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On Almost Kenmotsu Manifolds admitting Conformal Ricci-Yamabe Solitons

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ABSTRACT: This paper aims to characterize almost Kenmotsu Manifolds admitting conformal Ricci-Yamabe solitons. First, we studied $(\kappa, \mu)'$ and generalized $(\kappa, \mu)'-$ almost Kenmotsu manifolds which admits conformal Ricci-Yamabe soliton. We have shown that a $(\kappa, \mu)'$ almost Kenmotsu manifold M^{2n+1}admitting a conformal Ricci-Yamabe soliton is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ provided $2\lambda - \beta\tau \neq 4\alpha n\kappa - \left(p + \frac{2}{2n+1}\right)$. Also, we give conditions for the conformal pressure when the soliton is expanding, steady or shrinking. It is shown that if the manifold admits a conformal gradient Ricci-Yamabe soliton, then the potential vector field is a strict infinitesimal contact transformation. Finally, we construct an example of a 3−dimensional almost Kenmotsu manifold satisfying conformal Ricci-Yamabe soliton.

Key Words: Conformal Ricci-Yamabe soliton, conformal gradient Ricci-Yamabe soliton, locally isomorphic, contact transformation, almost kenmotsu manifolds.

Contents

1. Introduction

The theory of geometric flows shapes the foundation in understanding the geometric structures arise in Riemannian geometry. Hamilton [11] introduced the Ricci flow and used it to prove a three dimensional sphere theorem [12]. The Ricci flow plays the main role in forming the proof of the well known Poincaré conjecture. The Ricci soliton on a Riemannian manifold (M, g) is defined by

$$(\mathcal{L}_V g)(X_1, X_2) + 2Ric(X_1, X_2) = 2\lambda g(X_1, X_2),$$

for all vector fields X_1, X_2 on M. Here, $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along a potential vector field V, Ric is the Ricci tensor of M and $\lambda \in \mathbb{R}$. The Ricci soliton is a limiting solution of the Ricci flow: $\frac{\partial}{\partial t} g(t) = -Ric(t)$, where $g(0) = g$; and is said to be expanding, steady and shrinking according as λ is negative, zero and positive respectively. When the vector field V is the gradient of a smooth function ζ on M, that is, $V = \nabla\zeta$, then we say that the Ricci soliton is gradient.

Hamilton bring forth the idea of Yamabe flow to deal with the Yamabe problem on manifolds of positive conformal Yamabe invariant. The Yamabe soliton is a self-limiting solution to the Yamabe flow. The Yamabe soliton on a Riemannian manifold (M, g) is given by

$$(\mathcal{L}_V g)(X_1, X_2) = 2(\tau - \lambda)g(X_1, X_2),$$

for any vector fields X_1, X_2 on M, where τ is the scalar curvature of the manifold and $\lambda \in \mathbb{R}$. The Yamabe soliton is equivalent to Ricci soliton in dimension 2, but not for dimension $n > 2$.

Recently, in 2019, Güler and Crasmareanu [10] introduced and studied a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. The authors defined in [10] the Ricci-Yamabe map as:

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Definition 1.1 [10] A Riemannian flow on M is a smooth map:

$$g : I \subset \mathbb{R} \to Riem(M),$$

where I is a given open interval.

Definition 1.2 [10] The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by

$$RY^{(\alpha,\beta,g)}(t) := \frac{\partial g}{\partial t}(t) + 2\alpha Ric(t) + \beta R(t)g(t),$$

is called the $(\alpha,\beta)-$Ricci-Yamabe map of the Riemannian flow (M,g). If $RY^{(\alpha,\beta,g)} \equiv 0$, then $g(\cdot)$ will be called an $(\alpha,\beta)-$Ricci-Yamabe flow.

The $(\alpha,\beta)-$Ricci-Yamabe map can be Riemannian or semi-Riemannian or singular Riemannian flow since α and β can have positive and negative signature. The Ricci-Yamabe soliton emerges as the limit of solution of Ricci-Yamabe flow. A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. The Ricci-Yamabe soliton can be defined as the data $(g,V,\lambda,\alpha,\beta)$ satisfying

$$\mathcal{L}_V g + 2\alpha Ric = (2\lambda - \beta\tau)\, g, \tag{1.1}$$

where λ,α,β are real constants. If $V = D\zeta$, where D is the gradient operator and ζ, a smooth function on M, then the soliton is called gradient Ricci-Yamabe soliton (in short, GRYS) and then (1.1) can be written as

$$\nabla^2\zeta + \alpha Ric = \left(\lambda - \frac{1}{2}\beta\tau\right) g, \tag{1.2}$$

where $\nabla^2\zeta$ is the Hessian of ζ.
According as $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, then (M,g) is called Ricci-Yamabe shrinker, Ricci-Yamabe expander or Ricci-Yamabe steady soliton respectively.

