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Besov-Hankel norms in terms of the continuous Bessel wavelet transform

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ABSTRACT: Using the theory of Continuous Bessel wavelet transform in $L^p(\mathbb{R})$-spaces, we established the Parseval and inversion formulas for the $L^{p,\sigma}(\mathbb{R}^+)$- spaces. We investigate the continuity and boundedness properties of the Bessel wavelet transform in Besov-Hankel space. Our main results: are the characterization of Besov-Hankel space by using the Bessel wavelet coefficient.

Key Words: Besov-Hankel space, continuous Bessel wavelet transform, Hankel transform, Hankel convolution.

Contents

1. Introduction and Preliminaries

The Besov space $B^q_\alpha(L_p)$ is a set of functions f from L_p which have smoothness α. The parameter q gives a finer gradation of smoothness. Betancor and Rodriguez-Mesa extend the definition of Besov space with the help of Hankel transform and its properties to obtain a new function space called Besov-Hankel space. After that they were characterized the Besov-Hannkel spaces through the Bochner-Riesz mean and the partial Hankel integrals (see [1]- [7], [12]). Perrier and Basdevant [8] established the characterization of Besov spaces using continuous wavelet transform. Motivated by these two works, we characterized Besov-Hankel spaces by continuous Bessel wavelet transform.
The present paper is organized in the following manner: Section 1 is introductory, in which we recall some basic definitions and results like the Hankel transform and its properties, Besov-Hankel space, and continuous Bessel wavelet transform. In section 2, we derived Parseval's and Inversion formula for the continuous Bessel wavelet transform in $L^{p,\sigma}(\mathbb{R}^+)$. Additionally, we obtained continuity and boundedness properties of the Bessel wavelet transform in Besov-Hankel spaces. Section 3 pertains to the characterization of Besov-Hankel norms in terms of the continuous Bessel wavelet transform.
In this paper, as usual $L^{p,\sigma}(\mathbb{R}^+ = (0,\infty))$ denotes the weighted L^p- space with norm

$$\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} = \|f\|_{p,\sigma} = \left(\int_0^\infty |f(x)|^p d\sigma(x)\right)^{\frac{1}{p}}, (1 \le p < \infty), \tag{1.1}$$

$$\|f\|_{L^{\infty,\sigma}(\mathbb{R}^+)} = \text{ess sup}_{0<x<\infty}|f(x)| < \infty. \tag{1.2}$$

The Hankel transformation of the function $f \in L^{1,\sigma}(\mathbb{R}^+)$ is defined by

$$\hat{f}(x) = \int_0^\infty j(xt)f(t)d\sigma(t), \quad 0 \le x < \infty, \tag{1.3}$$

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where $\sigma(t) = \frac{t^{2\nu+1}}{2^{\nu+\frac{1}{2}}\Gamma(\nu+\frac{3}{2})}$, $j(x) = C_\nu x^{\frac{1}{2}-\nu} J_{\nu-\frac{1}{2}}(x)$, $C_\nu = 2^{\nu+\frac{1}{2}}\Gamma(\nu+\frac{1}{2})$ and $J_{\nu-\frac{1}{2}}$ denote the Bessel function of first kind of order $\nu - \frac{1}{2}$.
If $\hat{f} \in L^{1,\sigma}(\mathbb{R}^+)$ and $f \in L^{1,\sigma}(\mathbb{R}^+)$, then the inverse of Hankel transformation is given by

$$f(x) = \int_0^\infty j(xt)\hat{f}(t)d\sigma(t), \quad 0 < x < \infty. \tag{1.4}$$

Also, Parseval's formula of the Hankel transformation for $f, g \in L^{1,\sigma}(\mathbb{R}^+) \bigcap L^{2,\sigma}(\mathbb{R}^+)$ is given by

$$\int_0^\infty \hat{f}(x)\hat{g}(x)d\sigma(x) = \int_0^\infty f(u)g(u)d\sigma(u). \tag{1.5}$$

By denseness and continuity the Parseval's formula can be extended to all $f, g \in L^{2,\sigma}(\mathbb{R}^+)$. Hence Hankel transform is isometry on $L^{2,\sigma}(\mathbb{R}^+)$.

