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Fractional Order Iterative Boundary Value Problem

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ABSTRACT: In the present paper, we establish the existence and uniqueness results for an iterative differential equation involving Caputo fractional derivative of order $\alpha \in (1,2)$. We prove our main results by applying Schauder fixed point theorem. Some fractional inequalities are used to prove the extremal solution of our proposed model. As application, an example is given to illustrate the theoretical results.

Key Words: Iterative differential equation, Caputo derivative, maximal solutions, fractional differential inequalities.

Contents

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary propreties of various materials and processses. Fractional differential equations, in particular, are getting a lot of attention and importance. Highly remarkable monographs which provide tha main theorical tools for qualitative analysis of fractional differential equations and at the same time, show the interconnection as well as the contrast between integer differential models and fractional models (see [5,20]). Many works on the existence, uniqueness, stability, periodicity of solutions and optimal control for all kinds of fractional differential equations have been reported (see [12,13,15,17]). Iterative differential equation often arises in the modeling of a wide range of natural phenomena such as disease transmission models in epidemiology, two body problems of classical electrodynamics, population models, physical models, mechanical models and other numerous models (see [8,9,7]). This kind of equations which relates an unknown function, its derivatives and its iterates, is a special case of the so-called differential equations with state-dependent delays. More precisely, the following class of functional differential equations with state and time-varying delays (see [11]),

$$x^{(n)}(t) = f(t, x(t), x(t-\tau_1(t,x(t))), ..., x(t-\tau_n(t,x(t))))),$$

can be considered as the mother of our equation, where the complicated lags $\tau_i(t,x(t))$ give rise to appearance of the iterates. Almost the litterature related to this type of equations that presently exists includes first order equations. Unfortunately, very few papers have been devoted to study of order greather than one (see [9,14,6,18]). The difficulty in studying these equations which have distinctive characteristic lies in iterative terms that often hinder the application of usual methods. The study of

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iterative differential equations can be traced back to papers by Petuhov [18] and Eder [9]. In 1965, Petuhov [18] considered the existence of solutions of differential equation

$$x''(t) = \lambda x(x(t)),$$

under the condition that $x(t)$ maps the interval $[-T, T]$ into itself and that $x(0) = x(T) = \alpha$. He obtained conditions on λ and α for existence and uniqueness of solution. In 1984, Eder [9] used Banach contraction mapping priciple to find the existence and uniqueness of solution to the following iterative differential equation

$$x'(t) = f(x(x(t))),$$

with $x(t_0) = t_0$, where $t_0 \in [-1, 1]$. In 1990, Feçkan [10] obtained the local solution of the standard iterative differential equation

$$x'(t) = \phi(x(x(t))),$$

whit the initial condition $x(0) = 0$, where $\phi \in C^1(\mathbb{R})$. In 1993, Wang [21] studied the same equation associated to intial condition $x(\nu) = \nu$ by using the Schauder's fixed point theorem, where ν is the endpoint of the well defined interval. By using the nonexpansive operators, in 2010, Berinde [6] proposed the existence and and the convergence results to iterative differential equation studied by Feçkan and Wang. In 2018, Kaufmann [14] established the existence and uniqueness of solutions of the functional differential equation

$$x''(t) = f(t, x(t), x(x(t))),$$

by using Shauder's fixed point.
In this paper, we disscus the existence and uniqueness of the fractional iterative boundary value problem

$$\begin{cases} {}^C D^{\alpha}_{0+} x(t) = f(t, x^{[1]}(t), x^{[2]}(t), ..., x^{[m]}(t)), \quad t \in]0, T[, \\ x(0) = 0, \quad x(T) = T. \end{cases} \tag{1.1}$$

