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A Generalization of the Regular Function Modulo n

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ABSTRACT: A new generalization for von Neumann regular elements modulo n (regular elements modulo n) will be defined and studied. Also we survey general properties of the multiplicative function $V(n, m)$ which counts the number of n-regular elements in the ring $\mathbb{Z}_m$.

Key Words: n-regular elements, n-regular elements modulo m, Von Neumann regular function modulo n, summatory function.

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1. Introduction

Throughout this article, all rings will be assumed to be commutative and have unity 1. An element $a \in R$ is called von Neumann regular element if there exists $x \in R$ such that $a^2x = a$. A ring R is called von Neumann regular ring if all its elements are von Neumann regular elements. Von Neumann regular rings (elements) were studied extensively in the literature, see [2], [3] and [5]. In [2], Alkam et al. constructed a number theoretic function $V(n)$ that counts the number of von Neumann regular elements in the ring of integers modulo n, $\mathbb{Z}_n$, and studied it with algebraic tools. Other articles have looked at the function $V(n)$ using number-theoretic techniques, see [4].
Anderson et al. expanded the concept of a von Neumann regular element of a ring R to the concept of (m, n)-von Neumann regular element in [6]. The n-regular elements of the ring R are defined in this article, followed by the function $V(n, m)$, which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Several properties of the function $V(n, m)$ are discussed throughout the article. Finally, we introduce the generalization $F_n(m)$ of the function $F(m)$, then we relate it with the divisor function $\sigma(m)$.

2. n-regular elements of a ring

The n-regular elements of a ring R are defined in this section. Anderson et al. introduced the concept of an (m, n)-von Neumann regular element of a ring R in [6].

Definition 2.1. *Let R be a ring and let m, n be two positive integers. An element $a \in R$ is said to be (m, n)-von Neumann regular element (in short $(m, n) - vnr$) if there exists $b \in R$ such that $a^m b = a^n$.*

This article will focus on a special case of this definition, namely when $m = n + 1$, and we will call to the $(n + 1, n) - vnr$ element as the n-regular element. Each von Neumann regular element is clearly an n-regular element. Note: In [6], the term "n-regular" has been defined in a way that differs from ours.

Example 2.2. *In $\mathbb{Z}_4$, 2 is not a regular element, while 2 is a 2-regular element.*

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3. n-regular elements of the ring $\mathbb{Z}_m$

The arithmetic function $V(m)$, which counts the number of von Neumann regular elements in the ring $\mathbb{Z}_m$, was examined by Alkam et al. in [2] and To'th in [4]. In this section, we will generalize the function $V(m)$ to the function $V(n,m)$ which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Since each von Neumann regular element is an n-regular element, $V(m) \leq V(n,m)$.
The following definition separates the nilpotent elements of R, $Nil(R)$, into subsets each of certain nilpotincy.

Definition 3.1. *Let R be a ring and let n be a positive integer. Then $Nil_n(R) = \{x \in R : x^n = 0\}$. It is clear that $Nil(R) = \bigcup_{n\in\mathbb{N}} Nil_n(R)$.*

Lemma 3.2. *Let R be a local ring with maximal ideal M. Then the set of n-regular elements of R is $U(R)\bigcup Nil_n(R)$.*

Proof. Suppose that a is an n-regular element of the local ring R. Then there exists $b \in R$ such that $a^{n+1}b = a^n$. Hence either $a^n \in M$ or $a^n \notin M$. If $a^n \in M$, then $a \in M$ and a must be in $Nil_n(R)$. If $a^n \notin M$, then $a \in U(R)$. It is clear that $U(R)$ and $Nil_n(R)$ are subsets of the n-regular elements. □

Note that, if $x \in Nil_n(R)$, then $x^k = 0$ for any $k \geq n$.
The proof of the following theorem is straightforward by using Lemma [3.2] although we include the proof for the sake of completeness.

Theorem 3.3. *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative local rings and let n be a positive integer. Then the element $(x_\alpha)_{\alpha\in\Lambda} \in \{R_\alpha\}_{\alpha\in\Lambda}$ is an n-regular element if and only if for each $\alpha \in \Lambda$ either $x_\alpha \in U(R_\alpha)$ or $x_\alpha \in Nil_n(R_\alpha)$.*

