j)\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2,\quad j=1,\dots,m-1. \tag{3.14}$$

Applying equality (3.6), inequality (3.7) and equality (3.14), we obtain

$$\int\limits_{D}\left|\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_j}\right|ds_y\leq K_{\rho}(\lambda,x)\sigma^k e^{-\sigma x_m},\quad \sigma>1,\quad x\in\Omega. \tag{3.15}$$

Now, we estimate the integral $\int\limits_{D}\left|\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_m}\right|ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int\limits_{D}\left|\frac{\partial\Psi_{\sigma}(y,x;\lambda)}{\partial x_m}\right|ds_y\leq K_{\rho}(\lambda,x)\sigma^k e^{-\sigma x_m},\quad \sigma>1,\quad x\in\Omega. \tag{3.16}$$

From inequalities (3.15) and (3.16), bearing in mind (3.13), we obtain an estimate of (3.4). **Theorem 3.1 is proved.** □

Corollary 3.2. *For each $x\in\Omega$, the equalities are true*

$$\lim_{\sigma\to\infty}V_{\sigma}(x)=V(x),\quad \lim_{\sigma\to\infty}\frac{\partial V_{\sigma}(x)}{\partial x_j}=\frac{\partial V(x)}{\partial x_j},\quad j=1,\dots,m.$$

We define $\overline{G}_{\varepsilon}$ as

$$\overline{\Omega}_{\varepsilon}=\left\{(x_1,\dots,x_m)\in\Omega,\quad q>x_m\geq\varepsilon,\quad q=\max_{T}\psi(x'),\quad 0<\varepsilon<q\right\}.$$

Here $\psi(x')$-is a surface. It is easy to see that the set $\overline{\Omega}_{\varepsilon}\subset\Omega$ is compact.

Corollary 3.3. *If $x\in\overline{\Omega}_{\varepsilon}$, then the families of functions $\{V_{\sigma}(x)\}$ and $\left\{\frac{\partial V_{\sigma}(x)}{\partial x_j}\right\}$ converge uniformly for $\sigma\to\infty$, i.e.:*

$$V_{\sigma}(x)\rightrightarrows V(x),\quad \frac{\partial V_{\sigma}(x)}{\partial x_j}\rightrightarrows\frac{\partial V(x)}{\partial x_j},\quad j=1,\dots,m.$$

We should note that the set $E_{\varepsilon}=\Omega\backslash\overline{\Omega}_{\varepsilon}$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

Suppose that the boundary of the domain Ω consists of a hyper plane $y_m=0$ and a smooth surface Σ extending to infinity and lying in the layer

$$0<y_m<h,\quad h=\frac{\pi}{\rho},\quad \rho>0.$$

We assume that Σ is given by the equation

$$y_m=\psi(y'),\quad y'\in\mathbb{R}^{m-1},$$

where $\psi(y')$ satisfies the condition

$$\left|\psi'(y')\right| \leq M < \infty, \quad M = const.$$

We put

$$q = \max_{D} \psi(y'), \quad l = \max_{D} \sqrt{1 + \psi'^2(y')}.$$

Theorem 3.4. *Let $V(y) \in S_\rho(\Omega)$ satisfies in the condition (3.1), and on a smooth surface Σ the inequality*

$$|V(y)| \leq \delta, \quad 0 < \delta < 1. \tag{3.17}$$

Then the following relations are true

$$|V(x)| \leq K_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{q}} \delta^{\frac{x_m}{q}}, \quad \sigma > 1, \quad x \in \Omega, \tag{3.18}$$

$$\left|\frac{\partial V(x)}{\partial x_j}\right| \leq K_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{q}} \delta^{\frac{x_m}{q}}, \quad \sigma > 1, \quad x \in \Omega, \quad j = 1, \dots, m. \tag{3.19}$$

Proof. Let us first estimate inequality (3.18). Using the integral formula (2.23), we have

$$V(x) = \int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y + \int_{D} L_\sigma(y, x; \lambda))V(y)ds_y, \quad x \in \Omega. \tag{3.20}$$

We estimate the following

$$|V(x)| \leq \left|\int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y\right| + \left|\int_{D} L_\sigma(y, x; \lambda)V(y)ds_y\right|, \quad x \in \Omega. \tag{3.21}$$

Given inequality (3.17), we estimate the first integral of inequality (3.21).

$$\begin{aligned} \left|\int_{\Sigma} L_\sigma(y, x; \lambda)V(y)ds_y\right| &\leq \int_{\Sigma} |L_\sigma(y, x; \lambda)| \, |V(y)| \, ds_y \\ &\leq \delta \int_{\Sigma} |L_\sigma(y, x; \lambda)| \, ds_y, \quad x \in \Omega. \end{aligned} \tag{3.22}$$

To do this, we estimate the integrals $\int_{\Sigma} |\Psi_\sigma(y, x; \lambda)| \, ds_y$, $\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_j}\right| ds_y, (j = 1, \dots, m-1)$ and $\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_m}\right| ds_y$ on a smooth surface Σ.