Where $f : [0, T] \times \mathbb{R} \times ... \times \mathbb{R}$ is a continuouns function, $x^{[1]}(t) = x(t), x^{[2]}(t) = x(x(t)), ..., x^{[m]}(t) = x(x^{[m]-1}(t))$, $1 < \alpha < 2$ and ${}^C D^{\alpha}_{0+}$ is the Caputo derivative. We establish also some fractional differential inequalities which are an important and great mathematical topic for research, they have many applications, the most important ones are in establishing uniqueness of solutions in fractional differential equations and systems. Also they provide upper bounds of the solutions of the above equations. Moreover, the comparaison principle is proved. Note that due to the iterative term $x^{[i]}(t)$, $i = 1, 2, ..., m$ in order for solution to be well-defined, we require that the image of x be in the interval $[0, T]$; that is, we need $0 \le x(t) \le T$, for all $t \in [0, T]$. Our motivations came from the fact that very little know about fractional order iterative problems and that many dynamical systems are better characterized by the fractional derivatives. Furthermore, as far as know, this kind of problems has not been studied till know. Thus, it is worthwhile to contribute in filling this gap by continuing the investigation in this direction in order to enrich and complement the avalaible works in the litterature.
We arrang the rest of the paper as follows. In section 2, we present some definitions and propreties of fractional calculus related to our work. Section 3 contains the existence and uniqueness results which proved by using Schauder's and Banach fixed points. As an illustation, in section 4, we provide an example to show the feasibilty of our main results. Some fundamental fractional differential inequalities are proved in section 5. Section 6 is devoted to the existence of extremal solution. Finaly, a comparaison theorems, which will be useful for further study of qualitative behavior of solutions, are given in the last section.

2. Existence and uniquness results

We first convert the boundary value problem (1.1) to a fixed point problem. To do, applying the fractional integral to the both sides of the first equation of (1.1), we get

$$x(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[m]}(s)) ds.$$

Using the value conditions, we obtain $c_0 = 0$ and

$$c_1 = 1 - \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[m]}(s))ds.$$

Intersecting the value of c_0 and c_1, we have

$$\begin{aligned} x(t) =& t - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[m]}(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[m]}(s))ds. \end{aligned}$$

Then, x satisfies the integral equation

$$x(t) = t + \int_0^T G(t,s) f(s, x^{[1]}(s), x^{[2]}(s), ..., x^{[m]}(s))ds, \tag{2.1}$$

where,

$$G(t,s) = \begin{cases} -\frac{t}{T\Gamma(\alpha)}(T-s)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, & 0 \le s < t \le T, \\ -\frac{t}{T\Gamma(\alpha)}(T-s)^{\alpha-1}, & 0 \le t < s \le T. \end{cases} \tag{2.2}$$

Motivate by the above calculus, we can introduce the following lemma.

Lemma 2.1. *x is a solution of the boundary value problem (1.1) if only and if x satisfies the inetgral solution (2.1).*

Proof. It's clear that $x(0) = 0$ and $x(T) = T$. Applying the Caputo fractional derivative to (2.1), we obtain that

$${}_C D_{0+}^{\alpha} x(t) = f(t, x^{[1]}(t), x^{[2]}(t), ..., x^{[m]}(t)), \quad t \in]0, T[.$$

Which acheve the proof. □

By $X = C([0,T], \mathbb{R})$ we denote the Banach space of all continuous functions from $[0.T]$ into $\mathbb{R}$ equipped with the norm

$$\|x\| = \sup\{|x(t)|; t \in [0,1]\}.$$

We define the operator $\mathcal{A} : X \to X$ by

$$(\mathcal{A}\varphi)(t) = t + \int_0^T G(t,s) f(s, \varphi^{[1]}(s), \varphi^{[2]}(s), ..., \varphi^{[m]}(s))ds,$$

where $G(t,s)$ is the function defined by (2.2).

Recall that for solutions of the boundary value problem (1.1) to be well defined, we need $0 \le x(t) \le T$, for all $0 \le t \le T$. As such, if $x \in C([0,T])$ is a fixed point of $\mathcal{A}$ such that $0 \le (\mathcal{A}x)(t) \le T$, for all $0 \le t \le T$, then x is a solution of the boundary value problem (1.1). To guarantee $0 \le (\mathcal{A}x)(t) \le T$, we need the following lemma.