If $f, g \in L^{1,\sigma}(\mathbb{R}^+)$, then the convolution associated with the Hankel is defined as

$$(f\#g)(x) = \int_0^\infty f(x,y)g(y)d\sigma(y), \tag{1.6}$$

where the Hankel translation is given by

$$f(x,y) = \tau_y f(x) =: \int_0^\infty f(z)D(x,y,z)d\sigma(z), \;\; 0 < x, y < \infty, \tag{1.7}$$

and

$$\begin{aligned} D(x,y,z) &= \int_0^\infty j(xu)j(yu)j(zu)d\sigma(u) \\ &= 2^{3\nu-\frac{5}{2}}[\Gamma(\nu+\frac{1}{2})]^2 \left(\Gamma(\nu)\pi^{\frac{1}{2}}\right)^{-1}(xyz)^{-2\nu-1}[\Delta(xyz)]^{2\nu-2}, \end{aligned} \tag{1.8}$$

where $\Delta(x,y,z)$ denotes the area of a triangle. $D(x,y,z)$ is symmetric in x, y, z (see [13]).
From (1.4) and (1.8), we have

$$\int_0^\infty j(zu)D(x,y,z)d\sigma(z) = j(xu)j(yu), \;\; 0 < x, y < \infty, \;\; 0 \le u < \infty, \tag{1.9}$$

for $u = 0$, we get

$$\int_0^\infty D(x,y,z)d\sigma(z) = 1, \tag{1.10}$$

and

$$(\widehat{f\#g})(x) = \hat{f}(x)\hat{g}(x), \;\; 0 \le x < \infty. \tag{1.11}$$

Now, we recall some properties of Hankel convolution which are useful throughout the paper (see [10], [2], [12], [9]).

Lemma 1.1 *If $f \in L^{p,\sigma}(\mathbb{R}^+)$ for $1 \leq p < \infty$ then*

$$||\tau_y f(x)||_{L^{p,\sigma}(\mathbb{R}^+)} \leq ||f||_{L^{p,\sigma}(\mathbb{R}^+)}. \tag{1.12}$$

Lemma 1.2 *Let $f \in L^{p,\sigma}(\mathbb{R}^+)$ and $g \in L^{q,\sigma}(\mathbb{R}^+)$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then we have*

$$||f\#g||_{L^{r,\sigma}(\mathbb{R}^+)} \leq ||f||_{L^{p,\sigma}(\mathbb{R}^+)}||g||_{L^{q,\sigma}(\mathbb{R}^+)}. \tag{1.13}$$

Definition 1.1 (Besov-Hankel Space): Let $0 < \alpha < 1$ and $1 \leq p < \infty$. A mesurable function ϕ on $(0, \infty)$ belongs to $BH^{p,q}_{\alpha,\sigma}$ if $\phi \in L^{p,\sigma}(\mathbb{R}^+)$ and

$$\int_0^\infty \left(h^{-\alpha} w_p(\phi)(h)\right)^q \frac{dh}{h} < \infty \text{ for } 1 \leq q < \infty, \tag{1.14}$$

$$esssup_{h>0} \left(h^{-\alpha} w_p(\phi)(h)\right) < \infty \text{ for } q = \infty, \tag{1.15}$$

where $w_{p,\sigma}(\phi)(h) =: \|\tau_h \phi - \phi\|_{L^{p,\sigma}(\mathbb{R}^+)}$, $h \in (0, \infty)$.

Definition 1.2 (Bessel Wavelet): Using the properites of Hankel transform, Pathak and Dixit (see [2], [11], [14]) define the continuous Bessel wavelet for $\psi \in L^{p,\sigma}(\mathbb{R}^+)$, $1 \leq p < \infty$, $b \geq 0$ and $a > 0$ as

$$\begin{aligned} \psi_{b,a}(x) : &= D_a \tau_b \psi(x) \\ &= a^{-2\nu-1} \int_0^\infty \psi(z) D\left(\frac{b}{a}, \frac{x}{a}, z\right) d\sigma(z), \end{aligned} \tag{1.16}$$

where D_a denote the dilation operator.

The continuous Bessel wavelet transform of $f \in L^{2,\sigma}(\mathbb{R}^+)$ with respect to a wavelet $\psi \in L^{2,\sigma}(\mathbb{R}^+)$ is defined as

$$\begin{aligned} (B_\psi f)(b,a) &= \int_0^\infty f(x)\overline{\psi_{b,a}}(x) d\sigma(x) \\ &= a^{-2\nu-1} \int_0^\infty \int_0^\infty f(x)\overline{\psi(z)} D\left(\frac{b}{a}, \frac{x}{a}, z\right) d\sigma(z) d\sigma(x). \end{aligned} \tag{1.17}$$

Moreover , using (1.6), we have

$$(B_\psi f)(b,a) = (f\#\psi_a)(b), \tag{1.18}$$

where $\psi_a(t) = a^{-2\nu-1}\overline{\psi(t/a)}$.