Proof. $(x_\alpha)_{\alpha\in\Lambda}$ is n-regular element iff
there is $(y_\alpha)_{\alpha\in\Lambda}$ such that $(x_\alpha)^{n+1}_{\alpha\in\Lambda}(y_\alpha)_{\alpha\in\Lambda} = (x_\alpha)^n_{\alpha\in\Lambda}$ iff
$(x_\alpha^{n+1})_{\alpha\in\Lambda}(y_\alpha)_{\alpha\in\Lambda} = (x_\alpha^n)_{\alpha\in\Lambda}$ iff
$(x_\alpha^{n+1}y_\alpha)_{\alpha\in\Lambda} = (x_\alpha^n)_{\alpha\in\Lambda}$ iff
$x_\alpha^{n+1}y_\alpha = x_\alpha^n$ for each $\alpha \in \Lambda$ iff
x_α is n-regular element in R_α for each $\alpha \in \Lambda$ iff
for each $\alpha \in \Lambda$ either $x_\alpha \in U(R_\alpha)$ or $x_\alpha \in Nil_n(R_\alpha)$. □

It is known that for any prime number p and any positive integer α, $Nil(\mathbb{Z}_{p^\alpha}) = \langle\overline{p}\rangle = \overline{p}\mathbb{Z}_{p^\alpha} \cong \mathbb{Z}_{p^{\alpha-1}}$. Similarly, $\langle\overline{p}^2\rangle = \overline{p}\langle\overline{p}\rangle \cong \overline{p}\mathbb{Z}_{p^{\alpha-1}} \cong \mathbb{Z}_{p^{\alpha-2}}$. In general, for any integer k such that $1 \leq k \leq \alpha$, $\langle\overline{p}^k\rangle \cong \mathbb{Z}_{p^{\alpha-k}}$.
Now, We can use Lemma [3.2] to prove the following lemma

Lemma 3.4. *Let n be a positive integer. Then for any prime number p and any positive integer α, $V(n,p^\alpha) = \phi(p^\alpha) + p^{\alpha-\lceil\frac{\alpha}{n}\rceil}$.*

Proof. Since $\mathbb{Z}_{p^\alpha}$ is a local ring with maximal ideal $Nil(\mathbb{Z}_{p^\alpha}) = \langle\overline{p}\rangle$, $V(n,p^\alpha) = |U(\mathbb{Z}_{p^\alpha})| + |Nil_n(\mathbb{Z}_{p^\alpha})| = \phi(p^\alpha) + |\langle\overline{p}^{\lceil\frac{\alpha}{n}\rceil}\rangle| = \phi(p^\alpha) + p^{\alpha-\lceil\frac{\alpha}{n}\rceil}$. □

The following result is based on Lemma [3.4], where it highlights some special cases for n.

Corollary 3.5. *Let n be a positive integer. Then for any prime number p and any positive integer $\alpha \leq n$, $V(n,p^\alpha) = p^\alpha$.*

It is a well known fact that if the standard prime factorization of the positive integer m is $m = \prod_{i=1}^t p_i^{\alpha_i}$, then $\mathbb{Z}_m \cong \prod_{i=1}^t \mathbb{Z}_{p_i^{\alpha_i}}$. Hence by Theorem [3.3], we deduce that $V(n,m) = \prod_{i=1}^t V(n,p_i^{\alpha_i})$. Thus, $V(n,m)$ is multiplicative function with respect to m(that is, if m_1 and m_2 are relatively prime, then $V(n,m_1m_2) = V(n,m_1)V(n,m_2)$).
Some properties of the multiplicative function $V(n,m)$ are presented here. To prove the following theorem, we use some of the conclusions from [2] and [4]. Recall that, if $a|b$ and $gcd(a,\frac{b}{a}) = 1$, then a is said to be a unitary divisor of b, denoted by $a||b$.

Theorem 3.6. *Let n be a positive integer and let m be a positive integer with the standard prime factorization $m = \prod_{i=1}^{t} p_i^{\alpha_i}$, also let $k = \prod_{i=1}^{t} p_i^{\lceil \frac{\alpha_i}{n} \rceil}$. Then*

1. *If $n \geq \max\{\alpha_i\}_{i=1}^{t}$, then $V(n,m) = m$.*
2. *If $n \leq \min\{\alpha_i\}_{i=1}^{t}$, then $V(n,m) = \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}]$. As a special case, $V(m) = V(1,m) = \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + 1]$.*
3. *$V(n,m) = \frac{m}{k} V(k)$.*
4. *$V(n,m) = \sum\limits_{d||k} \frac{m}{k}\phi(d)$.*
5. *$V(n,m)$ is increasing with respect to both n and m (that is, if $n_1 \leq n_2$, then $V(n_1,m) \leq V(n_2,m)$, and if $m_1 \leq m_2$, then $V(n,m_1) \leq V(n,m_2)$).*
6. *$\frac{V(n,m)}{\phi(m)} = \sum\limits_{d||k} \frac{1}{\phi(d)}$.*

