Product of Generalized Derivations of Order 2 with Derivations Acting on Multilinear Polynomials with Centralizing Conditions

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ABSTRACT: Let $R$ be a prime ring with $\text{char}(R) \neq 2$. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$, $G$ be a nonzero generalized derivation of $R$ and $d$ a nonzero derivation of $R$. In this paper we describe all possible forms of $G$ in the case

$$G^2(f(ξ))d(f(ξ)) ∈ C$$

for all $ξ = (ξ_1, \ldots, ξ_n) ∈ R^n$.

Key Words: Prime ring, derivation, generalized derivation.

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1. Introduction

Throughout this paper, $R$ always denotes an associative prime ring, extended centroid $C$, and $U$ its Utumi quotient ring. It is proven that $C$ is a field, when $R$ is prime ring. Readers are provided with more details about $U$ and $C$. An additive map $d$ on $R$ is said to be derivation if:

$$d(xy) = d(x)y + xd(y) \quad \text{for all} \quad x, y ∈ R.$$

In [8] Brešar introduced a new notion by extending the concept of derivation, named generalized derivation. An additive map $F$ on $R$ is said to be generalized derivation if there exists a derivation $d$ on $R$ such that:

$$F(xy) = F(x)y + xd(y) \quad \text{for all} \quad x, y ∈ R.$$

The derivation $d$ involves in the definition of generalized derivation $F$ is called the associated derivation of $F$. A polynomial $f ∈ C[x_1, \ldots, x_n]$ is said to be multilinear if it is linear in every $x_i$, $1 ≤ i ≤ n$. During last three decades there has been a lot of studies on generalized derivation (see [1,3,5,6,7,10,11,12,14,17,23,24]) on different subsets of $R$.

In [19], Lee and Shiue showed that if $R$ is a prime ring, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$ and $d$ a nonzero derivation of $R$ such that $d(υ)u ∈ C$ for all $υ ∈ f(R)$, then $\text{char}(R) = 2$ and $R$ satisfies $s_4$.

In [5], Demir and Argaç considered a similar situation where the derivation is replaced by generalized derivation and the evaluations are taken over a non zero right ideal of $R$. More precisely they proved: Let $R$ be a noncommutative prime ring and $F$ is a generalized derivation on $R$ such that $F(υ)u ∈ C$ for all $υ ∈ f(ρ)$, where $ρ$ is a right ideal of $R$. Then $F(x) = ax$, where $a ∈ C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$, except when $\text{char}(R) = 2$ and $R$ satisfies $s_4$.

In [14], it is proved that if $F_1$ and $F_2$ are generalized derivations of a prime ring $R$ having $\text{char}(R) ≠ 2$, such that $F_1(x)F_2(x) = 0$ for all $x ∈ R$, then there exist elements $p, q ∈ U$ such that $F_1(x) = xp$ and

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$F_2(x) = qx$ for all $x \in R$ and $pq = 0$ except when at least one $F_i$ is zero. Moreover the above identity is studied by Carini et al. [3] by taking the multilinear polynomial and studied the structures of $F_1$ and $F_2$.

Further, Eroğlu and Argaç [10] determined all possible structures of $F$ by considering $F^2(u)u \in C$ for all $u \in f(R)$ and $F$ is a generalized derivation of $R$.

More recently, Yadav [24] described all possible forms of the maps when $F^2(u)d(u) = 0$ for all $u \in f(R)$, where $F$ is generalized derivation of $R$ and $d$ is a nonzero derivation of $R$. He proved the following:

Let $R$ be a noncommutative prime ring of $char(R) \neq 2$, $U$ be its Utumi quotient ring, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose that $d$ is a nonzero derivation of $R$ and $G$ is a generalized derivation on $R$. If

$$G^2(f(\xi))d(f(\xi)) = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then one of the following holds:

1. there exist $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;
2. there exist $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.