2. The Continuous Bessel Wavelet Transform in $L^{p,\sigma}(\mathbb{R}^+)$

Theorem 2.1 *Suppose that a wavelet $\psi \in L^{1,\sigma}(\mathbb{R}^+) \cap L^{2,\sigma}(\mathbb{R}^+)$ satisfies the admissibility condition*

$$A_\psi = \int_0^\infty \omega^{-2\nu-1} |\hat{\psi}(\omega)|^2 d\omega > 0,$$

where $\hat{\psi}$ denote the Hankel transform of ψ then continuous Bessel wavelet transform is a bounded linear operator

$$L^{p,\sigma}(\mathbb{R}^+) \to L^{2,\sigma}(\mathbb{R}^+, \frac{d\sigma(a)}{a^{2\nu+1}}) \times L^{p,\sigma}(\mathbb{R}^+),$$

moreover, for any $f \in L^{p,\sigma}(\mathbb{R}^+)$ *and* $1 < p < \infty$,

$$\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} \approx \left(\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{p}{2}} d\sigma(b)\right)^{\frac{1}{p}}. \tag{2.1}$$

Proof: Let S_p denote the space $L^{2,\sigma}(\mathbb{R}^+, \frac{d\sigma(a)}{a^{2\nu+1}}) \times L^{p,\sigma}(\mathbb{R}^+)$ associated to the norm

$$\|f\|_{S_p} = \left\{\int_0^\infty \left(\int_0^\infty |f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{p}{2}} d\sigma(b)\right\}^{\frac{1}{p}}.$$

If we take $p = 2$, then from Plancherel's theorem:

$$\begin{aligned} \|B_\psi f\|_{S_2} &= \left\{\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}}\right) d\sigma(b)\right\}^{\frac{1}{2}} \\ \|B_\psi f\|_{S_2} &= \sqrt{A_\psi}\|f\|_{L^{2,\sigma}}, \end{aligned}$$

where $A_\psi = \int_0^\infty \omega^{-2\nu-1}|\hat{\psi}(\omega)|^2 d\omega > 0$, if ψ is real. From singular integral theorem, the operators on $L^{2,\sigma}(\mathbb{R}^+, \frac{d\sigma(a)}{a^{2\nu+1}})$ holds inequality:

$$\|B_\psi f\|_{S_p} \leq C_p \|f\|_{L^{p,\sigma}(\mathbb{R}^+)} \text{ for } 1 < p \leq 2,$$

where the constant C_p depends only on p and ψ(see [15]). Due to duality the inequality is also valid for $1 < p < \infty$. It follows that

$$\left\{\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^2 \frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{p}{2}} d\sigma(b)\right\}^{\frac{1}{p}} \leq C_p \|f\|_{L^{p,\sigma}(\mathbb{R}^+)} \tag{2.2}$$

conversely suppose that $f \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{p,\sigma}(\mathbb{R}^+)$. Since continuous Bessel wavelet tranform is isomerty for every $g \in L^{2,\sigma}(\mathbb{R}^+) \cap L^{q,\sigma}(\mathbb{R}^+)$, we can write

$$\begin{aligned} \int_0^\infty \int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)} a^{-2\nu-1} d\sigma(a) d\sigma(b) &= A_\psi \langle f, g\rangle \\ \frac{1}{A_\psi}\int_0^\infty \int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)} a^{-2\nu-1} d\sigma(a) d\sigma(b) &= \int_0^\infty f(x)\overline{g(x)} d\sigma(x). \end{aligned} \tag{2.3}$$

Now,

$$\begin{aligned} \left|\int_0^\infty f(x)g(x)d\sigma(x)\right| &= \frac{1}{A_\psi}\left|\int_0^\infty \int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)} a^{-2\nu-1} d\sigma(a) d\sigma(b)\right| \\ &\leq \frac{1}{A_\psi}\int_0^\infty \int_0^\infty |B_\psi f(b,a)\overline{B_\psi g(b,a)} a^{-2\nu-1} d\sigma(a) d\sigma(b)|, \end{aligned}$$

using Schwarz inequality and then Holder's inequality, we have

$$\begin{aligned} \leq \ & \frac{1}{A_\psi}\left(\int_0^\infty \left(\int_0^\infty |B_\psi f(b,a)|^2 a^{-2\nu-1} d\sigma(a)\right)^{\frac{p}{2}} d\sigma(b)\right)^{\frac{1}{p}} \\ & \times \left(\int_0^\infty \left(\int_0^\infty |B_\psi g(b,a)|^2 a^{-2\nu-1} d\sigma(a)\right)^{\frac{q}{2}} d\sigma(b)\right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.
From equation (2.2), we get

$$\leq \frac{A_q}{A_\psi}\left(\int_0^\infty\left(\int_0^\infty |B_\psi f(b,a)|^2 a^{-2\nu-1}d\sigma(a)\right)^{\frac{p}{2}}d\sigma(b)\right)^{\frac{1}{p}}\|g\|_{L^{q,\sigma}(\mathbb{R}^+)},$$

where A_q is a constant depends on q and ψ only. Applying Density theorem on the above equation

$$\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} \leq A\left(\int_0^\infty\left(\int_0^\infty |B_\psi f(b,a)|^2 a^{-2\nu-1}d\sigma(a)\right)^{\frac{p}{2}}d\sigma(b)\right)^{\frac{1}{p}},$$

where $A=\frac{A_q}{A_\psi}$. This gives the required result of the theorem. □

2.1. Parseval's formula

Theorem 2.2 *Let us assume that $\phi_1\in L^{p,\sigma}(\mathbb{R}^+)$, $\phi_2\in L^{q,\sigma}(\mathbb{R}^+)$ with $1\leq p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. If ψ is a real wavelet then*