Lemma 2.2. *The function x is a solution of the boundary value probelm (1.1) if only and if x is a fixed point of $\mathcal{A}$ and $0 \le (\mathcal{A}x)(t) \le T$, for all $t \in [0,T]$.*

To establish our existence and uniqueness results we will need the following asumptions.

(**H1**) There exist $A > 0$ such that

$$|f(t, x_1, x_2, ..., x_m)| < A,$$

for all $x_1, x_2, ..., x_m \in \mathbb{R}$, $t \in [0,T]$ and $\dfrac{AT^{\alpha-1}}{\Gamma(\alpha+1)} < \dfrac{1}{\alpha+1}$.

(**H2**) There exist $L_1, L_2, ..., L_m > 0$ such that

$$|f(t, x_1, x_2, ..., x_m) - f(t, y_1, y_2, ..., y_m)| < \sum_{i=1}^{m} L_i \|x_i - y_i\|,$$

for all $t \in [0, T]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2, ..., m$.

We are now ready to presente our first result which will poved by Schauder's fixed point [19].

Theorem 2.3. *Assume that hypothesis **(H1)** holds. Then the fractional boundary value problem* (1.1) *has a solution defined on* $[0, T]$.

Proof. From the lemma 2.2, we first prove that $0 \leq (\mathcal{A}x)(t) \leq T$. To do, we consider $t \in [0, T]$, and we have

$$\begin{aligned}
(\mathcal{A}x)'(t) \quad &= 1 - \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(x), ..., x^{[m]}(s)) ds \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(x), ..., x^{[m]}(s)) ds \\
&\geq 1 - \frac{A}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds - \frac{A}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\
&\geq 1 - \frac{AT^{\alpha-1}}{\Gamma(\alpha+1)} (1+\alpha) > 0.
\end{aligned}$$

Consequently $\mathcal{A}x$ is increasing on $[0, T]$, which implies that $0 \leq (\mathcal{A}x)(t) \leq T$, for all $t \in [0, T]$. To complete the proof, it's suffice to show that $\mathcal{A}(S) \subset S$, where S is defined by

$$S = \{\varphi \in X; \quad \|\varphi\| \leq r\},$$

such that $1 + \dfrac{2AT^{\alpha-1}}{\Gamma(\alpha+1)} \leq \dfrac{r}{T}$. We have

$$\begin{aligned}
|\mathcal{A}x(t)| \quad &\leq T + \frac{AT^\alpha}{\Gamma(\alpha+1)} + \frac{AT^\alpha}{\Gamma(\alpha+1)} \\
&\leq T + \frac{2AT^\alpha}{\Gamma(\alpha+1)} \\
&\leq r.
\end{aligned}$$

Which, on taking the taking the norm of $t \in [0, T]$, implies that $\mathcal{A}(S) \subset S$. An application of Schauder's theorem yields a fixed point of the operator $\mathcal{A}$, which compelete the proof.

□

Now, we consider uniqueness of solution of the fractional boundary value problem (1.1). To this end, we need the Banach fixed point to show our second result [19].

Theorem 2.4. *Assume that **(H1)**, **(H2)** hold. Under the condition*

$$\sum_{i=1}^{m} L_i \leq \frac{\Gamma(\alpha+1)}{2T^\alpha},$$

the fractional boundary value problem (1.1) *has a unique solution on* $[0, T]$.

Proof. Since (**H1**) holds, we have $\mathcal{A}(S) \subset S$.

Let $x, y \in X$, $t \in [0,T]$, we have

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| \quad &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), ..., x^{[m]}(s)) - f(s, y(s), ..., y^{[m]}(s))| ds \\
&\quad + \frac{t}{\Gamma(\alpha)T} \int_0^T (T-s)^{\alpha-1} |f(s, x(s), ..., x^{[m]}(s)) - f(s, y(s), ..., y^{[m]}(s))| ds \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^m L_i \|x^{[i]} - y^{[i]}\| + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^m L_i \|x^{[i]} - y^{[i]}\| \\
&\leq \frac{2T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^m L_i \|x - y\|.
\end{aligned}$$

Since $\sum_{i=1}^m L_i \leq \dfrac{\Gamma(\alpha+1)}{2T^\alpha}$, then the operator $\mathcal{A}$ is a contraction mapping.