In this article we extend Yadav’s result [24] in central case. More precisely, we study the following:

**Theorem 1.1.** Let $R$ be a noncommutative prime ring of $char(R) \neq 2$, $U$ be its Utumi quotient ring, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose that $d$ is a nonzero derivation of $R$ and $G$ is a generalized derivation on $R$. If

$$G^2(f(\xi))d(f(\xi)) \in C$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then one of the following holds:

1. there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;
2. there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.

### 2. When derivations are inner

We dedicate this section to prove the main theorem in case both the generalized derivation $G$ and the derivation $d$ are inner, that is, there exist $a, b, c \in U$ such that $G(x) = ax + xb$ and $d(x) = [c, x]$ for all $x \in R$. Then $G^2(f(\xi))d(f(\xi)) \in C$ for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ implies

$$a^2 f(\xi) cf(\xi) + 2af(\xi)bcf(\xi) + f(\xi)b^2 cf(\xi) - a^2 f(\xi)^2 c - 2af(\xi)bf(\xi)c - f(\xi)b^2 f(\xi)c \in C.$$  

This gives

$$a' f(\xi) cf(\xi)^2 + 2af(\xi)pf(\xi)^2 + f(\xi)p' f(\xi)^2 + f(\xi)a' f(\xi)^2 c + 2f(\xi)af(\xi)bf(\xi)c + f(\xi)^2 b' f(\xi)c - a' f(\xi)^2 c f(\xi) - 2af(\xi)bf(\xi)c f(\xi) - f(\xi)b' f(\xi)c f(\xi) - f(\xi)a f(\xi) p f(\xi) - f(\xi)^2 p' f(\xi) = 0$$  

(2.1)

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, where $a' = a^2$, $b' = b^2$, $p = bc$ and $p' = b^2 c$.

**Proposition 2.1.** Let $C$ be a field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over $C$, $m \geq 2$. Suppose that $char(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$. If $a, b$ and $c \in R$ such that (2.1) holds for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then either $a$ or $b$ or $c$ are scalar matrices.

**Proof.** By our assumption (2.1) is a generalized polynomial identity of $R$. Suppose that all of $a, b$ and $c$ are not scalar matrices.

**Case-I:** Suppose that $C$ is infinite field.
As we assumed $a \notin C.I_n$ and $b \notin C.I_n$ and $c \notin C.I_n$. By [11, Lemma 1.5] there exists an invertible matrix $P$ in $M_m(C)$ such that $PaP^{-1}$, $PbP^{-1}$ and $PcP^{-1}$ have all non-zero entries. Clearly $R$ satisfies

\[ Pa'P^{-1}f(\xi)PcP^{-1}f(\xi)^2 + 2PaP^{-1}f(\xi)PpP^{-1}f(\xi)^2 + f(\xi)PpP^{-1}f(\xi)^2 + f(\xi)PaP^{-1}f(\xi)^2PcP^{-1} + 2f(\xi)PaP^{-1}f(\xi)PpP^{-1}f(\xi) + f(\xi)^2PpP^{-1}f(\xi)PcP^{-1} - PaP^{-1}f(\xi)^2PcP^{-1}f(\xi) - 2PaP^{-1}f(\xi)PbP^{-1}f(\xi)PcP^{-1}f(\xi) - f(\xi)PpP^{-1}f(\xi)PcP^{-1}f(\xi) - f(\xi)PdP^{-1}f(\xi)PcP^{-1}f(\xi) - 2f(\xi)PpP^{-1}f(\xi) - f(\xi)^2PpP^{-1}f(\xi) = 0 \quad (2.2) \]

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. By hypothesis $f(x_1, \ldots, x_n)$ is non central valued. Hence by [18] (see also [20]), there exist matrices $\xi_1, \ldots, \xi_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(\xi_1, \ldots, \xi_n) = \gamma e_{ij}$, with $i \neq j$. We replace this value of $f(\xi_1, \ldots, \xi_n)$ in (2.2), we get

\[ 2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1} - 2PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} - e_{ij}PpP^{-1}e_{ij}PcP^{-1}e_{ij} - e_{ij}PdP^{-1}e_{ij}PcP^{-1}e_{ij} - 2e_{ij}PaP^{-1}e_{ij}PpP^{-1}e_{ij} = 0 \quad (2.3) \]

