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On fractional calculus of the bivariate Mittag-Leffler function

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ABSTRACT: This article deals with a study of some fractional calculus properties of the bivariate Mittag-Leffler functions including the fractional integrals and derivatives and the singular integral equation involving the bivariate Mittag-Leffler functions in the kernel. Further, we introduce a fractional integral operator involving bivariate Mittag-Leffler functions in the kernel. Also, we discuss the links of our findings with known cases.

Key Words: bivariate Mittag-Leffler function, Riemann-Liouville fractional integrals and derivatives, generalized fractional calculus operators.

Contents

1. Introduction

The Mittag-Leffler function is an important function that is widely used in the subject of fractional calculus. Indeed solutions of various differential and integral equations involving fractional derivatives can be derived from Mittag-Leffler functions. Also, the Mittag-Leffler function can be seen in the solution of the same boundary value problems. For details, we refer to works by Gorenflo and Mainard [4], Kilbas and Saigo [6], Kilbas et al. [7], and Srivastava and Tomovski [19]
The Mittag-Leffler function is defined by [8]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, (\Re(\alpha) > 0, z \in \mathbb{C}), \tag{1.1}$$

where as usual notation $\Gamma(\lambda)$ for the Gamma function and $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, n \geq 0, \lambda \neq 0, -1, -2, ...,$ the Pochhammer symbol [16]. Here and in the following, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, and positive integers, respectively.
A generalization of (1.1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{1.2}$$

$$(\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0),$$

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is defined and studied by Wiman [21]. A generalization of (1.2) is introduced in terms of series representation by Prabhakar [10]:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.3}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0).$$

For $\gamma=1$, (1.3) coincides with (1.2), whereas for $\gamma=\beta=1$, (1.3) coincides with (1.1). Prabhakar [10] studied some properties of the integral operator

$$\mathrm{E}^{(\gamma)}_{\rho,\mu,w:a^+\varphi}(x)=\int_a^x (x-t)^{\mu-1}\ E_{\rho.\mu}[w(x-t)^{\rho}]\varphi(t)dt,(x>a), \tag{1.4}$$

$$(\rho,\mu,\gamma,w,\in\mathbb{C},\Re(\rho)>0,\Re(\mu)>0),$$

containing the function (1.3) in the kernel and applied the results obtained to prove the existence and uniqueness of the solution for the corresponding integral equation of the first kind

$$\int_a^x (x-t)^{\mu-1}\ E_{\rho,\mu}[w(x-t)^{\rho}]\varphi(t)dt=f(x),(x>a), \tag{1.5}$$

of finite interval $[a,b]$ of the axis $\mathbb{R}=(-\infty,\infty)$.
In [2], Bin-Saad et al. established some properties for the bivariate Mittag-Leffler function

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(m+n)!z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}, \tag{1.6}$$

$$(\alpha,\beta,\gamma,z_1,z_2\in\mathbb{C},\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0).$$

For $\beta=1$ and $z_1=0$ $(z_2=0)$, (1.6) coincides with (1.1), whereas for $z_1=0$ $(z_2=0)$, (1.6) coincides with (1.3):

$$E_{\alpha,1,\gamma}(z,0)=E_{\alpha}(z),\ E_{\alpha,\beta,\gamma}(z,0)=E_{\alpha,\beta}(z),\ E_{\alpha,\beta,\gamma}(0,z)=E_{\gamma,\beta}(z). \tag{1.7}$$

In our present study, we also propose to use the Riemann-Louville fractional derivative operator (see e.g. [5,9,13]:

$$\left(D^{\alpha}_{a^+}y\right)(x)=\left(\frac{d}{dx}\right)^n\left(I^{n-\alpha}_{a^+}y\right)(x), \tag{1.8}$$

$$(\alpha\in\mathbb{C},\Re(\alpha)>0,n=[\Re(\alpha)]+1),$$

defined for $a<x\leq b$, by ([13], Section 2.3 and 2.4), in terms of Riemann-Louville fractional calculus

$$\left(I^{\alpha}_{a^+}\ \varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \tag{1.9),}$$

$$(\alpha\in\mathbb{C},\Re(\alpha)>0,a<x\leq b).$$

Here the the operator $I^{\alpha}_{a^+}$ is defined on the space $\mathbb{L}(a,b)$ (see for detail [5,7,11]:

$$\mathbb{L}(a,b)=\left\{f(x):\parallel f\parallel_l=\int_a^b|f(t)|dt<\infty\right\}. \tag{1.10}$$