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty B_\psi\phi_1(b,a)\overline{B_\psi\phi_2(b,a)}a^{-2\nu-1}d\sigma(a)d\sigma(b) = \int_0^\infty \phi_1(x)\overline{\phi_2(x)}d\sigma(x), \tag{2.4}$$

where $A_\psi=\int_0^\infty \omega^{-2\nu-1}|\hat{\psi}(\omega)|^2d\omega>0$ and $\hat{\psi}$ denotes the Hankel transform.

Proof: Let us define bilinear transform $T: L^{p,\sigma}(\mathbb{R}^+)\times L^{q,\sigma}(\mathbb{R}^+)\to\mathbb{R}^+$ by

$$T(\phi_1,\phi_2)=\langle B_\psi\phi_1(b,a),B_\psi\phi_2(b,a)\rangle_{\left(\frac{d\sigma(a)}{a^{2\nu+1}},d\sigma(b)\right)}.$$

Now, applying Holder's inequality two times then

$$\begin{aligned}|T(\phi_1,\phi_2)| &= |\langle B_\psi\phi_1(b,a),B_\psi\phi_2(b,a)\rangle_{\left(\frac{d\sigma(a)}{a^{2\nu+1}},d\sigma(b)\right)}|\\ &\leq \int_0^\infty\left(\int_0^\infty|B_\psi\phi_1(b,a)|^2\frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{1}{2}}\left(\int_0^\infty|B_\psi\phi_2(b,a)|^2\frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{1}{2}}d\sigma(b)\\ &\leq\left(\int_0^\infty\left(\int_0^\infty|B_\psi\phi_1(b,a)|^2\frac{d\sigma(a)}{a^{2\nu+1}}\right)^{\frac{p}{2}}d\sigma(b)\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^\infty\left(\int_0^\infty|B_\psi\phi_2(b,a)|^2\frac{d\sigma(b)}{b^{2\nu+1}}\right)^{\frac{q}{2}}d\sigma(b)\right)^{\frac{1}{q}}\end{aligned}$$

from Theorem 2.1.

$$|T(\phi_1,\phi_2)|\leq C\|\phi_1\|_{L^{p,\sigma}(\mathbb{R}_+)}\|\phi_2\|_{L^{q,\sigma}(\mathbb{R}_+)}. \tag{2.5}$$

Moreover for all $\phi_1\in L^{2,\sigma}(\mathbb{R}^+)\cap L^{p,\sigma}(\mathbb{R}^+)$ and $\phi_2\in L^{2,\sigma}(\mathbb{R}^+)\cap L^{q,\sigma}(\mathbb{R}^+)$, we get

$$T(\phi_1,\phi_2)=\langle B_\psi\phi_1(b,a),B_\psi\phi_2(b,a)\rangle_{\left(\frac{d\sigma(a)}{a^{2\nu+1}},d\sigma(b)\right)}=A_\psi\langle\phi_1,\phi_2\rangle. \tag{2.6}$$

From equations (2.5), (2.6) and denseness of spaces the $L^{2,\sigma}(\mathbb{R}^+)\cap L^{p,\sigma}(\mathbb{R}^+)$ in $L^{p,\sigma}(\mathbb{R}^+)$ obtained required result. □

2.2. An inversion formula

Theorem 2.3 *Let us consider $\phi \in L^{p,\sigma}(\mathbb{R}^+)$ with $1 < p < \infty$ and ψ is a real wavelet. Then*

$$\phi(x) = \frac{1}{A_\psi} \int_0^\infty \int_0^\infty B_\psi \phi(b,a) \psi_{b,a}(x) \frac{d\sigma(a)}{a^{2\nu+1}} d\sigma(b). \tag{2.7}$$

The equality holds in $L^{p,\sigma}(\mathbb{R}^+)$ sense and the integral of right hand side have to be taken in the sense of distributions.