Now multiplying by $e_{ij}$ in (2.3) from right side, we get $2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} = 0$, this implies $e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} = 0$, as $char(R) \neq 2$. This is a contradiction as $PaP^{-1}$, $PbP^{-1}$ and $PcP^{-1}$ have all non-zero entries.

**Case-II:** When $C$ is finite field.

Let $K$ be an infinite field which is an extension of the field $C$. Let $\overline{R} = M_m(K) \cong R \otimes C K$. Since multilinear polynomial $f(x_1, \ldots, x_n)$ is non-central-valued on $R$, so it is also non-central-valued on $\overline{R}$. Consider the generalized polynomial

\[ \phi(\xi_1, \ldots, \xi_n) = a'f(\xi)c_1f(\xi)^2 + 2af(\xi)p_1f(\xi)^2 + f(\xi)p_1f(\xi)^2 + f(\xi) \]

which is a generalized polynomial identity for $R$. Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $\xi_1, \ldots, \xi_n$.

Hence the complete linearization of $\phi(\xi_1, \ldots, \xi_n)$ is a multilinear generalized polynomial $\Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n)$ in $2n$ indeterminates, moreover

\[ \Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n) = 2^n \phi(\xi_1, \ldots, \xi_n). \]

Clearly the multilinear polynomial $\Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n)$ is a generalized polynomial identity for $R$ and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\phi(\xi_1, \ldots, \xi_n) = 0$ for all $\xi_1, \ldots, \xi_n \in \overline{R}$ and then conclusion follows from above when $C$ was infinite.

\[ \Box \]

**Proposition 2.2.** Let $R$ be a prime ring of $char(R) \neq 2$, $C$ the extended centroid of $R$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$. If $R$ satisfies (2.1), then either $a$ or $b$ or $c$ are scalar matrices.

**Proof.** Since $R$ and $U$ satisfy the same generalized polynomial identities (see [4]), $U$ satisfies

\[ a'f(\xi)c_1f(\xi)^2 + 2af(\xi)p_1f(\xi)^2 + f(\xi)p_1f(\xi)^2 + f(\xi) \]

which is a generalized polynomial identity for $R$. Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $\xi_1, \ldots, \xi_n$.

Hence the complete linearization of $\phi(\xi_1, \ldots, \xi_n)$ is a multilinear generalized polynomial $\Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n)$ in $2n$ indeterminates, moreover

\[ \Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n) = 2^n \phi(\xi_1, \ldots, \xi_n). \]

Clearly the multilinear polynomial $\Theta(\xi_1, \ldots, \xi_n, s_1, \ldots, s_n)$ is a generalized polynomial identity for $R$ and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\phi(\xi_1, \ldots, \xi_n) = 0$ for all $\xi_1, \ldots, \xi_n \in \overline{R}$ and then conclusion follows from above when $C$ was infinite.

\[ \Box \]
for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Suppose that this is a trivial GPI for $U$. So,

$$
\begin{align*}
& a'f(\xi)cf(\xi)^2 + 2af(\xi)pf(\xi)^2 + f(\xi)p'f(\xi)^2 \\
& + f(\xi)a'f(\xi)^2c + 2f(\xi)af(\xi)bf(\xi)c + f(\xi)^2b'f(\xi)c \\
& - a'f(\xi)^2cf(\xi) - 2af(\xi)bf(\xi)cf(\xi) - f(\xi)b'f(\xi)cf(\xi) \\
& - f(\xi)a'f(\xi)cf(\xi) - 2f(\xi)af(\xi)pf(\xi) - f(\xi)^2p'f(\xi)
\end{align*}
$$