For the case $\varphi(x)=x^{\alpha},\alpha>-1$, we get the result

$$D^{\nu}_x x^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)}, \tag{1.11}$$

where $D^{\nu}_x=\left(\frac{d}{dx}\right)^{\nu},x>0,\alpha>-1,\nu\geq 0$ is not restricted to integer values. Also, we recall the relation

$$D^{\nu}_{a^+}\ I^{\nu}_{a^+}=\varphi, \tag{1.12}$$

$$(\alpha \in \mathbb{C}, \Re(\alpha) > 0, \varphi \in \mathbb{L}[a,b]),$$

and the Dirichlet formula [13]:

$$\int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx. \tag{1.13}$$

The present paper sequel to these earlier papers is motivated largely by the aforementioned work of Prabhakar [10] and Bin-Saad et al. [2] and is intended for the investigation of new bivariate Mittag- Leffler function with four parameters $E^{(\delta)}_{\alpha,\beta,\gamma}$. The paper is organized as follows. Section 2 deals with special cases of the special function $E^{(\delta)}_{\alpha,\beta,\gamma}$, its expansions, operational representations, and generating functions. Computation of the Riemann- Liouville fractional integral and derivative of $E^{(\delta)}_{\alpha,\beta,\gamma}$ are presented in Section 3. Section 4 is devoted to introducing and solving singular integral equation with $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel. In Section 5, we introduce an integral operator $\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ which contains $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel and investigate its transformation properties in the space of Lebesque summable continuous functions.

2. Bivariate Mittag-Leffler Function

We presented generalized bivariate Mittag-Leffler Function in [15], which has the following form:

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\delta)_{m+n}\; z_1^m z_2^n}{\Gamma(\alpha m + \gamma n + \beta)m!n!}, \tag{2.1}$$

$$(\alpha, \beta, \gamma, \delta, z_1, z_2 \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0).$$

The alternative representations

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2) = \sum_{m=0}^{\infty} (\delta)_m E^{(\delta+m)}_{\gamma,\beta+\alpha m}(z_2)\frac{z_1^m}{m!}, \tag{2.2}$$

and

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2) = \sum_{n=0}^{\infty} (\delta)_n E^{(\delta+n)}_{\alpha,\beta+\gamma n}(z_1)\frac{z_2^n}{n!}, \tag{2.3}$$

followed by series rearrangement and considering definition (1.3). According to the representations (2.2) and (2.3) and the definition of the generating function concept (see for example ([12], Chapter 8), we can say that the bivariate Mittag-Leffler $E^{(\delta)}_{\alpha,\beta,\gamma}$ has generated the set $E^{(\delta)}_{\alpha,\beta}$ and that $E^{(\delta)}_{\alpha,\beta,\gamma}$ is generating function for the Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta}$.
It is important to note that the bivariate series (2.1) is a special case of the generalized Wright hypergeometric function in two variables $S^{A:B,E}_{C:D;F}[x,y]$ introduced by Srivastava and Daoust ([17], p199(2.1)). Indeed, in this case, we have

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2) = \frac{1}{\Gamma(\delta)} S^{1:0,0}_{1:0;0}\left[\begin{array}{ll} [\delta:1,1]\,;--;--; & \\ & z_1, z_2 \\ [\beta:\alpha;\gamma]\,;--;--; & \end{array}\right].$$

According to the convergence conditions investigated by Srivastava and Daoust ([18], p199(2.1)) for the series $S^{A:B,E}_{C:D;F}[x,y]$, the series (2.1) converges for $\Re(\alpha) > 0$ and $\Re(\gamma) > 0$, $1+C+D-A-B > 0$ and $1+C+F-A-E > 0$.
We have the following relationships.

$$E^{(1)}_{\alpha,\beta,\gamma}(z_1,z_2) = E_{\alpha,\beta,\gamma}(z_1,z_2), \tag{2.4}$$

$$E^{(\delta)}_{\alpha,\beta,\gamma}(0,z) = E^{(\delta)}_{\gamma,\beta}(z), \tag{2.5}$$

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z,0)=E^{(\delta)}_{\alpha,\beta}(z). \tag{2.6}$$