Hence, we deduce by Banach contraction mapping principle that the operator has a unique fixed point which correspond to a unique solution of problem on $[0,T]$. This complete the proof. □

3. Fractional differential inequalities

In this section, we disscus a fundamental results relative to strict inequalities for the fractional boundary value problem (1.1). We begin by the following lemma which will be needed later.

Lemma 3.1. *[16] Let $m : [0,T] \to \mathbb{R}$ be locally Holder continuous function such that, for any $t_1 \in (0, +\infty)$, we have*

$$m(t_1) = 0, \quad \text{and} \quad m(t) \leq 0, \quad \text{for} \quad t \in [0, t_1].$$

Then it follows that

$$^C D_{0+}^\alpha m(t_1) \geq 0. \tag{3.1}$$

Let us now disscus a fundamental result relative to strict fractional differential inequalities.

Theorem 3.2. *Suppose that $z, y : [0,T] \to \mathbb{R}$ be locally Holder continuous, $f \in C([0,T] \times \mathbb{R} \times ... \times \mathbb{R}, [0,T])$, and*

$$(i) \quad {}^C D_{0+}^\alpha y(t) \leq f(t, y(t), ..., y^{[m]}(t)), \quad (ii) \quad {}^C D_{0+}^\alpha z(t) \geq f(t, z(t), ..., z^{[m]}(t)), \quad t \in [0,T],$$

with one of the inequalities being strict. Then

$$y^{[i]}(0) < z^{[i]}(0), \quad \textit{for all} \quad i = 1, 2, ..., m, \tag{3.2}$$

implies that

$$y^{[i]}(t) < z^{[i]}(t), \quad t \in [0,T], \quad \textit{for all} \quad i = 1, 2, ..., m. \tag{3.3}$$

Proof. Suppose that the conclusion (3.3) is false. Let us suppose that the inequality (ii) is strict. Since z, y be a Holder continuous functions, there exist $t_1 \in [0,T]$, such that $y^{[i]}(t_1) = z^{[i]}(t_1)$ and $y^{[i]}(t) < z^{[i]}(t), 0 \leq t \leq t_1$.

Setting $m(t) = y^{[i]}(t) - z^{[i]}(t), t \in [0, t_1]$, $i = 1, 2, .., m$, then $m(t_1) = 0$ and $m(t) \leq 0, t \in [0, t_1]$. Hence by lemma 3.1, we obtain ${}^C D_{0+}^\alpha m(t_1) \geq 0$, which yields

$$^C D_{0+}^\alpha y^{[i]}(t_1) \geq ^C D_{0+}^\alpha z^{[i]}(t_1), \quad i = 1, 2, ..., m$$

In particular, we have

$$f(t, y(t), ..., y^{[m]}(t)) \geq ^C D_{0+}^\alpha y(t_1) \geq ^C D_{0+}^\alpha z(t_1) > f(t, z(t), ..., z^{[m]}(t)).$$

This is a contradiction, since $y^{[i]}(t_1) = z^{[i]}(t_1)$, $i = 1, 2, ..., m$. Hence the conclusion (3.3) is valid and the proof is complete. □

The next result is for nonstrict fractional differential inequalities which requires a one sided Lipshitz type condition.