(2.5)

is zero element in $T = U \ast_{C} C\{\xi_1, \ldots, \xi_n\}$, the free product of $U$ and $C\{\xi_1, \ldots, \xi_n\}$, the free $C$-algebra in noncommuting indeterminates $\xi_1, \ldots, \xi_n$. This implies $\{1, c\}$ is linearly $C$-dependent, that is $c \in C$, as desired. Let us assume $c \notin C$, then by (2.5)

$$
\{f(\xi)a'f(\xi) + 2f(\xi)af(\xi)b + f(\xi)^2b'\}f(\xi)c = 0 \in T.
$$

(2.6)

This again implies that $\{1, b, b'\}$ is linearly $C$-dependent. There exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2b + \alpha_3b' = 0$. If $\alpha_3 = 0$, then $\alpha_2 \neq 0$ and hence $b \in C$, as desired. Thus we assume $\alpha_3 \neq 0$ and $b \notin C$. Then by (2.6)

$$
\{f(\xi)a'f(\xi) + 2f(\xi)af(\xi)b + \alpha f(\xi)^2b + \beta f(\xi)^2\}f(\xi)c = 0 \in T.
$$

(2.7)

Assume $a \notin C$, then $2f(\xi)af(\xi)b(\xi)c$ appears nontrivially in (2.7), which is a contradiction. So, either $a$ or $b$ or $c$ is central, as desired.

Next suppose that (2.4) is a non-trivial GPI for $Q$. Let $\overline{C}$ be the algebraic closure of $C$. We know that $U$ and $U \otimes_{C} \overline{C}$ satisfy the same GPIs. Since both $U$ and $U \otimes_{C} \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace $R$ by $U$ or $U \otimes_{C} \overline{C}$ according to $C$ finite or infinite and then applying Martindale’s theorem [21], we can say that $R$ is a primitive ring with nonzero socle $soc(R)$ and with $C$ as its associated division ring. Then, by Jacobson’s theorem [15, p.75], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Assume first that $V$ is finite dimensional over $C$, that is, $dim_{C}V = m$. By density of $R$, we have $R \cong M_{m}(C)$. Since $f(\xi_1, \ldots, \xi_n)$ is not central valued on $R$, $R$ must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.1, we get that either $a$ or $b$ or $c$ are in $C$. If $V$ is infinite dimensional over $C$, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_{m}(C)$ with $t = dim_{C}Ve$. Since $a_2, a_3, a_5$ are not in $C$, there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a, h_1] \neq 0, [b, h_2] \neq 0, [c, h_3] \neq 0$. By Litoff’s Theorem [13], there exists idempotent $e \in soc(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, h_1, h_2, h_3 \in eRe$. Since $R$ satisfies generalized identity

$$
eq \left\{a'f(e\xi_1e, \ldots, e\xi_ne)cf(e\xi_1e, \ldots, e\xi_ne)^2 + 2af(e\xi_1e, \ldots, e\xi_ne)pf(e\xi_1e, \ldots, e\xi_ne)^2 \\
+ f(e\xi_1e, \ldots, e\xi_ne)p'f(e\xi_1e, \ldots, e\xi_ne)^2 + f(e\xi_1e, \ldots, e\xi_ne)a'f(e\xi_1e, \ldots, e\xi_ne)^2c \\
+ 2f(e\xi_1e, \ldots, e\xi_ne)a'f(e\xi_1e, \ldots, e\xi_ne)b'f(e\xi_1e, \ldots, e\xi_ne)c \\
+ f(e\xi_1e, \ldots, e\xi_ne)^2b'f(e\xi_1e, \ldots, e\xi_ne)c - a'f(e\xi_1e, \ldots, e\xi_ne)^2cf(e\xi_1e, \ldots, e\xi_ne) \\
- 2af(e\xi_1e, \ldots, e\xi_ne)b'f(e\xi_1e, \ldots, e\xi_ne)cf(e\xi_1e, \ldots, e\xi_ne) \\
- f(e\xi_1e, \ldots, e\xi_ne)b'f(e\xi_1e, \ldots, e\xi_ne)cf(e\xi_1e, \ldots, e\xi_ne) \\
- 2f(e\xi_1e, \ldots, e\xi_ne)a'f(e\xi_1e, \ldots, e\xi_ne)cf(e\xi_1e, \ldots, e\xi_ne) \\
- 2f(e\xi_1e, \ldots, e\xi_ne)a'f(e\xi_1e, \ldots, e\xi_ne)pf(e\xi_1e, \ldots, e\xi_ne) \\
- f(e\xi_1e, \ldots, e\xi_ne)^2p'f(e\xi_1e, \ldots, e\xi_ne) \right\}e = 0,$$
By the same way as above we can prove the following propositions.