Given equality (3.6) and the inequality (3.7), we have

$$\int_{\Sigma} |\Psi_\sigma(y, x; \lambda)| \, ds_y \leq K_\rho(\lambda, x)\sigma^k e^{\sigma(q - x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.23}$$

To estimate the second integral, using equalities (3.6) and (3.9) as well as inequality (3.7), we obtain

$$\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(y, x; \lambda)}{\partial y_j}\right| ds_y \leq K_\rho(\lambda, x)\sigma^k e^{\sigma(q - x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.24}$$

To find the integral $\int\limits_{\Sigma} \left| \frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y$, using equality (3.7) and inequality (3.8), we obtain

$$\int\limits_{\Sigma} \left| \frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y \le K_\rho(\lambda,x)\sigma^k e^{\sigma(q-x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.25}$$

From (3.23)–(3.25) and applying (3.22), we obtain

$$\left| \int\limits_{\Sigma} L_\sigma(y,x;\lambda)V(y)ds_y \right| \le K_\rho(\lambda,x)\sigma^k \delta\, e^{\sigma(q-x_m)}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.26}$$

The following is known

$$\left| \int\limits_{D} L_\sigma(y,x;\lambda)V(y)ds_y \right| \le K_\rho(\lambda,x)\sigma^k M e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.27}$$

Now taking into account (3.26)–(3.27) and using (3.21)–(3.22), we have

$$|V(x)| \le \frac{K_\rho(\lambda,x)\sigma^k}{2}(\delta\, e^{\sigma q} + M)e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in \Omega. \tag{3.28}$$

Choosing σ from the equality

$$\sigma = \frac{1}{q}\ln\frac{M}{\delta}, \tag{3.29}$$

we obtain an estimate (3.18).

Now let us prove inequality (3.19). To do this, we find the partial derivative from the integral formula (2.23) with respect to the variable x_j, $j = 1,\dots,m$:

$$\begin{aligned} \frac{\partial V(x)}{\partial x_j} &= \int\limits_{\Sigma} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y + \int\limits_{D} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y \\ &= \frac{\partial V_\sigma(x)}{\partial x_j} + \int\limits_{D} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y, \quad x \in \Omega, \quad j = 1,\dots,m. \end{aligned} \tag{3.30}$$

Here

$$\frac{\partial V_\sigma(x)}{\partial x_j} = \int\limits_{\Sigma} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y. \tag{3.31}$$

We estimate the following

$$\begin{aligned} \left| \frac{\partial V(x)}{\partial x_j} \right| &\le \left| \int\limits_{\Sigma} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y \right| \\ &+ \left| \int\limits_{D} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y \right| \le \left| \frac{\partial V_\sigma(x)}{\partial x_j} \right| \\ &+ \left| \int\limits_{D} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y)ds_y \right|, \quad x \in \Omega, \quad j = 1,\dots,m. \end{aligned} \tag{3.32}$$

Given inequality (3.17), we estimate the first integral of inequality (3.32).

$$\left|\int\limits_{\Sigma} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y) ds_y\right| \le \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right| |V(y)|\, ds_y$$
$$\le \delta \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y, \quad x\in\Omega, \quad j=1,\dots,m. \tag{3.33}$$

To do this, we estimate the integrals $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y, (j=1,\dots,m-1)$ and $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$ on a smooth surface Σ.