A Conformal Ricci-Yamabe soliton (CRYS) on an almost Kenmotsu manifolds with dimension $(2n+1)$ can be defined as the data $(g,V,\lambda,\alpha,\beta)$ satisfying

$$\mathcal{L}_V g + 2\alpha Ric = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g \tag{1.3}$$

where Ric, $\mathcal{L}_V$ are the Ricci tensor and Lie derivative along a vector field V respectively. In [18], Wang proved that a $(\kappa,\mu)' - akm$ admitting Ricci soliton is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, the authors in [17] proved the same result for a gradient $\rho-$Einstein soliton. Also, the result is extended by the authors in ([6], [7]) for Ricci-Yamabe soliton and for conformal Ricci soliton. So, a logic question arise as:
Question. *Would the above result also holds true if a $(2n+1)$ dimensional $(\kappa,\mu)' - akm$ admits a conformal Ricci-Yamabe soliton or a conformal gradient Ricci-Yamabe soliton?*

We will give concrete answer to the question raised above in this paper. The paper is organized as follows: After Preliminaries, in section 3, following the methods of Dey and Majhi [7], we generalize their results on $(\kappa,\mu)'-$almost Kenmotsu manifolds admitting a conformal Ricci-Yamabe soliton. We proved that the manifold is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ provided $2\lambda - \beta\tau \neq 4\alpha n\kappa - \left(p + \frac{2}{2n+1}\right)$. We also proved that a generalized $(\kappa,\mu)'-$almost Kenmotsu manifold admitting a conformal Ricci-Yamabe soliton is η-Einstein. In section 4, we studied $(\kappa,\mu)'-$almost Kenmotsu manifold admitting a conformal gradient Ricci-Yamabe soliton and showed that the potential vector field is a strict infinitesimal contact transformation. Lastly, we construct an example of a $3-$dimensional $(\kappa,\mu)'-$almost Kenmotsu manifold which verifies our results.

2. Preliminaries

A differentiable manifold M^{2n+1} is said to have a (ϕ, ξ, η) structure or an almost contact structure if it admits a $(1,1)$ tensor field ϕ, a Reeb vector field ξ and a $1-$form η satisfying ([2], [3]):

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1 \tag{2.1}$$

where I is the identity endomorphism. If a manifold M^{2n+1} with the above structure admits a Riemannian metric g such that

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$$

for any vector fields X_1, X_2 on the manifold, then the manifold is known as an almost contact metric manifold. Almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied by many geometers (see [8], [9], [15]) and this type of manifolds are called almost Kenmotsu manifolds. A normal almost Kenmotsu manifold is called a Kenmotsu manifold [13]. Let us denote the distribution orthogonal to ξ by $\mathfrak{D}$ defined by [7]: $\mathfrak{D} = Ker(\eta) = Im(\phi).\mathfrak{D}$ is an integrable distribution on an almost Kenmotsu manifold as η is closed.

Many geometers studied and characterize almost Kenmotsu manifold admitting solitons and deduce some notion and conditions on the manifold (For details, see [1], [14], [16]) and also on Kenmotsu manifold [4]. Let M^{2n+1} be an almost Kenmotsu manifold (in short, akm). We denote by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [15]:

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi = \phi h = 0 \tag{2.2}$$

$$\nabla_{X_1}\xi = X_1 - \eta(X_1)\xi - \phi h X_1 \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2) \tag{2.4}$$

$$R(X_1, X_2)\xi = \eta(X_1)(X_2 - \phi h X_2) - \eta(X_2)(X_1 - \phi h X_1) + (\nabla_{X_2}\phi h)X_1 - (\nabla_{X_1}\phi h)X_2 \tag{2.5}$$

for any vector fields X_1, X_2. From (2.3), we see that

$$\nabla_\xi \xi = 0. \tag{2.6}$$

We define the $(1,1)$-type symmetric tensor field h' by $h' = h \circ \phi$ where h' is anti-commuting with ϕ and $h'\xi = 0$. Also,

$$h = 0 \iff h' = 0, \ h'^2 = (\kappa + 1)\phi^2 (\iff h^2 = (\kappa + 1)\phi^2) \tag{2.7}$$

On an almost Kenmotsu manifold, the authors in [8] defined for any $q \in M$ and $\kappa, \mu \in \mathbb{R}$ the $(\kappa, \mu)'-$nullity distribution as

$$\begin{aligned} N_q(\kappa, \mu)' = \{Z \in T_q(M) : R(X_1, X_2)X_3 = \kappa[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \\ + \mu[g(X_2, X_3)h'X_1 - g(X_1, X_3)h'X_2]\} \end{aligned} \tag{2.8}$$

This is called generalized nullity distribution when κ, μ are smooth functions.