Theorem 3.3. *Assume that the conditions of theorem 3.2 hold, with nonstict inequalities. Suppose that*

$$f(t, x_1, ..., x_m) - f(t, y_1, ..., y_m) \leq \sum_{i=1}^{m} L_i \frac{1}{1+t^\alpha}(x_i - y_i),$$

whenever $x_i \geq y_i, i = 1, 2, ..., m$ *and* $\sum_{i=1}^{m} L_i > 0.$

Then

$$y^{[i]}(0) \leq z^{[i]}(0), \quad i = 1, 2, ..., m,$$

implies that, provided $\Gamma(\alpha + 1) \leq \sum_{i=1}^{m} L_i$,

$$y^{[i]}(t) \leq z^{[i]}(t), \quad i = 1, 2, ..., m, \quad \textit{for all} \quad t \in [0, T]. \tag{3.4}$$

Proof. We set $z_\epsilon^{[i]}(t) = z^{[i]}(t) + \epsilon(1 + t^\alpha), i = 1, 2, .., m$, for small $\epsilon > 0$. So that, we have

$$z_\epsilon^{[i]}(0) > z^{[i]}(0), \quad \text{and} \quad z_\epsilon^{[i]}(t) > z^{[i]}(t), \quad i = 1, 2, ..., m, t \in [0, T].$$

In particular, we have $z_\epsilon(t) = z(t) + \epsilon(1 + t^\alpha)$, then

$$\begin{aligned} {}^C D_{0+}^\alpha z_\epsilon(t) &= {}^C D_{0+}^\alpha z(t) + \epsilon {}^C D_{0+}^\alpha (1 + t^\alpha) \\ &\geq f(t, z(t), ..., z^{[m]}(t)) + \epsilon\Gamma(\alpha + 1) \\ &\geq f(t, z_\epsilon(t), ..., z_\epsilon^{[m]}(t)) + \epsilon\Gamma(\alpha + 1) - \sum_{i=1}^{m} L_i \frac{1}{1+t^\alpha}(z_\epsilon - z) \\ &\geq f(t, z_\epsilon(t), ..., z_\epsilon^{[m]}(t)) + \epsilon(\Gamma(\alpha + 1) - \sum_{i=1}^{m} L_i) \\ &> f(t, z_\epsilon(t), ..., z_\epsilon^{[m]}(t)), \quad t \in [0, T]. \end{aligned}$$

Since $z_\epsilon^{[i]}(0) > z^{[i]}(0) \geq y^{[i]}(0)$, hence by application of the theorem 3.2 we get $y(t) < z_\epsilon(t)$ for all $t \in [0, T]$.

By the arbitarariness of $\epsilon > 0$, taking the limit $\epsilon \to 0$, we have $y(t) \leq z(t)$ for all $t \in [0, T]$. This complete the proof. □

4. Existence of maximal and minimal solutions

In this section, we shall prove the existence of maximal and minimal solutions for the fractional boundary value problem (1.1) on $[0, T]$. We need the following definition in what follows.

Definition 4.1. *A solution r of the fractional boundary value problem (1.1) is said to be maximal if for an other solution x of the problem (1.1) one has $x(t) \leq r(t)$, for all $t \in [0, T]$.*

Similarly, a solution ρ of the fractional boundary value problem (1.1) is said to be minimal if for an other solution x of the problem (1.1) one has $x(t) \geq \rho(t)$, for all $t \in [0, T]$.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications.

Given an arbitrarily small real number $\epsilon > 0$, consider the following boundary value problem

$$\begin{cases} {}^C D_{0+}^\alpha x(t) = f(t, x^{[1]}(t), x^{[2]}(t), ..., x^{[m]}(t)) + \epsilon, & t \in]0, T[, \\ x(0) = \epsilon, \quad x(T) = T. \end{cases} \tag{4.1}$$

Where, $x^{[1]}(t) = x(t), x^{[2]}(t) = x(x(t)), ..., x^{[m]}(t) = x(x^{[m]-1}(t))$, $1 < \alpha < 2$, ${}^C D_{0+}^\alpha$ is the Caputo derivative, and $f \in C([0, T] \times \mathbb{R} \times ... \times \mathbb{R}, \mathbb{R})$.

An existence theorem for the problem (4.1) can be stated as follows.