Proof: The proof followed from Theorem 2.2. □

Theorem 2.4 *Suppose $\psi \in L^{1,\sigma}(\mathbb{R}^+)$, then for $a > 0$, the operator $B_\psi : BH^{p,q}_{\alpha,\sigma} \to BH^{p,q}_{\alpha,\sigma}$ is continuous. Moreover, $||(B_\psi f)(b,a)||_{BH^{p,q}_{\alpha,\sigma}} \leq ||\psi||_{L^{1,\sigma}(\mathbb{R}^+)} ||f||_{BH^{p,q}_{\alpha,\sigma}}$.*

Proof: By the definition of continuous Bessel wavelet transform, we have

$$\begin{aligned}
(B_\psi f)(b,a) &= \int_0^\infty f(x)\overline{\psi_{b,a}(x)} d\sigma(x) \\
&= \int_0^\infty f(x) \left(\int_0^\infty D\left(\frac{b}{a}, \frac{x}{a}, z\right) \overline{\psi(z)} a^{-2\nu-1} d\sigma(z) \right) d\sigma(x) \\
&= \int_0^\infty \overline{\psi(z)} \left(\int_0^\infty D(b, x, az) f(x) d\sigma(x) \right) d\sigma(z) \\
&= \left\{ \int_0^\infty (\tau_{az} f)(b) \overline{\psi(z)} d\sigma(z) \right\}
\end{aligned} \tag{2.8}$$

Taking $L^{p,\sigma}(\mathbb{R}^+)$- norm and then using Minkowski's integral inequality, we have

$$\begin{aligned}
\|(B_\psi f)(b,a)\|_{L^{p,\sigma}(\mathbb{R}^+)} &= \| \int_0^\infty (\tau_{az} f)(b) \overline{\psi(z)} d\sigma(z) \|_{L^{p,\sigma}(\mathbb{R}^+)} \\
&\leq \int_0^\infty |\overline{\psi(z)}| d\sigma(z) \left(\int_0^\infty |(\tau_{az} f)(b)|^p d\sigma(b) \right)^{\frac{1}{p}} \\
&= \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)} \|\tau_{az} f\|_{L^{p,\sigma}(\mathbb{R}^+)} \\
&\leq \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)} \|f\|_{L^{p,\sigma}(\mathbb{R}^+)}
\end{aligned} \tag{2.9}$$

Using Minkowski's integral inequality and (2.8), we have

$$\begin{aligned}
\omega_{p,\sigma}(B_\psi f, h) &= \left(\int_0^\infty |\tau_h (B_\psi f)(b,a) - (B_\psi f)(b,a)|^p d\sigma(b) \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty |\tau_h \int_0^\infty (\tau_{az} f)(b) \overline{\psi(z)} d\sigma(z) - \int_0^\infty (\tau_{az} f)(b) \overline{\psi(z)} d\sigma(z)|^p d\sigma(b) \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty | \int_0^\infty (\tau_{az} \tau_h f)(b) \overline{\psi(z)} d\sigma(z) - \int_0^\infty (\tau_{az} f)(b) \overline{\psi(z)} d\sigma(z)|^p d\sigma(b) \right)^{\frac{1}{p}} \\
&\leq \int_0^\infty |\overline{\psi(z)}| d\sigma(z) \left(\int_0^\infty |\tau_{az} \tau_h f(b) - \tau_{az} f(b)|^p d\sigma(b) \right)^{\frac{1}{p}} \\
&= \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)} \|\tau_{az}(\tau_h f - f)\|_{L^{p,\sigma}(\mathbb{R}^+)} \\
&\leq \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)} \omega_{p,\sigma}(f, h)
\end{aligned} \tag{2.10}$$

Therefore, using (2.9) for $q<\infty$ we have the following estimate

$$\begin{aligned}
\|(B_\psi f)(b,a)\|_{BH^{p,q}_{\alpha,\sigma}} &= \|(B_\psi f)(b,a)\|_{L^{p,\sigma}(\mathbb{R}^+)} + \left(\int_0^\infty (h^{-\alpha}\omega_{p,\sigma}(B_\psi f,h))^q \frac{d\sigma(h)}{h}\right)^{\frac{1}{q}}\\
&\le \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)}\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} + \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)}\left(\int_0^\infty (h^{-\alpha}\omega_{p,\sigma}(f,h))^q \frac{d\sigma(h)}{h}\right)^{\frac{1}{q}}\\
&= \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)}\left(\|f\|_{L^{p,\sigma}(\mathbb{R}^+)} + \left(\int_0^\infty (h^{-\alpha}\omega_{p,\sigma}(f,h))^q \frac{dh}{h}\right)^{\frac{1}{q}}\right)\\
&= \|\psi\|_{L^{1,\sigma}(\mathbb{R}^+)}\|f\|_{BH^{p,q}_{\alpha,\sigma}}.
\end{aligned}$$