Let $X_1 \in \mathfrak{D}$ be the eigenvector of h' analogous to the eigenvalue γ. Then, from (2.7) it is obvious that $\gamma^2 = -(\kappa + 1)$, a constant. Therefore $\kappa \leq -1$ and $\gamma = \pm\sqrt{-\kappa - 1}$. We denote $[\gamma]'$ and $[-\gamma]'$, the corresponding eigenspaces akin to the non-zero eigenvalues γ and $-\gamma$ of h' respectively. It has been proved in [8] that in a $(\kappa, \mu)' - akm \ M^{2n+1}$ with $h' \neq 0, \ \kappa < -1, \ \mu = -2$ and $Spec(h') = \{0, \gamma, -\gamma\}$ with 0 as a simple eigenvalue and $\gamma = \sqrt{-\kappa - 1}$. also,

$$(\nabla_{X_1}h')X_2 = -g(h'X_1 + h'^2X_1, X_2)\xi - \eta(X_2)(h'X_1 + h'^2X_1) \tag{2.9}$$

Further, from [19] we have

$$QX_1 = -2nX_1 + 2n(\kappa + 1)\eta(X_1)\xi - 2nh'X_1 \tag{2.10}$$

From (2.10), the scalar curvature of M^{2n+1} is $2n(\kappa - 2n)$. From (2.8), we have

$$R(X_1, X_2)\xi = \kappa[\eta(X_2)X_1 - \eta(X_1)X_2] + \mu[\eta(X_2)h'X_1 - \eta(X_1)h'X_2] \tag{2.11}$$

where $\kappa, \mu \in \mathbb{R}$. Again, from above equation, we get

$$R(\xi, X_1)X_2 = \kappa[g(X_1, X_2)\xi - \eta(X_2)X_1] + \mu[g(h'X_1, X_2)\xi - \eta(X_2)h'X_1] \tag{2.12}$$

Contracting X_1 in (2.10) yields

$$S(X_2, \xi) = 2n\kappa\eta(X_2) \tag{2.13}$$

Using (2.3) and (2.6), we have

$$(\nabla_{X_1}\eta)X_2 = g(X_1, X_2) - \eta(X_1)\eta(X_2) + g(h'X_1, X_2) \tag{2.14}$$

In [5], the authors define an infinitesimal contact transformation on M as

Definition 2.1 *A potential vector field V is infinitesimal contact transformation on an almost contact metric manifold if $\mathcal{L}_V\eta = \zeta\eta$ for some function ζ. In particular, if $\mathcal{L}_V\eta = 0$, then V is said to be strict infinitesimal contact transformation.*

3. Conformal Ricci-Yamabe Soliton (CRYS) on $(\kappa, \mu)'$−almost Kenmotsu manifolds

In this section, we study $(\kappa, \mu)' - akm$ and generalized $(\kappa, \mu)' - akm$ admitting a conformal Ricci-Yamabe soliton.

Lemma 3.1 *[7] In a $(\kappa, \mu)' - akm$ M^{2n+1} with $h' \neq 0$, we have*

$$(\nabla_{X_3}Ric)(X_1, X_2) - (\nabla_{X_1}Ric)(X_2, X_3) - (\nabla_{X_2}Ric)(X_1, X_3) = -4n(\kappa + 2)g(h'X_1, X_2)\eta(X_3).$$

Lemma 3.2 *[7] In a $(\kappa, \mu)' - akm$ M^{2n+1}, $(\mathcal{L}_{X_1}h')X_2 = 0$ for any $X_1, X_2 \in [\gamma]'$ or $X_1, X_2 \in [-\gamma]'$, where $Spec(h') = \{0, \gamma, -\gamma\}$*

Theorem 3.1 *A conformal Ricci-Yamabe soliton (CRYS) with $\alpha, \beta > 0$ on a $(2n+1)-$ dimensional $(\kappa, \mu)' - akm$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, provided that $2\lambda - \beta\tau \neq 4\alpha n\kappa - (p + \frac{2}{2n+1})$ or the $CRYS$ is (i) expanding, (ii) steady or (iii) shrinking, according to whether the conformal pressure p is*
(1) $p < 2n\beta(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1}$
(2) $p = 2n\beta(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1}$
(3) $p > \frac{2n\beta(2n+1)^2 + 4\alpha n(2n+1) - 2}{2n+1}$

Proof: From the soliton equation (1.3), we have

$$(\mathcal{L}_V g)(X_1, X_2) + 2\alpha Ric(X_1, X_2) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X_1, X_2) \tag{3.1}$$

Taking covariant derivative of (3.1) along any vector field X_3 yields

$$(\nabla_{X_3}\mathcal{L}_V g)(X_1, X_2) = -2\alpha(\nabla_{X_3}Ric)(X_1, X_2) \tag{3.2}$$

Using the result of Yano [20], we have

$$(\mathcal{L}_V\nabla_{X_1}g - \mathcal{L}_{X_1}\nabla_V g - \nabla_{[V,X_1]}g)(X_2, X_3) = -g((\mathcal{L}_V\nabla)(X_1, X_2), X_3) - g((\mathcal{L}_V\nabla)(X_1, X_3), X_2)$$

Since g is parallel with respect to the Levi-civita connection ∇, then the above relation becomes

$$(\nabla_{X_1}\mathcal{L}_V g)(X_2, X_3) = g((\mathcal{L}_V\nabla)(X_1, X_2), X_3) + g((\mathcal{L}_V\nabla)(X_1, X_3), X_2) \tag{3.3}$$