Corollary 2.1. Let $\beta,\delta,z_1,z_2\in\mathbb{C},\Re(\beta)>0$ and $n\in\mathbb{N}$. Then

$$E^{(\delta)}_{n,\beta,n}(z_1,z_2)=\frac{1}{\Gamma(\beta)}\,{}_1F_n\left[\delta;\frac{\beta}{n},\frac{\beta+1}{n},\cdots,\frac{\beta+n-1}{n};\frac{z_1+z_2}{n^n}\right], \tag{2.7}$$

where ${}_pF_q$ is the generalized hypergeometric series ([12], Chapter 5).
proof Proof. We have

$$E^{(\delta)}_{n,\beta,n}(z_1,z_2)=\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\delta)_{s+k}\ z_1^s\ \ z_2^k}{\Gamma(ns+nk+\beta)s!k!}.$$

Now, by letting $s\mapsto s-k$ and using the result (see e.g. [16]):

$$(\lambda)_{mn}=m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n,$$

we get the right-hand side of the assertion (2.7).

Corollary 2.2. Let $\alpha,\beta,\delta,z_1,z_2\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0$. Then

$$E^{(\delta)}_{\alpha,\beta,\alpha}(z_1z_2,z_2)=E^{(\delta)}_{\alpha,\beta}(z_1z_2+z_2). \tag{2.8}$$

Proof. We have

$$E^{(\delta)}_{\alpha,\beta,\alpha}(z_1z_2,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\ z_1^m,z_2^{n+m}}{\Gamma(\alpha m+\alpha n+\beta)m!n!}.$$

Then letting $n\mapsto n-m$ and considering the Mittag-Leffler function $E^{(\gamma)}_{\alpha,\beta}$ defined by (1.3), we get (2.8).

Remark 2.1 Given the relation (2.3) and the result (2.8), we obtain the following interesting relation between two versions of the Mittag-Leffler function (1.3)

$$E^{(\delta)}_{\alpha,\beta}(z_1z_2+z_2)=\sum_{n=0}^{\infty}(\delta)_nE^{(\delta+n)}_{\alpha,\beta+\gamma n}(z_1z_2)\frac{z_2^n}{n!}. \tag{2.9}$$

Corollary 2.3. Let $\alpha,\beta,\lambda,\sigma,z_1,z_2\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\lambda),\Re(\sigma)\}>0$. Then

$$E^{(\lambda+\sigma)}_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{n=0}^{\infty}(\lambda)_nE^{(\lambda+\sigma+n)}_{\alpha,\beta+\gamma n}(z_1)\times{}_2F_1\left[\begin{array}{cc}-n,\sigma; & \\ & 1\\ 1-\lambda-n; & \end{array}\right]\frac{z_2^n}{n!}, \tag{2.10}$$

where ${}_2F_1$ is the Gaussian hypergeometric function (see [16]).
Proof. On putting $\delta=\lambda+\sigma$ in (2.3) and using the classical formula of Nörlund for Pochhammer symbol (see [1], Section 1, Chapter 3)

$$(a+b)_k=\sum_{m=0}^{k}\binom{k}{m}(a)_{k-m}(b)_k, \tag{2.11}$$

we obtain (2.10).
According to the inverse operator D^t_{-1} we can rewrite $E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2)$ in the series form

$$E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\ z_1^mz_2^n}{m!n!}\,(tD_t)^{-(\alpha m+\gamma n)}\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}\right\}, \tag{2.12}$$

which further yields the Rodrigues-type relation

$$t^{\beta-1}E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2)=\Big(1-z_1\,(tD_t)^{-\alpha}-z_2\,(tD_t)^{-\gamma}\Big)^{-\delta}\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}\right\}. \tag{2.13}$$

Again, by exploiting the same procedure leading to (2.13), we can establish the following operational and exponential representation

$$\begin{aligned}&u^{\beta-1}v^{\delta-1}E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2)\\&=\exp\Big[z_1\,(uD_u)^{-\alpha}\,(D_v v)+z_2\,(uD_u)^{-\gamma}\,(D_v v)\Big]\left\{\frac{u^{\beta-1}v^{\delta-1}}{\Gamma(\beta)}\right\}.\end{aligned} \tag{2.14}$$

Finally, by exploiting the results (2.13) and (2.14), we can derive generating functions for the Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}(z_1,z_2)$. Indeed, by starting from identity (2.13), multiplying both the sides of (2.13) by $\frac{u^n}{n!}$, replacing δ by n, $(n=1,2,\cdots)$, and then taking the sum, we obtain the result

$$e^{u\left[1-z_1(tD_t)^{-\alpha}-z_2(tD_t)^{-\gamma}\right]}\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}\right\}=t^{\beta-1}\sum_{n=0}^{\infty}E^{(n)}_{\alpha,\beta,\gamma}(z_1,z_2)\frac{u^n}{n!}. \tag{2.15}$$