Proof. Since $V(n,m)$ is multiplicative in m, we can deduce (1) and (2) using Lemma [3.4]. To prove (3),

$$\begin{aligned} V(n,m) &= \prod_{i=1}^{t}[\phi(p_i^{\alpha_i}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}] \\ &= \prod_{i=1}^{t}[(p_i^{\alpha_i} - p_i^{\alpha_i - 1}) + p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}] \\ &= \prod_{i=1}^{t} p_i^{\alpha_i - \lceil \frac{\alpha_i}{n} \rceil}[(p_i^{\lceil \frac{\alpha_i}{n} \rceil} - p_i^{\lceil \frac{\alpha_i}{n} \rceil - 1}) + 1] \\ &= \frac{m}{k} V(k). \end{aligned}$$

To prove (4), you can use (3) and the property $V(k) = \sum\limits_{d||k} \phi(d)$ that is found in [2] and [4].

The proof of (5) is straightforward.

To prove (6), combine (3) and the fact, if $m = \prod_{i=1}^{t} p_i^{\alpha_i}$ and $n = \prod_{i=1}^{t} p_i^{\beta_i}$ are positive integers, then $\frac{\phi(m)}{\phi(n)} = \frac{m}{n}$, as well as the property $\frac{V(k)}{\phi(k)} = \sum\limits_{d||k} \frac{1}{\phi(d)}$ that is found in [2] and [4].

□

Example 3.7. *Take $m = 2^4.3^3$ and $n = 3$, then $k = 2^2.3$ and $\phi(m) = 144$. Thus, $V(k) = 9$ and $V(3,m) = 324 = \frac{2^4.3^3}{2^2.3}.9$.*
Also, the values of d such that $d||k$ are $1, 3, 4$ and 12. So, $\sum\limits_{d||k} \frac{1}{\phi(d)} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{9}{4} = \frac{324}{144}$.

4. Sum of n-regular elements modulo m

In this section, we will use the number-theoretic consideration to prove some results concerning the n-regular elements modulo m. Firstly, consider $Reg(n,m) = \{a \in \mathbb{Z} : 1 \leq a \leq m, a\, is\, n-regular\, (mod\, m)\}$ denotes the set of all n-regular elements modulo m. Then $V(n,m) = |Reg(n,m)|$.

Theorem 4.1. $V(n,m) = \sum\limits_{\substack{d||m \\ t||k \\ d\,and\,t\,have \\ common\,prime \\ divisors}} \frac{d}{t}\phi(\frac{m}{d})$

Proof. Let $y_i = \frac{p_i^{\alpha_i}}{p_i^{\lceil \frac{\alpha_i}{n} \rceil}\phi(p_i^{\alpha_i})}$, $1 \le i \le t$, and $y = \prod_{i=1}^{t} y_i = \frac{m}{k\phi(m)}$. Then

$$\begin{aligned}\phi(m)[y + \sum_{1\le i\le t} \frac{y}{y_i} + \sum_{1\le i<j\le t} \frac{y}{y_i y_j} + \cdots + \frac{y}{\prod_{i=1}^{t} y_i}] &= \prod_{i=1}^{t} \phi(p_i^{\alpha})(y_i+1) \\ &= \prod_{i=1}^{t} V(n, p_i^{\alpha}) \\ &= V(n,m).\end{aligned}$$

□

Tóth in [4] gave a formula for the sum of regular elements $(mod\, m)$, $S(m) = \frac{m(V(m)+1)}{2}$, Tóth's formula was analogous to the formula $\sum_{\substack{1\le a\le m \\ gcd(a,m)=1}} a = \frac{m\phi(m)}{2}$.

The following theorem gives a formula for $S(n,m)$, which is analogous to $S(m) = \frac{m(V(m)+1)}{2}$.

Theorem 4.2. *For the positive integers n and m, $S(n,m) = \frac{m(V(n,m)+1)}{2}$.*

Proof. $S(n,m) = \sum_{a\in Reg(n,m)} a$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{(a,m)=d \\ (a,k)=t}} a$$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{(\frac{a}{d},\frac{m}{d})=1 \\ (a,k)=t}} a$$

$$= \sum_{\substack{d||m \\ t||k}} \sum_{\substack{j=1 \\ (j,\frac{m}{d})=1 \\ (a,k)=t}}^{\frac{m}{d}} jd$$

$$= \sum_{\substack{d||m \\ t||k}} t \sum_{i=1}^{\frac{d}{t}} i \left(\sum_{\substack{j=1 \\ (j,\frac{m}{d})=1 \\ (a,k)=t}}^{\frac{m}{d}} j \right)$$