Since $\mathcal{L}_V\nabla$ is symmetric, then from (3.3), we obtain

$$g((\mathcal{L}_V\nabla)(X_1, X_2), X_3) = \frac{1}{2}(\nabla_{X_1}\mathcal{L}_V g)(X_2, X_3) + \frac{1}{2}(\nabla_{X_2}\mathcal{L}_V g)(X_1, X_3) - \frac{1}{2}(\nabla_{X_3}\mathcal{L}_V g)(X_1, X_2) \tag{3.4}$$

Utilizing (3.2) in (3.4), we get

$$g((\mathcal{L}_V\nabla)(X_1,X_2),X_3)=\alpha[(\nabla_{X_3}Ric)(X_1,X_2)-(\nabla_{X_1}Ric)(X_2,X_3)-(\nabla_{X_2}Ric)(X_1,X_3) \tag{3.5}$$

In view of Lemma 3.1, (3.5) can be written as

$$g((\mathcal{L}_V\nabla)(X_1,X_2),X_3)=-4n\alpha(\kappa+2)g(h'X_1,X_2)\eta(X_3)$$

which implies

$$(\mathcal{L}_V\nabla)(X_1,X_2)=-4n\alpha(\kappa+2)g(h'X_1,X_2)\xi. \tag{3.6}$$

Taking $X_2=\xi$ in (3.6), we get

$$(\mathcal{L}_V\nabla)(X_1,\xi)=0$$

From the above expression, we can have $\nabla_{X_2}(\mathcal{L}_V\nabla)(X_1,\xi)=0$, which results in

$$(\nabla_{X_2}\mathcal{L}_V\nabla)(X_1,\xi)+(\mathcal{L}_V\nabla)(\nabla_{X_2}X_1,\xi)+(\mathcal{L}_V\nabla)(X_1,\nabla_{X_2}\xi)=0 \tag{3.7}$$

Using $(\mathcal{L}_V\nabla)(X_1,\xi)=0$, (3.5) and (2.3) in (3.7) we get

$$(\nabla_{X_2}\mathcal{L}_V\nabla)(X_1,\xi)=4n\alpha(\kappa+2)(g(h'X_1,X_2)+g(h'^2X_1,X_2))\xi. \tag{3.8}$$

It is known that [20]

$$(\mathcal{L}_VR)(X_1,X_2)X_3=(\nabla_{X_1}\mathcal{L}_V\nabla)(X_2,X_3)-(\nabla_{X_2}\mathcal{L}_V\nabla)(X_1,X_3),$$

Taking $X_2=X_3=\xi$ in (3.8) and utilizing the result in the foregoing equation, we have

$$(\mathcal{L}_VR)(X_1,\xi)\xi=0 \tag{3.9}$$

Now, setting $X_2=\xi$ in (3.1) and in view of (3.18), we obtain

$$(\mathcal{L}_Vg)(X_1,\xi)=\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]\eta(X_1) \tag{3.10}$$

Taking Lie differentiation of $g(X_1,\xi)=\eta(X_1)$ along V and using (3.10) yields

$$(\mathcal{L}_V\eta)X_1-g(X_1,\mathcal{L}_V\xi)=\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]\eta(X_1) \tag{3.11}$$

Putting $X_1=\xi$ in the foregoing expression, we get

$$\eta(\mathcal{L}_V\xi)=\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right] \tag{3.12}$$

From (2.11) we have

$$R(X_1,\xi)\xi=\kappa(X_1-\eta(X_1)\xi)-2h'X_1 \tag{3.13}$$

Utilizing (3.11)-(3.13) and (2.11)-(2.12), we get

$$\begin{aligned}(\mathcal{L}_VR)(X_1,\xi)\xi=&\ \kappa\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right](X_1-\eta(X_1)\xi)-2(\mathcal{L}_Vh')X_1\\&-2\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]h'X_1-2n\eta(X_1)h'(\mathcal{L}_V\xi)\\&-2g(h'X_1,\mathcal{L}_V\xi)\xi.\end{aligned} \tag{3.14}$$

Equating (3.9) and (3.14) and then taking an inner product with X_2, we obtain

$$\begin{aligned}\kappa\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right][g(X_1,X_2)-\eta(X_1)\eta(X_2)]-2g((\mathcal{L}_Vh')X_1,X_2)\\ -2\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(h'X_1,X_2)-2n\eta(X_1)g(h'(\mathcal{L}_V\xi),X_2)\\ -2g(h'X_1,\mathcal{L}_V\xi)\eta(X_2)=0\end{aligned} \tag{3.15}$$

Replacing X_1 by ϕX_1 in the foregoing expression, we obtain

$$\begin{aligned}\kappa\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X_1,X_2)-2g((\mathcal{L}_Vh')\phi X_1,X_2)\\ -2\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(h'\phi X_1,X_2)=0.\end{aligned} \tag{3.16}$$

Letting $X_1\in[-\gamma]'$ and $V\in[\gamma]'$, then $\phi X_1\in[\gamma]'$. Thus (3.16) yields

$$\begin{aligned}(\kappa-2\gamma)\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X_1,X_2)\\ -2g((\mathcal{L}_Vh')\phi X_1,X_2)=0.\end{aligned} \tag{3.17}$$