Similarly, if we employ the identity (2.14), we get the following generating function

$$\begin{aligned}&\exp\Big[z_1\,(uD_u)^{-\alpha}\,(D_v v)+z_2\,(uD_u)^{-\gamma}\,(D_v v)+tv\Big]\left\{\frac{u^{\beta-1}}{v\Gamma(\beta)}\right\}\\&=u^{\beta-1}v^{-1}\sum_{n=0}^{\infty}E^{(n)}_{\alpha,\beta,\gamma}(z_1,z_2)\frac{(vt)^n}{n!}.\end{aligned} \tag{2.16}$$

3. Fractional calculus approach

In this section, we establish the Riemann-Liouville fractional integrals and derivatives of the bivariate Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$.

Theorem 3.1. Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(w_1),\Re(w_2)\}>0$. Then the following fractional integral formula holds.

$$\begin{aligned}&{}_xI^{\lambda}_{a^+}\Big[(x-a)^{\beta-1}\,E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma})\Big]\\&=(x-a)^{\beta+\lambda-1}E^{(\delta)}_{\alpha,\beta+\lambda,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma}).\end{aligned} \tag{3.1}$$

Proof. Upon interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (2.1) and (1.9), we find

$$\begin{aligned}&{}_xI^{\lambda}_{a^+}\Big[(x-a)^{\beta-1}\,E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma})\Big]\\&=\int_a^x\frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\,w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}(t-a)^{\alpha m+\gamma n+\beta-1}dt,\ (x>a),\\&=\frac{1}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\,w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}\int_a^x(x-t)^{\lambda-1}(t-a)^{\alpha m+\gamma n+\beta-1}dt,\\&=(x-a)^{\beta+\lambda-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\,w_1^m w_2^n(x-a)^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+\lambda)m!n!},\end{aligned}$$

which gives us the desired result (3.1).

Theorem 3.2. Let $\alpha,\beta,\gamma,\delta\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta-\lambda),\Re(w_1),\Re(w_2)\}>0$. Then the following fractional integral formula holds.

$$\begin{aligned}&{}_xI^{\lambda}_{a+}\left[(x-a)^{\delta-\lambda-1}\;E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a),w_2(x-a))\right]\\&=\frac{\Gamma(\delta-\lambda)}{\Gamma(\delta)}(x-a)^{\delta-1}E^{(\delta-\lambda)}_{\alpha,\beta,\gamma}(w_1(x-a),w_2(x-a)).\end{aligned}\tag{3.2}$$

Proof. We refer to the proof of Theorem (3.1).

Corollary 3.1. As a consequence of (2.6) and Theorem 3.2, we have

$$\begin{aligned}&{}_xI^{\lambda}_{a+}\left[(x-a)^{\delta-\lambda-1}\;E^{(\delta)}_{\alpha,\beta}(w_1(x-a))\right]\\&=\frac{\Gamma(\delta-\lambda)}{\Gamma(\delta)}(x-a)^{\delta-1}E^{(\delta-\lambda)}_{\alpha,\beta}(w_1(x-a)).\end{aligned}\tag{3.3}$$

where $\alpha,\beta,\delta\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\delta-\lambda),\Re(w_1)\}>0$.

We now proceed to find the fractional derivative of the bivariate Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$

Theorem 3.3. Let $\alpha,\beta,\gamma,\delta\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(w_1),\Re(w_2)\}>0$. Then the following fractional derivative formula holds.

$$\begin{aligned}&{}_xD^{\lambda}_{a+}\left[(x-a)^{\beta-1}\;E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma})\right]\\&=(x-a)^{\beta+\lambda-1}E^{(\delta)}_{\alpha,\beta-\lambda,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma}).\end{aligned}\tag{3.4}$$

Proof. Upon using (1.8), interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (2.1) and (1.9), we find

$$\begin{aligned}&{}_xD^{\lambda}_{a+}\left[(x-a)^{\beta-1}\;E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma})\right]\\&={}_xD^{k}_{a+}\,{}_xI^{k-\lambda}_{a+}\left[(x-a)^{\beta-1}\;E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha},w_2(x-a)^{\gamma})\right]\\&=\frac{1}{\Gamma(k-\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\;w_1^mw_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}{}_xD^{k}_{a+}\int_a^x(x-t)^{k-\lambda-1}(t-a)^{\alpha m+\gamma n+\beta-1}dt,\\&=(x-a)^{\beta-\lambda-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\;w_1^mw_2^n(x-a)^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\beta-\lambda)m!n!},\end{aligned}$$

which gives us the desired result (3.4).