Theorem 4.2. *Assume that (**H1**) holds. Then for every small $\epsilon > 0$ the fractional boundary value problem (4.1) has a solution defined on $[0, T]$.*

Proof. Since
$$\frac{AT^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+1) < 1.$$
Then for every $\epsilon > 0$, we have
$$\frac{AT^{\alpha-1}}{\Gamma(\alpha+1)}((\alpha+1) - \epsilon(1-\alpha)) < 1.$$
Now the rest of the proof is similar to theorem 2.3. □

Our main existence theorem for maximal solution for the fractional boundary value problem (1.1) is the following.

Theorem 4.3. *Assume that (**H1**) and the conditions of theorem 3.2 hold. If*
$$\frac{AT^{\alpha-1}}{\Gamma(\alpha+1)}((\alpha+1) - \epsilon_0(1-\alpha)) < 1.$$
Then the fractional boundary value problem (1.1) has a maximal solution on $[0, T]$.

Proof. Let $\{\epsilon_n\}_{n=0}^{\infty}$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Where ϵ_0 is a positive real number satisfying the inequality
$$\frac{AT^{\alpha-1}}{\Gamma(\alpha+1)}((\alpha+1) - \epsilon_0(1=\alpha)) < 1.$$
By theorem 4.2, there exists a solution $r(t, \epsilon_n)$ defined on $[0, T]$ of the fractional boundary value problem
$$\begin{cases} {}^{C}D_{0+}^{\alpha} = f(t, x^{[1]}(t), x^{[2]}(t), ..., x^{[m]}(t)) + \epsilon_n, & t \in]0, T[, \\ x(0) = \epsilon_n, \quad x(T) = T. \end{cases} \tag{4.2}$$
Then any solution u of the problem (1.1) satisfies
$${}^{C}D_{0+}^{\alpha} u(t) \leq f(t, u(t), ..., u^{[m]}(t)),$$
and any solution of auxiliary problem (4.2) satisfies
$${}^{C}D_{0+}^{\alpha} r(t, \epsilon_n) = f(t, r(t, \epsilon_n), ..., r^{[m]}(t, \epsilon_n)) + \epsilon_n > f(t, r(t, \epsilon_n), ..., r^{[m]}(t, \epsilon_n)).$$
Where $u(0) = 0 < \epsilon_n = r(0, \epsilon_n)$. Then
$$u^{[i]}(0) = 0 < \epsilon_n = r^{[i]}(0, \epsilon_n), \quad i = 1, 2, ..., m.$$

Implies that
$$u(t) \leq r(t, \epsilon_n), \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad t \in [0, T]. \tag{4.3}$$

Since $\epsilon_2 = r(0, \epsilon_2) < \epsilon_1 = r(0, \epsilon_1)$, Then
$$r^{[i]}(0, \epsilon_2) < r^{[i]}(0, \epsilon_1), i = 1, 2, ..., m.$$
Hence by theorem 3.2, we infer that
$$r^{[i]}(t, \epsilon_2) \leq r^{[i]}(t, \epsilon_1), \quad \text{for all} \quad t \in [0, T], i = 1, 2, ..., m.$$
In paricular
$$r(t, \epsilon_2) \leq r(t, \epsilon_1), \quad \text{for all} \quad t \in [0, T].$$

Therefore, $r(t,\epsilon_n)$ is a decreasing sequence of positive real numbers and the limits

$$r(t) = \lim_{n\longrightarrow\infty} r(t,\epsilon_n) \tag{4.4}$$

exists.