This complete the proof of theorem. □

3. Characterization of Besov-Hankel space

Theorem 3.1 *Let $f\in BH^{p,q}_{\alpha,\sigma}(\mathbb{R}^+)$ $(p,q>1,\alpha\notin\mathbb{Z})$ and analysing wavelet ψ has $[\alpha]+1$ cancellations and $(z^{\alpha-[\alpha]}\psi)\in L^{1,\sigma}(\mathbb{R}^+)$, then the wavelet coefficient of function f holds following conditions:*

$$\begin{array}{ll}
\text{if } q<\infty, & \int_0^\infty \left[a^{-\alpha}\|B_\psi(.,a)\|_{L^{p,\sigma}(\mathbb{R}^+)}\right]^q \frac{da}{a}<\infty,\\
\text{if } q=\infty, & a\to a^{-\alpha}\|B_\psi(.,a)\|_{L^{p,\sigma}(\mathbb{R}^+)}\in L^\infty(\mathbb{R}^+).
\end{array}$$

Moreover the function $a\to a^{-\alpha}\|B_\psi(.,a)\|_{L^{p,\sigma}(\mathbb{R}^+)}\in L^q(\mathbb{R}^+,\frac{da}{a})$ and we have:

$$\|a^{-\alpha}\|B_\psi(.,a)\|_{L^{p,\sigma}(\mathbb{R}^+)}\|_{L^{q,\sigma}(\mathbb{R}^+,\frac{da}{a})} \le \|z^{\alpha-[\alpha]}\psi\|_{L^{1,\sigma}(\mathbb{R}^+)}\times\|h^{\alpha-[\alpha]}\omega_{p,\sigma}(f,h)\|_{L^{q,\sigma}((\mathbb{R}^+,\frac{dh}{h})} \tag{3.1}$$

Proof: By the definition of continuous Bessel wavelet transform, we have

$$\begin{aligned}
B_\psi(b,a) &= \int_0^\infty f(x)\overline{\psi_{b,a}(x)}d\sigma(x)\\
&= \int_0^\infty f(x)\left(\int_0^\infty a^{-2\nu-1}D\left(\frac{b}{a},\frac{x}{a},z\right)\overline{\psi(z)}d\sigma(z)\right)d\sigma(x)\\
&= \int_0^\infty \overline{\psi(z)}\left(\int_0^\infty a^{-2\nu-1}D\left(\frac{b}{a},\frac{x}{a},z\right)f(x)d\sigma(x)\right)d\sigma(z)\\
&= \int_0^\infty \overline{\psi(z)}\left(\int_0^\infty D\left(b,x,az\right)f(x)d\sigma(x)\right)d\sigma(z)\\
&= \left\{\int_0^\infty (\tau_{az}f)(b)\overline{\psi(z)}d\sigma(z) - \int_0^\infty f(b)\overline{\psi(z)}d\sigma(z)\right\}\\
&= \int_0^\infty \overline{\psi(z)}\left((\tau_{az}f)(b)-f(b)\right)d\sigma(z).
\end{aligned}$$

Taking $L^{p,\sigma}(\mathbb{R}^+)-$ norm of the wavelet coefficient

$$\|B_\psi(b,a)\|_{L^{p,\sigma}(\mathbb{R}^+)} = \int_0^\infty\left\{|\int_0^\infty \overline{\psi(z)}\left((\tau_{az}f)(b)-f(b)\right)d\sigma(z)|^p\right\}^{\frac{1}{p}}d\sigma(b).$$

Using Minkowski inequality of integrability for $p\ne\infty$

$$\|B_\psi(b,a)\|_{L^{p,\sigma}(\mathbb{R}^+)} \le \int_0^\infty\left\{\int_0^\infty |(\tau_{az}f)(b)-f(b)|d\sigma(b)|^p\right\}^{\frac{1}{p}}|\psi(z)|d\sigma(z). \tag{3.2}$$

Suppose that $q<\infty$ and integrating w.r.t. a, we get

$$\int_0^\infty \left[a^{-\alpha}\|B_\psi(b,a)\|_{L^{p,\sigma}}\right]^q\frac{da}{a} \le \int_0^\infty\left[a^{-\alpha}\int_0^\infty |\psi(z)|\omega_p(f,az)d\sigma(z)\right]^q\frac{da}{a}.$$

Again using Minkowski integrabilty inequality

$$\int_0^\infty \left[a^{-\alpha}\|B_\psi(b,a)\|_{L^{p,\sigma}}\right]^q \frac{da}{a} \leq \left[\int_0^\infty |\psi(z)|d\sigma(z)\left\{\int_0^\infty \left(a^{-\alpha}\omega_p(f,az)\right)^q \frac{da}{a}\right\}^{\frac{1}{q}}\right]^q.$$