Theorem 3.4. Let $\alpha,\beta,\gamma,\delta\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(w_1),\Re(w_2)\}>0$,. Then the following fractional derivative formula holds.

$$\begin{aligned}&{}_xD^{\lambda}_{a+}\left[(x-a)^{\delta+\lambda-1}\;E^{(\delta)}_{\alpha,\beta,\gamma}(w_1(x-a),w_2(x-a))\right]\\&=\frac{\Gamma(\delta+\lambda)}{\Gamma(\delta)}(x-a)^{\delta-1}E^{(\delta+\lambda)}_{\alpha,\beta,\gamma}(w_1(x-a),w_2(x-a)).\end{aligned}\tag{3.5}$$

Proof. We refer to the proof of Theorem 3.3.

Corollary 3.2. As a consequence of (2.5) and Theorem 3.4, we have

$${}_xD^{\lambda}_{a+}\left[(x-a)^{\delta+\lambda-1}\;E^{(\delta)}_{\alpha,\beta}(w_1(x-a))\right]$$

$$= \frac{\Gamma(\delta+\lambda)}{\Gamma(\delta)}(x-a)^{\delta-1}E^{(\delta+\lambda)}_{\alpha,\beta+\lambda,}(w_1(x-a)), \tag{3.6}$$

where $\alpha,\beta,\delta\in\mathbb{C}, \{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(w_1),\Re(w_2)\}>0,.$

Theorem 3.5. Let $\alpha,\beta,\gamma,\delta,\lambda,w_1,w_2\in\mathbb{C}, \{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda)\}>0, n\in\mathbb{N}$. Then the following formulas hold.

$$D^{\lambda}_x\left[x^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right] = x^{\beta-\lambda-1}\ E^{(\delta)}_{\alpha,\beta-\lambda,\gamma}(w_1x^{\alpha},w_2x^{\gamma}). \tag{3.7}$$

In particular

$$D^{k}_x\left[x^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right] = x^{\beta-\lambda-1}\ E^{(\delta)}_{\alpha,\beta-k,\gamma}(w_1x^{\alpha},w_2x^{\gamma}). \tag{3.8}$$

Proof. We have

$$D^{\lambda}_x\left[x^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}\left(D^{\lambda}_x\ x^{\beta+\alpha m+\gamma n-1}\right),$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta-\lambda)m!n!}x^{\beta+\alpha m+\gamma n-\lambda-1},$$

$$= x^{\beta-\lambda-1}\ E^{(\delta)}_{\alpha,\beta-\lambda,\gamma}(w_1x^{\alpha},w_2x^{\gamma}).$$

which proves (3.7). The relation (3.8) follows from (3.7) when $\lambda=k,(k\in\mathbb{N})$.

4. Solution of a Singular integral equation with $E^{(\delta)}_{\alpha,\beta,\gamma}$ in kernel

In this section, we solve a singular integral equation with the generalized bivariate Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel. We denote the Laplace transform of a function f ([16], p.218) by

$$\mathbb{L}\left[f(t)\right](p) = \breve{f}(p) = \int_0^{\infty} e^{-pt}f(t)dt, (\Re(p)>0). \tag{4.1}$$

Theorem 4.1. Let $\alpha,\beta,\gamma,\delta,w_1,w_2\in\mathbb{C},\ \{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)\}>0$. Then

$$\mathbb{L}\left[x^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((wx)^{\alpha},(wx)^{\gamma}\right)\right](p) = p^{-\beta}\ \left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-\delta}. \tag{4.2}$$