**Proposition 2.3.** Let $R$ be a prime ring of $\text{char}(R) \neq 2$, $C$ the extended centroid of $R$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$. If $c$ and $k \in R$ such that

$$f(\xi)kc(f(\xi))^2 - f(\xi)kf(\xi)c - f(\xi)^2kc(\xi) + f(\xi)^2kf(\xi)c = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then either $k \in C$ or $c \in C$.

**Proposition 2.4.** Let $R$ be a prime ring of $\text{char}(R) \neq 2$, $C$ the extended centroid of $R$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$. If $c$ and $k \in R$ such that

$$kf(\xi)c(f(\xi))^2 - kf(\xi)^2cf(\xi) - f(\xi)kf(\xi)c + f(\xi)kf(\xi)c = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then either $k \in C$ or $c \in C$.

**Lemma 2.5.** Let $R$ be a noncommutative prime ring of $\text{char}(R) \neq 2$, $U$ be its Utumi quotient ring, $C$ be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$. Suppose for some $a, b, c \in U$, $G(x) = ax + xb$, and $d(x) = [c, x]$ for all $x \in R$ with $c \notin C$. If

$$G^2(f(\xi))d(f(\xi)) \in C$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then one of the following holds:

1. $G(x) = (a + b)x$ for all $x \in R$ with $(a + b)^2 = 0$;
2. $G(x) = x(a + b)$ for all $x \in R$ with $(a + b)^2 = 0$.

**Proof.** By the hypothesis, we have

$$(a^2f(\xi) + 2af(\xi)b + f(\xi)b^2)(cf(\xi) - f(\xi)c) \in C$$

(2.8)

that is

$$[(a^2f(\xi) + 2af(\xi)b + f(\xi)b^2)(cf(\xi) - f(\xi)c), f(\xi)] = 0$$

(2.9)

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Then by Proposition 2.2, either $a \in C$ or $b \in C$ or $c \in C$. Since $c \notin C$, so either $a \in C$ or $b \in C$.

If $a \in C$, it follows hypothesis as

$$f(\xi)(a + b)^2(cf(\xi) - f(\xi)c) \in C$$
that is
\[ f(\xi)(a + b)^2cf(\xi)^2 - f(\xi)(a + b)^2f(\xi)cf(\xi) - f(\xi)^2(a + b)^2cf(\xi) + f(\xi)^2(a + b)^2f(\xi)c = 0 \]
for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Then by Proposition 2.3, \((a + b)^2 \in C\). If \( b \in C \), it follows hypothesis as
\[ (a + b)^2f(\xi)(cf(\xi) - f(\xi)c) \in C \]
that is
\[ (a + b)^2f(\xi)cf(\xi)^2 - (a + b)^2f(\xi)cf(\xi) - f(\xi)(a + b)^2f(\xi)cf(\xi) + f(\xi)(a + b)^2f(\xi)^2c = 0 \]
for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Then by Proposition 2.4, \((a + b)^2 \in C\). Thus in both the above cases we have \((a + b)^2 \in C\) and hence we can write \((a + b)^2x = G^2(x) = x(a + b)^2\) for all \( x \in f(R) \).