Given equality (3.6), inequality (3.7) and equality (3.14), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y \le K_\rho(\lambda,x)\sigma^k e^{\sigma(q-x_m)}, \quad \sigma>1, \quad x\in\Omega. \tag{3.34}$$

Now, we estimate the integral $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y \le K_\rho(\lambda,x)\sigma^k e^{\sigma(q-x_m)}, \quad \sigma>1, \quad x\in\Omega. \tag{3.35}$$

From (3.34) and (3.35), bearing in mind (3.33), we obtain

$$\left|\int\limits_{\Sigma} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y) ds_y\right| \le K_\rho(\lambda,x)\sigma^k \delta e^{\sigma(q-x_m)}, \quad \sigma>1, \quad x\in\Omega,$$
$$j=1,\dots,m. \tag{3.36}$$

The following is known

$$\left|\int\limits_{D} \frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j} V(y) ds_y\right| \le K_\rho(\lambda,x)\sigma^k M e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega,$$
$$j=1,\dots,m. \tag{3.37}$$

Now taking into account (3.36)–(3.37), bearing in mind (3.32), we have

$$\left|\frac{\partial V(x)}{\partial x_j}\right| \le \frac{K_\rho(\lambda,x)\sigma^k}{2}(\delta e^{\sigma q}+M)e^{-\sigma x_m}, \quad \sigma>1, \quad x\in\Omega,$$
$$j=1,\dots,m. \tag{3.38}$$

Choosing σ from the equality (3.29) we obtain an estimate (3.19). **Theorem 3.4 is proved.** □

Assume that $V(y)\in S_\rho(\Omega)$ and instead of $V(y)$ on Σ its continuous approximations $f_\delta(y)$ are given, respectively, with error $0<\delta<1$ then

$$\max_{\Sigma} |V(y)-f_\delta(y)| \le \delta. \tag{3.39}$$

We put

$$V_{\sigma(\delta)}(x) = \int\limits_{\Sigma} L_\sigma(y,x;\lambda) f_\delta(y) ds_y, \quad x\in\Omega. \tag{3.40}$$

Theorem 3.5. *Let $V(y) \in S_\rho(\Omega)$ on the part of the plane $y_m = 0$ satisfies in the condition (3.1).*

Then the following estimates is true

$$\left|V(x) - V_{\sigma(\delta)}(x)\right| \leq K_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{q}} \delta^{\frac{x_m}{q}}, \quad \sigma > 1, \quad x \in \Omega, \tag{3.41}$$

$$\left|\frac{\partial V(x)}{\partial x_j} - \frac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq K_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{q}} \delta^{\frac{x_m}{q}}, \quad \sigma > 1, \quad x \in \Omega, \quad j = 1, \dots, m. \tag{3.42}$$

Proof. From the integral formulas (2.23) and (3.40), we have

$$V(x) - V_{\sigma(\delta)}(x) = \int\limits_{\partial\Omega} L_\sigma(y, x; \lambda) L(y) ds_y$$

$$- \int\limits_{\Sigma} L_\sigma(y, x; \lambda) f_\delta(y) ds_y = \int\limits_{\Sigma} L_\sigma(y, x; \lambda) V(y) ds_y$$

$$+ \int\limits_{D} L_\sigma(y, x; \lambda) V(y) ds_y - \int\limits_{\Sigma} L_\sigma(y, x; \lambda) f_\delta(y) ds_y$$

$$= \int\limits_{\Sigma} L_\sigma(y, x; \lambda) \left\{V(y) - f_\delta(y)\right\} ds_y + \int\limits_{D} L_\sigma(y, x; \lambda) L(y) ds_y.$$

and

$$\frac{\partial V(x)}{\partial x_j} - \frac{\partial V_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial\Omega} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} V(y) ds_y$$

$$- \int\limits_{\Sigma} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} f_\delta(y) ds_y = \int\limits_{\Sigma} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} V(y) ds_y$$

$$+ \int\limits_{D} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} V(y) ds_y - \int\limits_{\Sigma} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} f_\delta(y) ds_y$$

$$= \int\limits_{\Sigma} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} \left\{V(y) - f_\delta(y)\right\} ds_y + \int\limits_{D} \frac{\partial L_\sigma(y, x; \lambda)}{\partial x_j} V(y) ds_y,$$

$$j = 1, \dots, m.$$

Using conditions (3.1) and (3.39), we estimate the following:

$$\left|V(x)-V_{\sigma(\delta)}(x)\right| = \left|\int\limits_{\Sigma} L_\sigma(y,x;\lambda)\left\{V(y)-f_\delta(y)\right\}ds_y\right|$$

$$+\left|\int\limits_{D} L_\sigma(y,x;\lambda)V(y)ds_y\right| \le \int\limits_{\Sigma}\left|L_\sigma(y,x;\lambda)\right|\left|\left\{V(y)-f_\delta(y)\right\}\right|ds_y$$