Proof. We have

$$\mathbb{L}\left[x^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((wx)^{\alpha},(wx)^{\gamma}\right)\right](p)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}(wx)^{\alpha m+\gamma n}}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\int_0^{\infty}e^{-px}x^{\alpha m+\gamma n+\beta-1}dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}}{m!n!}p^{\alpha m+\gamma n+\beta}w^{\alpha m+\gamma n}$$

$$= p^{-\beta}\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-\delta},$$

which is the desired result.
Next, we establish an integral formula involving the product of two bivariate Mittag-Leffler functions.
Theorem 4.2. Let $\alpha,\beta,\gamma,\delta,\lambda,\sigma,w\in\mathbb{C}$ such that $\{\ \Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\sigma)\}>0$. Then

$$\int_0^x (x-t)^{\beta-1}E^{(\delta)}_{\alpha,\beta,\gamma}\left[(w(x-t))^{\alpha},(w(x-t))^{\gamma}\right]\times t^{\lambda-1}E^{(\sigma)}_{\alpha,\lambda,\gamma}\left[(wt))^{\alpha},(wt)^{\gamma}\right]dt$$

$$= t^{\beta+\lambda-1}\ E^{(\delta+\sigma)}_{\alpha,\beta+\lambda,\gamma}\left[(wt)^{\alpha},(wt)^{\gamma}\right]. \tag{4.3}$$

Proof. With the help of convolution theorem for the Laplace transform (see [20]):

$$\mathbb{L}\left[\int_0^x f(x-t)g(t)dt\right](p)=\mathbb{L}\left[f(x)\right](p)\ \mathbb{L}\left[g(x)\right](p),$$

we have

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha},(w(x-t))^{\gamma}\right)\times t^{\lambda-1}\ E^{(\sigma)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha},(wt)^{\gamma}\right)\right](p).$$

$$=\mathbb{L}\left[(t^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((wt)^{\alpha},(wt)^{\gamma}\right)\right](p)\times\ \cdot\mathbb{L}\left[t^{\lambda-1}E^{(\sigma)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha},(wt)^{\gamma}\right)\right](p).$$

Now, from Theorem 4.1

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha},(w(x-t))^{\gamma}\right)\times t^{\lambda-1}\ E^{(\sigma)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha},(wt)^{\gamma}\right)\right](p)$$

$$=p^{-(\beta+\lambda)}\ \left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-(\delta+\sigma)}.$$

Hence

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(\delta)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha},(w(x-t))^{\gamma}\right)\times t^{\lambda-1}\ E^{(\sigma)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha},(wt)^{\gamma}\right)\right](p)$$

$$=\mathbb{L}\left[t^{\beta+\lambda-1}\ E^{(\delta+\sigma)}_{\alpha,\beta+\lambda,\gamma}\left((wt))^{\alpha},(wt)^{\gamma}\right)\right](p). \tag{4.4}$$

Finally, taking the inverse Laplace transform of (4.5), the result follows.

Now, let us consider the following convolution equation involving the bivariate Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel.

$$\int_0^x (x-t)^{\beta-1}E^{(\delta)}_{\alpha,\beta,\gamma}\left[(w(x-t))^{\alpha},(w(x-t))^{\gamma}\right]\cdot\phi(t)dt=\psi(x),(\Re(\beta)>-1). \tag{4.5}$$

Theorem 4.3. The singular integral equation (4.6) admits a locally integrable solution

$$\phi(x)=\int_0^x (x-t)^{\nu-\beta-1}E^{(\delta)}_{\alpha,\nu-\beta,\gamma}\left[(w(x-t))^{\alpha},(w(x-t))^{\gamma}\right]\cdot{}_tI^{-\nu}_{0^+}\psi(t)dt, \tag{4.6}$$

provided that ${}_tI^{-\nu}_{0^+}\varphi(t)$ exists for $\Re(\nu)>\Re(\beta+1)$ and locally integrable for $0<t<\mu<\infty$.
Proof. Applying the Laplace transform on both sides of (4.5), using the convolution theorem as well as Theorem 4.1, we get

$$p^{-\beta}\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}}{m!n!}\left(\frac{w}{p}\right)^{\alpha m+\gamma n}\ \mathbb{L}\left[\phi(t)\right](p)=\mathbb{L}\left[\psi(t)\right](p), \tag{4.7}$$

which under the assumptions that $\left|\left(\frac{w}{p}\right)^{\alpha}+\left(\frac{w}{p}\right)^{\gamma}\right|<1$, can be written in the form

$$p^{-\beta}\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-\delta}\mathbb{L}\left[\phi(t)\right](p)=\mathbb{L}\left[\psi(t)\right](p), \tag{4.8}$$

Therefore, we have

$$\mathbb{L}\left[\phi(t)\right](p)=\left\{\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{\delta}p^{\beta-\nu}\right\}\left\{p^{\nu}\mathbb{L}\left[\psi(t)\right](p)\right\}. \tag{4.9}$$