Now we show that the convergence in (4.4) is uniform on $[0,T]$. To do, it's enough to show that the sequence $r(t,\epsilon_n)$ is equicontinuous on $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. Then

$$\begin{aligned}|r(t_1,\epsilon_n) - r(t_2,\epsilon_n)| \le & |t_1 - t_2| + \frac{|t_1-t_2|}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,r(s,\epsilon_n),...,r^{[m]}(s,\epsilon_n))|ds \\ & +\epsilon_n\frac{|t_1-t_2|}{\Gamma(\alpha+1)}T^{\alpha-1} + \int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}||f(s,r(s,\epsilon_n),...,r^{[m]}(s,\epsilon_n))|ds \\ & + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,r(s,\epsilon_n),...,r^{[m]}(s,\epsilon_n))|ds \\ \le & t_2 - t_1 + \frac{t_2-t_1}{\Gamma(\alpha)T}\frac{AT^\alpha}{\alpha} + \epsilon_n\frac{t_2-t_1}{\Gamma(\alpha+1)}T^{\alpha-1} + A(\frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)}) + 2A(\frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}).\end{aligned}$$

When $t_2 - t_1 \to 0$, we have $|r(t_1,\epsilon_n) - r(t_2,\epsilon_n)| \to 0$, for all $n \in \mathbb{N}$.

Therefore, $r(t,\epsilon_n) \to r(t)$, uniformaly for all $t \in [0,T]$.

Next we show that $r(t)$ is solution of the fractional boundary value problem (1.1) defined on $[0,T]$.

Since $r(t,\epsilon_n)$ is solution of the fractinal boundary value problem (4.2), we have

$$\begin{aligned}r(t,\epsilon_n) \quad & = t + \epsilon_n - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,r(s,\epsilon_n),...,r^{[m]}(s,\epsilon_n))ds - \epsilon\frac{tT^{\alpha-1}}{\Gamma(\alpha+1)} \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,r(s,\epsilon_n),...,r^{[m]}(s,\epsilon_n))ds.\end{aligned}$$

Since the convergence in (4.4) is uniform, one can obtain that

$$\begin{aligned}r(t) = & t - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,r(s),r^{[2]}(s),...,r^{[m]}(s))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,r(s),r^{[2]}(s),...,r^{[m]}(s))ds.\end{aligned}$$

Which implies that $r(t)$ is solution of (1.1). By using (4.3) we obtaion that $u(t) \le r(t)$, for all $t \in [0,T]$.

Hence the fractional boundary value problem has a maximal solution on $[0,T]$.

□

5. Comparaison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to fractional boundary value problem (1.1). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to our problem on $[0,T]$.

Theorem 5.1. *Assume that **(H1)** holds. Suppose that there exist $L_i > 0, i = 1,2,...,m$, such that*

$$f(t,x_1,...,x_n) - f(t,y_1,...,y_n) < \sum_{i=1}^m \frac{1}{1+t^\alpha}(x_i - y_i),$$

for all $x_i, y_i \in \mathbb{R}$ with $x_i \ge y_i$, for all $i = 1,2,...,m$, and $t \in [0,T]$, where $\sum_{i=1}^m L_i \le \Gamma(1+\alpha)$.

Furthermore, if there exist a function $u \in C([0,T],\mathbb{R})$ such that

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) \le f(t,u(t),...,u^{[m]}(t)), & t \in [0,T], \\ u(0) = 0 \quad u(T) = T. \end{cases}$$

Then

$$u(t) \le r(t)$$

for all $t \in [0,T]$, where $r(t)$ is a maximal solution of tha fractional boundary value problem (1.1).

Proof. Let ϵ be arbitrarily small. By theorem 4.3, $r(t,\epsilon)$ is a maximal solution of the problem (4.1) so that the limit

$$r(t) = \lim_{\epsilon \to 0} r(t,\epsilon),$$

is uniform on $[0,T]$ and the function r is maximal solution of the problem (1.1). Hence

$$\begin{cases} {}^{C}D^{\alpha}_{0+}r(t,\epsilon) = f(t,r(t,\epsilon),...,r^{[m]}(t,\epsilon)) + \epsilon, & t \in [0,T], \\ r(0,\epsilon) = \epsilon \quad r(T,\epsilon) = T. \end{cases}$$

It follows that

$$\begin{cases} {}^{C}D^{\alpha}_{0+}r(t,\epsilon) > f(t,r(t,\epsilon),...,r^{[m]}(t,\epsilon)), & t \in [0,T], \\ r(0,\epsilon) > u(0). \end{cases}$$