Applying change of variable $h = az$

$$\begin{aligned}
&= \left[\int_0^\infty z^\alpha|\psi(z)|d\sigma(z)\left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\sigma}(f,h)\right)^q \frac{dh}{h}\right\}^{\frac{1}{q}}\right]^q \\
&\leq \left\{\int_0^\infty z^\alpha|\psi(z)|d\sigma(z)\right\}^q \times \left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\sigma}(f,h)\right)^q \frac{dh}{h}\right\} \\
&= \left\{\int_0^\infty |z^\alpha\psi(z)|d\sigma(z)\right\}^q \times \left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\sigma}(f,h)\right)^q \frac{dh}{h}\right\} \\
&< \infty.
\end{aligned} \tag{3.3}$$

If $q = \infty$ the hypothesis on f says that $h^{-\alpha}\omega_{p,\sigma}(f,h) \in L^\infty(\mathbb{R}^+)$, so

$$\|B_\psi(b,a)\|_{L^{p,\sigma}(\mathbb{R}^+)} \leq a^\alpha\|h^{-\alpha}\omega_{p,\sigma}(f,h)\|_{L^\infty(\mathbb{R}^+)}\int_0^\infty |z^\alpha\psi(z)|d\sigma(z). \tag{3.4}$$

The theorem has been proved for $0 < \alpha < 1$. □

The converse of Theorem 3.1. is true, indeed the behavior at small scales of the Bessel wavelet coefficients characterizes Besov-Hankel spaces.

Theorem 3.2 *Suppose $\alpha > 0$, α not an integer and a function ψ is a real $C^{[\alpha]+1}$- regular analysing wavelet with all derivatives rapidly decreasing. If $f, f', f'', f''', ..., f^{[}\alpha] \in L^{p,\sigma}(\mathbb{R}^+)$ $(1 < p < \infty)$, and if $a^{-\alpha}\|(B_\psi f)(a,\cdot)\|_{L^{p,\sigma}(\mathbb{R}^+)} \in L^{q,\sigma}(\mathbb{R}^+, \frac{da}{a})$ $(1 \leq q < \infty)$, then $f \in BH^{p,q}_{\alpha,\sigma}$ and we have*

$$\begin{aligned}
\|h^{-(\alpha-[\alpha]}w_{p,\sigma}(f^{([\alpha])},h)\|_{L^{q,\sigma}(\mathbb{R}^+)} &\leq \frac{1}{A_\psi}\left(\frac{2}{(\alpha-[\alpha])}\|\psi^{[\alpha]}\|_{L^{1,\sigma}(\mathbb{R}^+)} + \frac{1}{1-(\alpha-[\alpha])}\|\psi^{[\alpha]+1}\|_{L^{1,\sigma}(\mathbb{R}^+)}\right) \\
&\quad \times\|a^{-\alpha}\|(B_\psi f)(a,\cdot)\|_{L^{p,\sigma}(\mathbb{R}^+)}\|_{L^{q,\sigma}(\mathbb{R}^+,\frac{da}{a})}
\end{aligned} \tag{3.5}$$

Proof: Let $f \in L^{p,\sigma}(\mathbb{R}^+)$. By inversion formula of Bessel wavelet transform

$$f(x) = \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)\psi_{a,b}(x)d\sigma(b) \tag{3.6}$$

and

$$\tau_h f(x) = \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)\tau_h\psi_{a,b}(x)d\sigma(b). \tag{3.7}$$

Then

$$\begin{aligned}
\tau_h f(x) - f(x) &= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)\left\{\tau_h\psi_{a,b}(x) - \psi_{a,b}(x)\right\}d\sigma(b) \\
&= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)\left\{\tau_{\frac{h}{a}}\tau_{\frac{b}{a}}\psi(\frac{x}{a}) - \tau_{\frac{b}{a}}\psi(\frac{x}{a})\right\}d\sigma(b) \\
&= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)a^{-2\nu-1}D(\frac{b}{a},\frac{x}{a},y)d\sigma(b) \\
&\qquad \times\int_0^\infty \left\{\tau_{\frac{h}{a}}\psi(y) - \psi(y)\right\}d\sigma(y) \\
&= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\tau_{\frac{y}{a}}(B_\psi f)(a,\frac{x}{a})\int_0^\infty \left\{\tau_{\frac{h}{a}}\psi(y) - \psi(y)\right\}d\sigma(y)
\end{aligned}$$

Taking $L^{p,\sigma}$- norm on both side, we have

$$w_{p,\sigma}(f,h) = \frac{1}{A_\psi}\left\{\int_0^\infty |\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\tau_{\frac{y}{a}}(B_\psi f)(a,\frac{x}{a})\int_0^\infty \left\{\tau_{\frac{h}{a}}\psi(y)-\psi(y)\right\}d\sigma(y)|^p d\sigma(x)\right\}^{\frac{1}{p}}$$

applying Minkowski's inequality

$$\le \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\left\{\int_0^\infty |\tau_{\frac{y}{a}}(B_\psi f)(a,\frac{x}{a})|^p d\sigma(x)\right\}^{\frac{1}{p}}$$

$$= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(t)}{t^{2\nu}}\|(B_\psi f)(\frac{h}{t},\cdot)\|_{L^{p,\sigma}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)$$