$$+\int\limits_{D}\left|L_\sigma(y,x;\lambda)\right|\left|V(y)\right|ds_y \le \delta\int\limits_{\Sigma}\left|L_\sigma(y,x;\lambda)\right|ds_y$$

$$+M\int\limits_{D}\left|L_\sigma(y,x;\lambda)\right|ds_y.$$

and

$$\left|\frac{\partial V(x)}{\partial x_j}-\frac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}\right| = \left|\int\limits_{\Sigma}\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\left\{V(y)-f_\delta(y)\right\}ds_y\right|$$

$$+\left|\int\limits_{D}\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}V(y)ds_y\right| \le \int\limits_{\Sigma}\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|\left\{U(y)-f_\delta(y)\right\}\right|ds_y$$

$$+\int\limits_{D}\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|V(y)\right|ds_y \le \delta\int\limits_{\Sigma}\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y$$

$$+M\int\limits_{D}\left|\frac{\partial L_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y, \quad j=1,\dots,m.$$

Now, repeating the proof of Theorems 3.1 and 3.4, we obtain

$$\left|V(x)-V_{\sigma(\delta)}(x)\right| \le \frac{K_\rho(\lambda,x)\sigma^k}{2}(\delta\, e^{\sigma q}+M)e^{-\sigma x_m},$$

$$\left|\frac{\partial V(x)}{\partial x_j}-\frac{V_{\sigma(\delta)}(x)}{\partial x_j}\right| \le \frac{K_\rho(\lambda,x)\sigma^k}{2}(\delta\, e^{\sigma q}+M)e^{-\sigma x_m}, \quad j=1,\dots,m.$$

From here, choosing σ from equality (3.29), we obtain an estimates (3.41) and (3.42).
Theorem 3.5 is proved. □

Corollary 3.6. *For each $x\in\Omega$, the following equalities are true*

$$\lim_{\delta\to 0} V_{\sigma(\delta)}(x) = V(x), \quad \lim_{\delta\to 0}\frac{\partial V_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial V(x)}{\partial x_j}, \quad j=1,\dots,m.$$

Corollary 3.7. *If $x\in\overline{\Omega}_\varepsilon$, then the families of functions $\left\{V_{\sigma(\delta)}(x)\right\}$ and $\left\{\dfrac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ are convergent uniformly for $\delta\to 0$, i.e.:*

$$V_{\sigma(\delta)}(x) \rightrightarrows V(x), \quad \frac{\partial V_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial V(x)}{\partial x_j}, \quad j=1,\dots,m.$$

4. Conclusion

In this paper, based on previous papers, we explicitly found a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation in a $m-$dimensional unbounded domain. We note that for solving applicable problems, the approximate values of $V(x)$ and $\dfrac{\partial V(x)}{\partial x_j}$, $x \in \Omega$, $j = 1, \ldots, m$ should be found.

In this paper, we have built a family of vector-functions $V(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\dfrac{\partial V(x, f_\delta)}{\partial x_j} = \dfrac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}$, $(j = 1, \ldots, m)$ depend on a parameter σ. Also, we prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $V_{\sigma(\delta)}(x)$ and $\dfrac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}$ are convergent to a solution $V(x)$ and its derivative $\dfrac{\partial V(x)}{\partial x_j}$, $x \in \Omega$ at point $x \in \Omega$. A family of vector-functions $V_{\sigma(\delta)}(x)$ and $\dfrac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method to find the approximate solution of the problem.

Thus, functionals $V_{\sigma(\delta)}(x)$ and $\dfrac{\partial V_{\sigma(\delta)}(x)}{\partial x_j}$ determine the regularization of the solution of problems (1.1) and (2.1).

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Davron Aslonqulovich Juraev,
Department of scientific research, innovation and training of scientific and pedagogical staff,
University of Economics and Pedagogy, I. Karimov 13, 180100, Karshi,
Uzbekistan;
Department of Natural Science Disciplines,
Higher Military Aviation School of the Republic of Uzbekistan, Jaykhun 54, 180117, Karshi,
Uzbekistan;
Department of Mathematics,
Anand International College of Engineering, Near Kanota, Agra Road, Jaipur-303012, Rajasthan,
India.
E-mail address: juraevdavron12@gmail.com; davronjuraev81@gmail.com