Taking the inverse Laplace transform of both sides of (4.9) and with the aid of the following property ([20], p.217, Eq. (3.8))

$$p^{\mu}\mathbb{L}\left[f(t)\right](p) = \mathbb{L}\left[{}_tI_{0^+}^{-\mu}\ f(t)\right](p),$$
$$(\mu, p \in \mathbb{C}; \Re(p) > 0),$$

which holds true for suitable f, we thus find

$$\phi(x) = \int_0^x (x-t)^{\nu-\beta-1} E_{\alpha,\nu-\beta,\gamma}^{(\delta)}\left[(w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right] \cdot {}_tI_{0^+}^{-\nu}\psi(t)dt.$$

5. An integral operator with $E_{\alpha,\beta,\gamma}^{(\delta)}$ in kernel

In this section, we consider the integral operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}$ defined by

$$\left(\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}\varphi\right)(x)$$
$$= \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta,\gamma}^{(\delta)}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt, (x>a), \tag{5.1}$$

with $\alpha, \beta, \gamma, \delta, w_1, w_2 \in \mathbb{C},\ \{\Re(\beta), \Re(\alpha), \Re(\gamma), Re(\delta)\} > 0$.
In particular, for $w_2 = 0$, (5.1) gives the integral operator by Prabhakar [10]

$$\left(\mathbf{E}_{\alpha,\beta,w:a^+}^{(\delta)}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{(\delta)}\left(w(x-t)^{\alpha}\right)\varphi(t)dt, (x>a), \tag{5.2}$$

with generalized Mittag-Leffler function defined by (1.3) in the kernel,
which further, for $\delta = \alpha = 1$ reduces to

$$\left(\mathbf{E}_{\alpha,\beta,w:a^+}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}\left(w_1(x-t)\right)\varphi(t)dt, (x>a).$$

We now show that the integral operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}$ is bounded on the space $L(a,b)$.

Theorem 5.1. Let $\alpha, \beta, \gamma, \delta, w_1, w_2 \in \mathbb{C}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$, then the operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}$ is bounded on $L(a,b)$ and

$$\parallel \mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}\varphi \parallel_l \leq A \parallel \varphi \parallel_l, \tag{5.3}$$

where

$$A = (b-a)^{\Re(\beta)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|(\delta)_{m+n}|}{|\Gamma(\alpha m+\gamma n+\beta)|[\Re(\alpha)m+\Re(\gamma)n+\Re(\beta)]}$$
$$\frac{|w_1(b-a)^{\Re(\alpha)}|^m}{m!} \quad \frac{|w_2(b-a)^{\Re(\gamma)}|^n}{n!}. \tag{5.4}$$

Proof. Using (1.9) and (2.1) and interchanging the order of integration and applying the Dirichlet formula (1.13), we find that

$$\parallel \mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}^{(\delta)}\varphi \parallel_l$$
$$= \int_a^b \left|\int_a^x (x-t)^{\beta-1} E_{\alpha,\beta,\gamma}^{(\delta)}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt\right| dx,$$
$$\leq \int_a^b \left\{\int_t^b (x-t)^{\Re(\beta)-1}\left|E_{\alpha,\beta,\gamma}^{(\delta)}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\right| dx\right\} |\varphi(t)|\, dt$$
$$= \int_a^b \left\{\int_0^{b-t} u^{\Re(\beta)-1}\left|E_{\alpha,\beta,\gamma}^{(\delta)}\left(w_1u^{\alpha}, w_2u^{\gamma}\right)\right| du\right\} |\varphi(t)|\, dt$$

$$\leq \int_a^b \left\{ \int_0^{b-a} u^{\Re(\beta)-1} \left| E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du \right\} |\varphi(t)|\, dt. \tag{5.5}$$

Using (2.1), carrying out term-by-term integration, and taking into account (5.4), we obtain

$$\int_0^{b-a} u^{\Re(\beta)-1} \left| E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du$$

$$\leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|(\delta)_{m+n}| \quad |w_1|^m \ |w_2|^n}{|\Gamma(\alpha m + \gamma n + \beta)| \quad m!n!} \times \int_0^{b-a} u^{\Re(\beta)+\Re(\alpha)m+\Re(\gamma)n-1} du = A, \tag{5.6}$$

and (5.5) yields (5.4), which completes the proof of the theorem.