Since $V,\phi X_1\in[\gamma]'$, using Lemma 3.2 we have $(\mathcal{L}_Vh')\phi X_1=0$. Therefore (3.17) becomes

$$(\kappa-2\gamma)\left[2\lambda-\beta\tau-4n\alpha\kappa-\left(p+\frac{2}{2n+1}\right)\right]g(\phi X_1,X_2)=0$$

which implies either $\kappa=2\gamma$ or $2\lambda=\beta\tau+4\alpha n\kappa+\left(p+\frac{2}{2n+1}\right)$

Case I: If $\kappa=2\gamma$, then from $\gamma^2=-(\kappa+1)$, we get $\gamma=-1$ and thus $\kappa=-2$. Therefore, from Proposition 4.2 of [8], we have

$$R(X_{1_\gamma},X_{2_\gamma})X_{3_\gamma}=0$$

and

$$R(X_{1_\gamma},X_{2_\gamma})X_{3_\gamma}=-4[g(X_{2_{-\gamma}},X_{3_{-\gamma}})X_{1_{-\gamma}}-g(X_{1_{-\gamma}},X_{3_{-\gamma}})X_{2_{-\gamma}}]$$

for any $X_{1_\gamma},X_{2_\gamma},X_{3_\gamma}\in[\gamma]'$ and $X_{1_{-\gamma}},X_{2_{-\gamma}},X_{3_{-\gamma}}\in[-\gamma]'$. Also, $\mu=-2$, it follows from Proposition 4.3 of [8] that $\kappa(X_1,\xi)=-4$ for any $X_1\in[-\gamma]'$ and $\kappa(X_1,\xi)=0$ for any $X_1\in[\gamma]'$. Again, we see that $\kappa(X_1,X_2)=-4$ for any $X_1,X_2\in[-\gamma]'$ and $\kappa(X_1,X_2)=0$ for any $X_1,X_2\in[\gamma]'$. As is shown in [8], the distribution $[\xi]\oplus[\gamma]'$ is integrable with totally geodesic leaves and the distribution $[-\gamma]'$ is integrable with totally umbilical leaves by $\mathbb{H}=-(1-\gamma)\xi$, where $\mathbb{H}$ is the mean curvature tensor field for the leaves of $[-\gamma]'$ immersed in M^{2n+1}. Here, $\gamma=-1$, then both the orthogonal distributions $[\xi]\oplus[\gamma]'$ and $[-\gamma]'$ are integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

Case II: Let $2\lambda=\beta\tau+4n\alpha\kappa+(p+\frac{2}{2n+1})$. Since $\tau=2n(\kappa-2n)$ in an *akm* of dimension $2n+1$, we get

$$2\lambda=2\beta n(\kappa-2n)+4\alpha n\kappa+\left(p+\frac{2}{2n+1}\right)$$

Now the CRYS is expanding, steady or shrinking depending on whether $\lambda<0$, $\lambda=0$ or $\lambda>0$ respectively. Therefore the CRYS is expanding when

$$p<-4\alpha n\kappa-2\beta n(\kappa-2n)-\frac{2}{2n+1},$$

steady when

$$p=2\beta n(2n-\kappa)-4\alpha n\kappa-\frac{2}{2n+1},$$

and shrinking when

$$p > 2\beta n(2n - \kappa) - 4\alpha n\kappa - \frac{2}{2n+1},$$

the last expression is obtained by taking $\kappa = -1$ which completes the proof. □

Theorem 3.2 *If $(g, \xi, \lambda, \alpha, \beta)$ is a CRYS in a generalized $(\kappa, \mu)' - akm$ M^{2n+1}, then the manifold is $\eta-$Einstein provided $2n\alpha - 1 \neq 0$ and the expression for λ is given by*

$$\lambda = \frac{p}{2} + \frac{1}{2n+1} + (2\alpha + \beta)n\kappa - 2n^2\beta \tag{3.18}$$

Proof: From the soliton equation (1.3) we have

$$(\mathcal{L}_\xi g)(X_1, X_2) + 2\alpha Ric(X_1, X_2) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X_1, X_2) \tag{3.19}$$

Using (2.3), we get

$$(\mathcal{L}_\xi g)(X_1, X_2) = 2g(X_1, X_2) - 2\eta(X_1)\eta(X_2) - 2g(\phi h X_1, X_2) \tag{3.20}$$

Utilizing (3.20) in (3.19), we obtain

$$\begin{aligned} &2g(X_1, X_2) - 2\eta(X_1)\eta(X_2) - 2g(\phi h X_1, X_2) \\ &+ 2\alpha Ric(X_1, X_2) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g(X_1, X_2) \end{aligned} \tag{3.21}$$

From (2.10), we get

$$g(\phi h X_1, X_2) = \frac{1}{2n} Ric(X_1, X_2) + g(X_1, X_2) - (\kappa + 1)\eta(X_1)\eta(X_2) \tag{3.22}$$