Considering \( G^2(f(\xi)) = f(\xi)(a + b)^2 \), our hypothesis \( G^2(f(\xi))d(f(\xi)) \in C \) gives \( f(\xi)[(a + b)^2c, f(\xi)] \in C \) for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Then by [19] we have \((a + b)^2c \in C\). This implies \((a + b)^2 = 0\) as \( c \notin C \).

Thus we arrive either \( G(x) = (a + b)x \) or \( x(a + b) \), with \((a + b)^2 = 0\). These are our required conclusions.

\[ \square \]

3. Proof of the main theorem

In light of the notion in [17, Theorem 3], generalized derivation \( G \) has its form \( G(x) = ax + \delta(x) \) for some \( a \in U \) and \( \delta \) is a derivation on \( U \).

Now if we consider \( f(\xi_1, \ldots, \xi_n) \) be a noncentral multilinear polynomial over the field \( C \) and \( d \) is a derivation on \( R \).

We shall use the notation
\[ f(\xi_1, \ldots, \xi_n) = \xi_1\xi_2 \cdots \xi_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma \xi_{\sigma(1)}\xi_{\sigma(2)} \cdots \xi_{\sigma(n)} \]
for some \( \alpha_\sigma \in C \), and \( S_n \) denotes the symmetric group of degree \( n \).

Then we have
\[ d(f(\xi_1, \ldots, \xi_n)) = df(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n), \]
where \( df(\xi_1, \ldots, \xi_n) \) be the polynomials obtained from \( f(\xi_1, \ldots, \xi_n) \) replacing each coefficients \( \alpha_\sigma \) with \( d(\alpha_\sigma) \). Similarly, by calculation, we have
\[ d^2(f(\xi_1, \ldots, \xi_n)) = df(\xi_1, \ldots, \xi_n) + 2 \sum_i f^d(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n) \]
\[ + \sum_i f(\xi_1, \ldots, d^2(\xi_i), \ldots, \xi_n) \]
\[ + \sum_{i \neq j} f(\xi_1, \ldots, d(\xi_i), \ldots, d(\xi_j), \ldots, \xi_n). \]

By hypothesis, we have
\[ (G(a)f(\xi) + 2a\delta(f(\xi)) + \delta^2(f(\xi)))d(f(\xi)) \in C \]
for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). By [18], we have
\[ (G(a)f(\xi) + 2a\delta(f(\xi)) + \delta^2(f(\xi)))d(f(\xi)) \in C \]
for all \( \xi = (\xi_1, \ldots, \xi_n) \in U^n \).

If \( d \) and \( \delta \) both are inner derivations, then by Proposition 2.2, we have our conclusions of Main Theorem. Thus, to prove our Main Theorem, we need to consider the case when not both \( d \) and \( \delta \) are inner. Indeed we have to consider the two following embedded cases.
• $d$ and $\delta$ are linearly $C$-independent modulo inner derivations of $U$.
• $d$ and $\delta$ are linearly $C$-dependent modulo inner derivations of $U$.

**Case-1:** When $d$ and $\delta$ are linearly $C$-independent modulo inner derivations of $U$.

By (3.1) $U$ satisfies

$$
\left\{ \begin{array}{l}
F(a)f(\xi_1, \ldots, \xi_n) + 2a \{ f^\delta(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, \delta(\xi_i), \ldots, \xi_n) \} \\
+ \{ f^{\delta^2}(\xi_1, \ldots, \xi_n) + 2 \sum_i f^{\delta}(\xi_1, \ldots, \delta(\xi_i), \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, \delta^2(\xi_i), \ldots, \xi_n) \\
+ \sum_{i \neq j} f(\xi_1, \ldots, \delta(\xi_i), \ldots, \delta(\xi_j), \ldots, \xi_n) \} \{ f^d(\xi_1, \ldots, \xi_n) \\
+ \sum_i f(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n) \} \in C.
\end{array} \right.
$$