Note that for $w_2 = 0$, (5.4) gives the known result ([14], p.542(4.10))

$$\| \mathbf{E}^{(\gamma)}_{\alpha,\beta,w:a^+} \varphi \|_l \leq D \| \varphi \|_l, \tag{5.7}$$

where

$$D = (b-a)^{\Re(\beta)} \sum_{m=0}^{\infty} \frac{|(\delta)_m| \ |w_1 (b-a)^{\Re(\alpha)}|^m}{|\Gamma(\alpha m + \beta)| [\Re(\alpha)m + \Re(\beta)] m!}. \tag{5.8}$$

Next, we show that the integral operator $\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded in the space $C[a,b]$ of continuous function h on $[a,b]$ with a finite norm

$$\| h \|_C = \max_{a \leq x \leq b} |h(x)|. \tag{5.9}$$

Theorem 5.2. Let $\alpha, \beta, \gamma, \delta, w_1, w_2 \in \mathbb{C}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $b > a$, then the operator $\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded on $C[a,b]$ and

$$\| \mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+} \varphi \|_C \leq A \| \varphi \|_C, \tag{5.10}$$

where A is given by (5.4).
Proof. Using (5.1) and (5.9), we have for any $x \in [a,b]$ and $\varphi \in C[a,b]$

$$\left| \left(\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+} \varphi \right)(x) \right|$$

$$= \left| \int_a^x (x-t)^{\beta-1} E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 (x-t)^\alpha, w_2 (x-t)^\gamma\right) \varphi(t) dt \right|$$

$$\leq \int_a^x \left| (x-t)^{\beta-1} E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 (x-t)^\alpha, w_2 (x-t)^\gamma\right) \right| |\varphi(t)|\, dt$$

$$\leq \| \varphi \|_C \int_a^x (x-t)^{\Re(\beta)-1} \left| E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 (x-t)^\alpha, w_2 (x-t)^\gamma\right) \right| dt$$

$$= \| \varphi \|_C \int_0^{x-a} u^{\Re(\beta)-1} \left| E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du$$

$$\leq \| \varphi \|_C \int_0^{b-a} u^{\Re(\beta)-1} \left| E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du$$

$$= A \| \varphi \|_C,$$

where A is the same as in (5.4).

The following Lemma is an application of the previous results.

Lemma 5.1. Let $\alpha, \beta, \gamma, \delta, w_1, w_2 \in \mathbb{C}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and$b > a$, then

$$\left(\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}(t-a)^{\lambda-1}\right)(x)$$
$$= \Gamma(\lambda)(x-a)^{\beta+\lambda-1} E^{(\delta)}_{\alpha,\beta+\lambda,\gamma}\left[w_1(x-a)^{\alpha}, w_2(x-a)^{\gamma}\right]. \tag{5.11}$$

Proof. Making use of (2.1) and the operator (5.1), term-by-term integrating and applying the formula [3, 1.5(1) and 1.5(5)] for the Beta and Gamma functions, it gives

$$\left(\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}(t-a)^{\lambda-1}\right)(x)$$
$$= \int_a^x (x-t)^{\beta-1} E^{(\delta)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)(t-a)^{\lambda-1} dt,$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\delta)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta) m! n!} \ \int_a^x (x-a)^{\beta+\alpha m+\gamma n-1}(t-a)^{\lambda-1},$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\delta)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta) m! n!} B(\beta+\alpha m+\gamma n, \lambda)(x-a)^{\beta+\lambda+\alpha m+\gamma n-1},$$

which yields (5.11) according to the definition (2.1).

6. Conclusions

We used the Riemann-Liouville fractional integral and derivative operators to investigate several properties of the bivariate Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$. We established the fractional integration and derivation of the function $E^{(\delta)}_{\alpha,\beta,\gamma}$. Based on careful analysis of function $E^{(\delta)}_{\alpha,\beta,\gamma}$, we succeed in dominating the solution of the singular integral equation involving the function $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel. Further, we introduced an integral operator $\mathbf{E}^{(\delta)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ with $E^{(\delta)}_{\alpha,\beta,\gamma}$ in the kernel and derived its transformation properties in the space of Lebesgue summable continuous function spaces. Future work using these Mittag-Leffler function $E^{(\delta)}_{\alpha,\beta,\gamma}$ is expected to include, for example, numerical approximation of the functions by solving fractional differential equations, numerical approximation of the operators by Bernstein-polynomial techniques, asymptotic analysis of the functions, and many more properties waiting to be proved mathematically and then applied in practice.

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