Now substituting (3.22) in (3.21), we get

$$Ric(X_1, X_2) = \frac{n[2\lambda - 2n\beta(\kappa - 2n) - (p + \frac{2}{2n+1})]}{2n\alpha - 1} g(X_1, X_2) - \frac{2n\kappa}{2n\alpha - 1}\eta(X_1)\eta(X_2) \tag{3.23}$$

which implies that the manifold is $\eta-$Einstein. Now taking $X_1 = X_2 = \xi$ in (3.23) yields (3.18). This completes the proof. □

4. Conformal Gradient Ricci-Yamabe Soliton on $(\kappa, \mu)'-$almost Kenmotsu manifolds

From (1.3), we get the conformal gradient Ricci-Yamabe soliton equation by considering the vector field V to be a gradient of some smooth function ζ on the manifold as

$$\nabla\nabla\zeta + \alpha Ric = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] g \tag{4.1}$$

Lemma 4.1 *If $(g, D\zeta, \lambda, \alpha, \beta)$ is a conformal gradient Ricci-Yamabe soliton (CGRYS) with $\alpha \neq 0$ on a $(\kappa, \mu)' - akm$ M^{2n+1}, then following relation:*

$$R(X_1, X_2)D\zeta = \alpha[2n(\kappa + 2)(\eta(X_1)h'X_2 - \eta(X_2)h'X_1)]$$

holds. Here, ζ is a smooth function such that $V = D\zeta$, where D is the gradient operator.

Proof: From (4.1), we have

$$\nabla_{X_1} D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] X_1 - \alpha Q X_1 \tag{4.2}$$

Covariant derivative of the above relation along X_2 yields

$$\nabla_{X_2}\nabla_{X_1} D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] \nabla_{X_2} X_1 - \alpha \nabla_{X_2} Q X_1 \tag{4.3}$$

Interchanging X_1 and X_2 in the above equation yields

$$\nabla_{X_1}\nabla_{X_2} D\zeta = [\lambda - \frac{\beta\tau}{2} - (\frac{p}{2} + \frac{1}{2n+1})]\nabla_{X_1} X_2 - \alpha \nabla_{X_1} Q X_2 \tag{4.4}$$

Again, from (4.2), we get

$$\nabla_{[X_1,X_2]} D\zeta = \left[\lambda - \frac{\beta\tau}{2} - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] (\nabla_{X_1} X_2 - \nabla_{X_2} X_1) - \alpha Q(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) \tag{4.5}$$

Utilizing (4.3)-(4.5) in the equation

$$R(X_1, X_2)D\zeta = \nabla_{X_1}\nabla_{X_2} D\zeta - \nabla_{X_2}\nabla_{X_1} D\zeta - \nabla_{[X_1,X_2]} D\zeta$$

results in

$$R(X_1, X_2)D\zeta = \alpha[(\nabla_{X_2} Q)X_1 - (\nabla_{X_1} Q)X_2] \tag{4.6}$$

Now, utilizing (2.3), (2.9), (2.10) and (2.14) we obtain

$$\begin{aligned}(\nabla_{X_2} Q)X_1 &= \nabla_{X_2} X_1 - Q(\nabla_{X_2} X_1)\\ &= 2n(\kappa+1)[g(X_1, X_2) - \eta(X_1)\eta(X_2) + g(h'X_1, X_2)]\xi\\ &+ 2n(\kappa+1)\eta(X_1)(X_2 - \eta(X_2)\xi - \phi h X_2) + 2ng(h'X_2 + h'^2 X_2, X_1)\xi\\ &+ 2n\eta(X_1)(h'X_2 + h'^2 X_2)\end{aligned} \tag{4.7}$$

Interchanging X_1 and X_2 in (4.7) yields the expression for $(\nabla_{X_1} Q)X_2$. Then, on simplification and using (2.7), (4.6) becomes

$$R(X_1, X_2)D\zeta = \alpha[2n(\kappa+2)(\eta(X_1)h'X_2 - \eta(X_2)h'X_1)]$$

which completes the proof. □

Theorem 4.1 *A $(\kappa,\mu)' - akm$ M^{2n+1} with $h' \neq 0$ admitting a conformal gradient Ricci-Yamabe soliton with $\alpha \neq 0$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ provided V is not pointwise collinear with the Reeb vector field, otherwise the manifold does not admit a conformal gradient Ricci-Yamabe soliton.*

Proof: Setting $X_1 = \xi$ in Lemma 4.1 and then taking inner product with X_1 yields

$$g(R(\xi, X_2)D\zeta, X_1) = \alpha 2n(\kappa+2)g(h'X_2, X_1) \tag{4.8}$$

Again from (2.12), we have

$$\begin{aligned}g(R(\xi, X_2)D\zeta, X_1) &= -g(R(\xi, X_2)X_1, D\zeta)\\ &= -\kappa g(X_1, X_2)(\xi\zeta) + \kappa\eta(X_1)(X_2\zeta) + 2g(h'X_1, X_2)(\xi\zeta) - 2\eta(X_1)((h'X_2)\zeta)\end{aligned} \tag{4.9}$$