for all $\xi_1, \ldots, \xi_n \in U$. Since $d$ and $\delta$ are not inner, by Kharchenko’s theorem [16], $U$ satisfies

$$
\left\{ \begin{array}{l}
F(a)f(\xi_1, \ldots, \xi_n) + 2a \{ f^\delta(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n) \} \\
+ \{ f^{\delta^2}(\xi_1, \ldots, \xi_n) + 2 \sum_i f^{\delta}(\xi_1, \ldots, x_i, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) \\
+ \sum_{i \neq j} f(\xi_1, \ldots, x_i, \ldots, x_j, \ldots, \xi_n) \} \{ f^d(\xi_1, \ldots, \xi_n) \\
+ \sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n) \} \in C.
\end{array} \right.
$$

In particular $U$ satisfies the blended component

$$
\left[ \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n) f(\xi_1, \ldots, \xi_n) \right] = 0. \quad (3.2)
$$

Putting $y_i = [q', \xi_i]$ for each $i \in \{1, \ldots, n\}$, for some $q' \notin C$ and $z_1 = \xi_1, z_2 = \ldots, z_n = 0$, we get

$$
[q' f(\xi_1, \ldots, \xi_n)] f(\xi_1, \ldots, \xi_n), f(\xi_1, \ldots, \xi_n) = 0
$$

that is

$$
[q', f(\xi_1, \ldots, \xi_n)]_2 f(\xi_1, \ldots, \xi_n) = 0
$$

this implies

$$
[q', f(\xi_1, \ldots, \xi_n)]_2 = 0 \quad \text{as} \quad f(\xi_1, \ldots, \xi_n) \notin C
$$

for all $\xi_1, \ldots, \xi_n \in U$. Then by [22] we get $q' \in C$, which is a contradiction.

**Case-2:** When $d$ and $\delta$ are linearly $C$-dependent modulo inner derivations of $U$.

In this case we get $\alpha, \beta \in C$ and $q \in U$ such that $\alpha d + \beta \delta = ad_q$. It is clear from the context that $(\alpha, \beta) \neq (0, 0)$. So with out loss of generality we arrive the following two subcases:

**Sub-case-i:** When $\alpha = 0$.

Then we get $\delta(x) = [p, x]$, where $p = \beta^{-1} q$. It is obvious that $d$ is not inner, otherwise we get contradiction. Now from (3.1) we have

$$
(a^2 f(\xi_1, \ldots, \xi_n) + 2a' f(\xi_1, \ldots, \xi_n)b + f(\xi_1, \ldots, \xi_n)b^2)df(df(\xi_1, \ldots, \xi_n)) \in C
$$
for all $\xi_1, \ldots, \xi_n \in U$ and $a' = a + p, b' = -p \in U$. Now from above we can write
\begin{align*}
(a^2 f(\xi_1, \ldots, \xi_n) + 2a' f(\xi_1, \ldots, \xi_n)b' + f(\xi_1, \ldots, \xi_n)b^2).
(f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n)) \in C.
\end{align*}

(3.3)

Since $d$ is not inner, by Kharchenko’s theorem [16]
\begin{align*}
(a^2 f(\xi_1, \ldots, \xi_n) + 2a' f(\xi_1, \ldots, \xi_n)b' + f(\xi_1, \ldots, \xi_n)b^2).
(f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)) \in C.
\end{align*}

In particular $U$ satisfies the blended component
\begin{align*}
\left[(a^2 f(\xi_1, \ldots, \xi_n) + 2a' f(\xi_1, \ldots, \xi_n)b' + f(\xi_1, \ldots, \xi_n)b^2).
\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n), f(\xi_1, \ldots, \xi_n)\right] = 0.
\end{align*}

(3.4)

Replacing $y_i$ by $[q, \xi_i]$, for some $q \notin C$ in (3.4) we have
\begin{align*}
\left[(a^2 f(\xi_1, \ldots, \xi_n) + 2a' f(\xi_1, \ldots, \xi_n)b' + f(\xi_1, \ldots, \xi_n)b^2)
[q, f(\xi_1, \ldots, \xi_n)], f(\xi_1, \ldots, \xi_n)\right] = 0.
\end{align*}

(3.5)

Which is similar as (2.9) of Lemma 2.5, so from there we get our conclusions (1) and (2) of main theorem.