Which implies that

$$u^{[i]}(0) < r^{[i]}(0,\epsilon), \quad \text{for all} \quad i = 1,2,...,m.$$

By application of theorem (3.3) we have

$$u^{[i]}(t) < r^{[i]}(t,\epsilon), \quad \text{for all} \quad i = 1,2,..,m.$$

In paricular,

$$u(t) < r(t,\epsilon).$$

We acheve the proof of the theorem by taking $\epsilon \to 0$. □

Theorem 5.2. *Assume that **(H1)** holds. Suppose that there exist $L_i > 0, i = 1,2,...,m$, such that*

$$f(t,x_1,...,x_n) - f(t,y_1,...,y_n) < \sum_{i=1}^{m} \frac{1}{1+t^{\alpha}}(x_i - y_i),$$

for all $x_i, y_i \in \mathbb{R}$ with $x_i \ge y_i$, for all $i = 1,2,...,m$, and $t \in [0,T]$, with $\sum_{i=1}^{m} L_i \le \Gamma(1+\alpha)$.

Furthermore, if there exist a function $v \in C([0,T],\mathbb{R})$ such that

$$\begin{cases} D^{\alpha}v(t) \ge f(t,v(t),...,v^{[m]}(t)), & t \in [0,T], \\ v(0) > 0 \quad v(T) = T. \end{cases}$$

Then

$$\rho(t) \le v(t)$$

for all $t \in [0,T]$, where $\rho(t)$ is a minimal solution of tha fractional boundary value problem (1.1).

Note that the proof is similar and can be obtained with the same arguments with appropriate modifications.

6. An illustrative example

In this section, We present an example to better illustrate our existence and uniqueness results. Consider the fractional iterative differential equation

$$ {}^C D^{\frac{3}{2}} 0^+ x(t) = -\frac{1}{3} - \frac{1}{16}\cos(x(t)) + \frac{1}{20}\sin(x^{[2]}(t)) + \frac{1}{24}\cos(x^{[3]}(t)), \quad t \in [0, \pi], \tag{6.1} $$

associated to the boundary condtions

$$ x(0) = 0, \quad x(\pi) = \pi. \tag{6.2} $$

We have

$$ f(t, x, y, z) = -\frac{1}{3} - \frac{1}{16}\cos(x) + \frac{1}{20}\sin(y) + \frac{1}{24}\cos(z). $$

Then

$$ |f(t, x, y, z)| \le A, \quad \text{where} \quad A = 0,4875. $$

Since $\frac{A\pi^{\alpha-1}}{\Gamma(\alpha+1)} = 0,26 < \frac{1}{\alpha+1} = \frac{2}{5} = 0,4$, then the hypothesis (**H1**) holds. By theorem 2.3, the problem (6.1)-(6.2) has a solution on $[0, T]$.

By the mean value theorem, we have

$$ \begin{aligned} |f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| &\le \frac{1}{16}|x_1 - y_1| + \frac{1}{20}|x_2 - y_2| + \frac{1}{24}|x_3 - y_3| \\ &\le \sum_{i=1}^{3} L_i \|x_i - y_i\|, \end{aligned} $$

where $L_1 = \frac{1}{16}$, $L_2 = \frac{1}{20}$ and $L_3 = \frac{1}{24}$.

Since $\sum_{i=1}^{3} L_i = 0,154166667 \le \frac{\Gamma(\alpha+1)}{2\pi^\alpha} = 0,29841551$. Then all asumptions of theorem (2.4) hold. Hence the fractional boundary value problem (6.1)-(6.2) has a unique solution on $[0, T]$, which is a maximal solution.

7. Conclusion

In this paper, we have looked at fractional boundary value problems in the context of Caputo derivative. Due to Schauder's fixed point and some results of fractional calculus, the existence and uniqueness of solution of our problem are established. Also, we give a necessary condition for the existence of maximal solution.

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