Equating (4.8) and (4.9) we get

$$\begin{aligned}-\kappa g(X_1, X_2)(\xi\zeta) + \kappa\eta(X_1)(X_2\zeta) + 2g(h'X_1, X_2)(\xi\zeta)\\ - 2\eta(X_1)((h'X_2)\zeta) = 2n(\kappa+2)\alpha g(h'X_2, X_1)\end{aligned}$$

Antisymmetrizing the above relation results in

$$\kappa\eta(X_1)(X_2\zeta)-\kappa\eta(X_2)(X_1\zeta)-2\eta(X_1)((h'X_2)\zeta)+2\eta(X_2)((h'X_1)\zeta)=0 \tag{4.10}$$

Putting $X_1=\xi$ in (4.10) yields

$$\kappa(X_2\zeta)-\kappa(\xi\zeta)\eta(X_2)-2((h'X_2)\zeta)=0$$

which implies

$$\kappa[(D\zeta)-(\xi\zeta)\xi]-2h'(D\zeta)=0 \tag{4.11}$$

Operating on h' in the above equation and using (2.7), we get

$$h'(D\zeta)=-\frac{2(\kappa+1)}{\kappa}[d\zeta-(\xi\zeta)\xi] \tag{4.12}$$

Utilizing (4.12) in (4.11) yields

$$(\kappa+2)^2[D\zeta-(\xi\zeta)\xi]=0$$

which implies either $\kappa=-2$ or $D\zeta=(\xi\zeta)\xi$. Let us consider these two cases in the following:
Case I: For $\kappa=-2$, then by the same argument made in Case I of Theorem 3.1, the manifold M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.
Case II: For $V=D\zeta=(\xi\zeta)\xi$, then V is pointwise collinear with the Reeb vector field ξ. Then, Differentiating $D\zeta=(\xi\zeta)\xi$ covariantly along X_1 and using (2.3), we obtain

$$\nabla_{X_1}D\zeta=(X_1(\xi\zeta))\xi+(\xi\zeta)(X_1-\eta(X_1)\xi-\phi hX_1) \tag{4.13}$$

Equating (4.2) and (4.13), we obtain

$$\alpha QX_1=\left(\left[\lambda-\frac{\beta\tau}{2}-\left(\frac{p}{2}+\frac{1}{2n+1}\right)\right]-(\xi\zeta)\right)X_1+((\xi\zeta)\eta(X_1)-X_1(\xi\zeta))\xi+(\xi\zeta)\phi hX_1 \tag{4.14}$$

Comparing (2.10) and the above equation, we get

$$\left[\lambda-\frac{\beta\tau}{2}-\left(\frac{p}{2}+\frac{1}{2n+1}\right)\right]-\xi\zeta=-2n\alpha \tag{4.15}$$

$$(\xi\zeta)\eta(X_1)-X_1(\xi\zeta)=2n(\kappa+1)\eta(X_1) \tag{4.16}$$

$$(\xi\zeta)\phi h=-2nh' \tag{4.17}$$

Utilizing (4.17) in (4.15) and (4.16) we get

$$\lambda=\frac{\beta\tau}{2}+\left(\frac{p}{2}+\frac{1}{2n+1}\right)+4n(1-\alpha) \tag{4.18}$$

and

$$2n\eta(X_1)=2n(\kappa+1)\eta(X_1) \tag{4.19}$$

for any vector field X_1 which implies $\kappa=0$ which is a contradiction as $\kappa\leq-1$. This completes the proof. □

Moreover, if $V=D\zeta=(\xi\zeta)\xi=d\xi$, where $d=\xi\zeta$ is a smooth function on the manifold M^{2n+1}. By (2.3), we have

$$(\mathcal{L}_{d\xi}g)(X_1,X_2)=(X_1d)\eta(X_2)+(X_2d)\eta(X_1)+2d[g(X_1,X_2)-\eta(X_1)\eta(X_2)-g(\phi hX_1,X_2)] \tag{4.20}$$

Utilizing (4.20) in (3.1), we obtain

$$\begin{aligned}&(X_1d)\eta(X_2)+(X_2d)\eta(X_1)+2d[g(X_1,X_2)-\eta(X_1)\eta(X_2)\\&-g(\phi hX_1,X_2)]=\left[2\lambda-\beta\tau-\left(p+\frac{2}{2n+1}\right)\right]g(X_1,X_2)-2\alpha Ric(X_1,X_2)\end{aligned} \tag{4.21}$$

Setting $X_1 = X_2 = \xi$ in the foregoing relation and utilizing (3.18) yields

$$2(\xi d) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] - 4n\alpha\kappa \tag{4.22}$$

Again, considering the orthonormal basis of the tangent space $\{E_j\}$ at each point of the manifold and setting $X_1 = X_2 = E_j$ in (4.21) and then summing over j results as

$$2(\xi d) = \left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right](2n+1) - 2\alpha\tau - 4nd \tag{4.23}$$

Since $\alpha, \beta, \lambda, \tau, p$ are all constants, so, from (4.22) and (4.23) d is also a constant. Therefore, (4.22) results in

$$\left[2\lambda - \beta\tau - \left(p + \frac{2}{2n+1}\right)\right] = 4n\alpha\kappa \tag{4.24}$$

Further, since d is constant, we get $\mathcal{L}_V\xi = 0$. Utilizing (4.24) in (3.10) yields $(\mathcal{L}_V\eta)X_1 = 0$ for any vector field X_1. Now, substituting $\mathcal{L}_V\xi = 0$ and (4.24) in (3.15) results in $(\mathcal{L}_V h')X_1 = 0$ for any vector field X_1, which implies that V leaves h' invariant.