**Sub-case-ii:** When $\alpha \neq 0$.

Then we have $d = \mu \delta + ad_c$, for some $\mu \in C$ and $c \in U$. Here $\delta$ never be an inner derivation, otherwise both $d$ and $\delta$ will be inner, a contradiction. Then from (3.1) we have
\begin{align*}
(G(a)f(\xi_1, \ldots, \xi_n) + 2a\delta(f(\xi_1, \ldots, \xi_n)) + \delta^2(f(\xi_1, \ldots, \xi_n)))
(\mu\delta(f(\xi_1, \ldots, \xi_n)) + [c, f(\xi_1, \ldots, \xi_n)]) \in C
\end{align*}

(3.6)

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. This is a differential identity containing the terms of the type $\delta$ and $\delta^2$. As, $\delta$ and $\delta^2$ are outer, by Kharchenko’s theorem [16] $\delta(\xi)$ and $\delta^2(\xi)$ can be replaced by $x_i$ and $y_i$ respectively in (3.6). And hence $U$ satisfies the blended component
\begin{align*}
(\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)) (\mu \sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)) \in C,
\end{align*}

that is
\begin{align*}
\left[\mu \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n), f(\xi_1, \ldots, \xi_n)\right] = 0.
\end{align*}

(3.7)

Replacing $y_i$ by $[q, \xi_i]$, where $q \notin C$ and $x_1 = \xi_1, x_2 = \ldots = x_n = 0$ in (3.7) we get
\begin{align*}
\left[\mu[q, f(\xi_1, \ldots, \xi_n)], f(\xi_1, \ldots, \xi_n), f(\xi_1, \ldots, \xi_n)\right] = 0,
\end{align*}

that is
\begin{align*}
\left[[\mu q, f(\xi_1, \ldots, \xi_n)], f(\xi_1, \ldots, \xi_n)\right] f(\xi_1, \ldots, \xi_n) = 0,
\end{align*}

that is
\begin{align*}
[\mu q, f(\xi_1, \ldots, \xi_n)]_2 = 0,
\end{align*}
for all $\xi_1, \ldots, \xi_n \in R$, as $f(\xi_1, \ldots, \xi_n)$ is noncentral. So by [22] we get $\mu q \in C$, this says $\mu = 0$. Then from (3.6) we get
\[
\left( G(a) f(\xi_1, \ldots, \xi_n) + 2a \delta(f(\xi_1, \ldots, \xi_n)) \ight)
+ \delta^2(f(\xi_1, \ldots, \xi_n)) [c, f(\xi_1, \ldots, \xi_n)] \in C,
\]
for all $\xi_1, \ldots, \xi_n \in U$. Again from above putting the expressions of $\delta(f(\xi_1, \ldots, \xi_n))$ and $\delta^2(f(\xi_1, \ldots, \xi_n))$ we will find a blended component satisfied by $U$ as follows:
\[
\sum_i f(\xi_1, \ldots, \delta^2(\xi_i), \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C.
\]
As it is mentioned earlier that $\delta$ is outer, then by applying Kharchenko’s theorem [16], we replace $\delta^2(\xi_i)$ by $y_i$ in (3.9) we get the following:
\[
\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C.
\]
In particular $y_1 = \xi_1$ and $y_2 = \cdots y_n = 0$ we get
\[
f(\xi_1, \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C
\]
for all $\xi_1, \ldots, \xi_n \in R$. Then from [19] we get $c \in C$. Finally we get $\mu = 0$ and $c \in C$, which implies $d = 0$, a contradiction.
This completes the proof.

References

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