$$= k(1+2+3+\ldots+\tfrac{m}{k}) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} t \sum_{i=1}^{\frac{d}{t}} i \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= k(\tfrac{m}{k}+1)\tfrac{m}{2k} + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} ti \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= \tfrac{m}{2}(\tfrac{m}{k}+1) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} d \left(\tfrac{m}{2d}\phi(\tfrac{m}{d})\right)$$

$$= \tfrac{m}{2} \left((\tfrac{m}{k}+1) + \sum_{\substack{d||m \\ t||k \\ d<m \\ t<k}} \sum_{i=1}^{\frac{d}{t}} \phi(\tfrac{m}{d}) \right)$$

$$= \frac{m}{2}\left(1 + \sum_{\substack{d||m \\ t||k}} \phi(\tfrac{m}{d})\tfrac{d}{t}\right)$$
$$= \frac{m}{2}\left(1 + V(n,m)\right)$$ □

Example 4.3. *For* $m = 16$, $n = 3$, $V(n,m) = 12$ *and* $R(n,m) = \{1,3,4,5,7,8,9,11,12,13,15,16\}$. *Then* $\sum_{a \in Reg(n,m)} a = 104 = \frac{16}{2}(1+12)$.

5. The summatory function $F_n(m)$

The summatory function $F(m) = \sum_{d|m} V(d)$ is calculated in [2]. In this section, we will calculate $F_n(m)$, the generalized form of the summatory function $F(m)$. Let $F_n(m) = \sum_{d|m} V(n,d)$. Since $V(n,m)$ is multiplicative concerning m, $F_n(m)$ is also multiplicative concerning m, hence $F_n(m)$ is completely characterized by its values on powers of primes. Recall that the functions $\sigma(n)$ and $\sigma_k(n)$ are defined as the sum, or the sum of k-th powers, of the divisors of n respictively.

Theorem 5.1. *Let p be a prime number and let α and n be positive integers. Then*

1. *If $\alpha \leq n$, then $F_n(p^\alpha) = \sigma(p^\alpha)$.*
2. *If $\alpha > n$, then $F_n(p^\alpha) = p^\alpha + 1 + \sigma(p^{n-1})\sigma_{n-1}(p^{q-1}) + p^{q(n-1)}\sigma(p^{r-1})$, where q, r are the quotient and remainder when we divide α by n.*

Proof. 1. In Corollary [3.5] if $\alpha \leq n$, then $V(n,p^\alpha) = p^\alpha$. Hence, the result is clear.

2. If $\alpha > n$, then $F_n(p^\alpha) = \sum_{d|p^\alpha} V(n,d)$

$$= \sum_{k=0}^{\alpha} V(n,p^k)$$
$$= \sum_{k=0}^{\alpha} \phi(p^k) + p^{k-\lceil \frac{k}{n} \rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{\alpha} p^{k-\lceil \frac{k}{n} \rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{n} p^{k-\lceil \frac{k}{n} \rceil} + \sum_{k=n+1}^{2n} p^{k-\lceil \frac{k}{n} \rceil} + \sum_{k=2n+1}^{3n} p^{k-\lceil \frac{k}{n} \rceil} + \cdots + \sum_{k=(q-1)n+1}^{qn} p^{k-\lceil \frac{k}{n} \rceil} + \sum_{k=qn+1}^{\alpha} p^{k-\lceil \frac{k}{n} \rceil}$$
$$= p^\alpha + 1 + \sum_{k=1}^{n} p^{k-1} + \sum_{k=n+1}^{2n} p^{k-2} + \sum_{k=2n+1}^{3n} p^{k-3} + \cdots + \sum_{k=(q-1)n+1}^{qn} p^{k-q} + \sum_{k=qn+1}^{\alpha} p^{k-(q+1)}$$
$$= p^\alpha + 1 + \left(\sum_{k=0}^{n-1} p^k\right)\left[1 + p^{n-1} + p^{2(n-1)} + \cdots + p^{(q-1)(n-1)}\right] + p^{q(n-1)} \sum_{k=0}^{r-1} p^k$$
$$= p^\alpha + 1 + \sigma(p^{n-1})\sigma_{n-1}(p^{q-1}) + p^{q(n-1)}\sigma(p^{r-1}).$$
□

.

6. Conclusions

This article defines the n-regular elements of the ring R, as well as the function $V(n,m)$, which counts the number of n-regular elements in the ring $\mathbb{Z}_m$. Throughout the article, we have established that the set of n-regular elements of the ring R is $U(R)\bigcup Nil_n(R)$. Also, we show that the function $V(n,m)$ is multiplicative. The sum of the n-regular elements modulo m is calculated using number theoretic considerations. Also, we introduce the summatory function $F_n(m) = \sum_{d|m} V(n,d)$ and we find the relationship between $F_n(m)$ and the divisor function $\sigma(m)$.

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