Thus, we can state the following:

Corollary 4.1 *On a $(\kappa,\mu)' - akm$ M^{2n+1} with $\kappa \neq -2$ admitting a conformal gradient RYS, then V is a constant multiple of ξ which further implies the potential vector field V is a strict infinitesimal contact transformation and leaves h' invariant.*

5. Example of a 3-dimensional $(\kappa,\mu)' - akm$ satisfying CRYS

Consider a 3-dimensional manifold $M^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ where (x_1, x_2, x_3) are the standard coordinates in $\mathbb{R}^3$. Let us take E_1, E_2 and E_3 to be the three vector fields in $\mathbb{R}^3$ which satisfies

$$[E_1, E_2] = E_2,\ [E_1, E_3] = E_3 \text{ and } [E_i, E_j] = 0\ \forall\ i, j = 2, 3$$

and let g be the Riemannian metric such that

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j,\ \forall\ i, j = 1, 2, 3 \end{cases}$$

Here, we take E_1 as the Reeb vector field. Suppose ϕ is the $(1,1)-$tensor field and η be the $1-$form defined such that

$$\phi(E_1) = 0,\ \phi(E_2) = E_3,\ \phi(E_3) = 2E_2$$
$$\eta(X_4) = g(X_4, E_1) \text{ for any } X_4 \in T(M^3)$$

Moreover, let $h'E_1 = 0,\ h'E_2 = E_2,\ h'E_3 = E_3$. By the linearity of ϕ and g, we have

$$\eta(E_1) = 1,\ \phi^2(X_4) = -X_4 + \eta(X_4)E_1,\ g(\phi X_4, \phi X_5) = g(X_4, X_5) - \eta(X_4)\eta(X_5)$$

The following relations are obtained directly by using Koszul's formula:

$$\nabla_{E_1}E_1 = 0,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = 0,$$
$$\nabla_{E_2}E_1 = -E_2,\ \nabla_{E_2}E_2 = E_1 - \frac{1}{2},\ \nabla_{E_2}E_3 = 0,$$
$$\nabla_{E_3}E_1 = -E_3,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = E_1 - \frac{1}{2}.$$

From the above relations, we get that

$$\nabla_{X_1}E_1 = -\phi^2X_1 + h'X_1$$

for any $X_1 \in T(M^3)$. Thus, the structure (ϕ, E_1, η, g) is an almost contact metric structure so that M^3 is an almost Kenmotsu manifold (akm) of dimension 3. Utilizing the above results, we can calculate the components of the curvature tensor R as follows:

$$\begin{aligned}
R(E_1, E_2)E_1 &= R(E_3, E_2)E_3 = 4E_2 \\
R(E_1, E_2)E_2 &= R(E_1, E_3)E_3 = -4E_1 \\
R(E_1, E_3)E_1 &= R(E_2, E_3)E_2 = 4E_3 \\
R(E_2, E_1)E_1 &= R(E_2, E_3)E_3 = -4E_2 \\
R(E_3, E_1)E_1 &= R(E_3, E_2)E_2 = -4E_3 \\
R(E_2, E_2)E_2 &= R(E_3, E_1)E_3 = 4E_1
\end{aligned}$$

In view of the above results obtained for the curvature tensor R, we observe that the Reeb vector field E_1 belongs to the $(\kappa, \mu)'-$nullity distribution where $\kappa = -2$ and $\mu = -2$. Thus, from the formula $\gamma^2 = -(\kappa + 1)$, we get $\gamma = \pm 1$. Considering $\gamma = -1$, thus, by the same argument made in Case I of Theorem 3.1, we conclude that M^3 is locally isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$.
Using the curvature tensor formula, we have

$$R(X_1, X_2)X_3 = -4[g(X_2, X_3)X - g(X_1, X_3)X_2]$$

From the foregoing equation, we obtain

$$Ric(X_2, X_3) = -8g(X_2, X_3)$$

which implies $\tau = -24$. Now, we can easily see that

$$(\mathcal{L}_{E_1} g)(E_1, E_1) = 0, \ (\mathcal{L}_{E_1} g)(E_2, E_2) = (\mathcal{L}_{E_1} g)(E_3, E_3) = -2.$$

Consider $V = E_1$ and then tracing the soliton equation (1.3), we get

$$\lambda = \frac{p}{2} + \frac{1}{3} - 4(2\alpha + 3\beta) - \frac{2}{3}$$

Hence, $(g, E_1, \lambda, \alpha, \beta)$ is a CRYS on M^3. Therefore, Theorem 3.1 is verified.

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