Now,consider $0<\alpha<1$ and using Minkowski's inequality

$$\begin{aligned}\left\{\int_0^\infty \frac{dh}{h}h^{-\alpha q}w_{p,\sigma}(f,h)^q\right\}^{\frac{1}{q}} &\le \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(t)}{t^{2\nu}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\\ &\quad\times\left\{\int_0^\infty \frac{dh}{h}h^{-\alpha q}\|(B_\psi f)(\frac{h}{t},\cdot)\|^q_{L^{p,\sigma}}\right\}^{\frac{1}{q}}\\ &= \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(t)}{t^{2\nu+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\\ &\quad\times\left\{\int_0^\infty \frac{da}{a}a^{-\alpha q}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}}\right\}^{\frac{1}{q}}\\ &= \frac{C}{A_\psi}\int_0^\infty \frac{dt}{t^{1+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\\ &\quad\times\left\{\int_0^\infty \frac{da}{a}a^{-\alpha q}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}}\right\}^{\frac{1}{q}}.\end{aligned}$$

Using Lemma 1.1, we obtain

$$\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\le 2\|\psi\|_{L^{1,\sigma}} \tag{3.8}$$

and

$$\begin{aligned}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y) &= \int_0^\infty |\int_0^t \frac{d}{dz}\tau_z\psi(y)dz|d\sigma(y)\\ &= \int_0^\infty |\int_0^t (\tau_z\psi(y))'\,dz|d\sigma(y)\\ &\le \int_0^t\int_0^\infty |(\tau_z\psi(y))'\,dz|d\sigma(y)\\ &\le \|\psi'\|_{L^{1,\sigma}}t.\end{aligned} \tag{3.9}$$

Here, we observe that

$$\begin{aligned}\int_0^\infty \frac{dt}{t^{1+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y) &= \int_0^1 \frac{dt}{t^{1+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\\ &\quad+\int_1^\infty \frac{dt}{t^{1+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y)-\psi(y)|d\sigma(y)\\ &\le 2\|\psi\|_{L^{1,\sigma}}\int_1^\infty \frac{dt}{t^{1+\alpha}}+2\|\psi'\|_{L^{1,\sigma}}\int_0^1 \frac{dt}{t^\alpha}\end{aligned}$$

$$= \frac{2}{\alpha}\|\psi\|_{L^{1,\sigma}} + \frac{1}{1-\alpha}\|\psi'\|_{L^{1,\sigma}},$$

which proved the result for $0 < \alpha < 1$.
If $1 < \alpha < 2$, by the hypothesis $f' \in L^p(\mathbb{R}^+)$ and ψ is a C^1- regular function with ψ' rapidly decreasing at infinity. From equations (3.6) and (3.7),we have the equality

$$\tau_h f'(x) - f'(x) = \frac{1}{A_\psi}\int_0^\infty \frac{d\sigma(a)}{a^{2\nu+1}}\int_0^\infty (B_\psi f)(a,b)\left\{\tau_h \psi'_{a,b}(x) - \psi'_{a,b}(x)\right\} d\sigma(b)$$

Calculate in similar manner as above for f' gives the following estimation

$$\begin{aligned}\left\{\int_0^\infty \frac{dh}{h} h^{-(\alpha-1)q} w_{p,\sigma}(f,h)^q\right\}^{\frac{1}{q}} &\leq \frac{1}{A_\psi}\int_0^\infty \frac{dt}{t^{1+\alpha}}\int_0^\infty |\tau_{\frac{h}{a}}\psi(y) - \psi(y)| d\sigma(y) \\ &\quad \times \left\{\int_0^\infty \frac{dh}{h}(a)^{-\alpha q}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}}\right\}^{\frac{1}{q}} \\ &\leq \frac{1}{A_\psi}\left(\frac{2}{(\alpha-1)}\|\psi'\|_{L^{1,\sigma}} + \frac{1}{1-(\alpha-1)}\|\psi''\|_{L^{1,\sigma}}\right) \\ &\quad \times \left\{\int_0^\infty \frac{dh}{h} a^{-\alpha q}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}}\right\}^{\frac{1}{q}}\end{aligned}$$

this proves that $f' \in BH^{p,q}_{\alpha-1,\sigma}$ the hypothesis $a \to a^{-\alpha}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}} \in L^{q,\sigma}(\mathbb{R}^+)$ implies $a \to a^{-(\alpha-1)}\|(B_\psi f)(a,\cdot)\|^q_{L^{p,\sigma}} \in L^{q,\sigma}(\mathbb{R}^+)$ then $f \in BH^{p,q}_{\alpha-1,\sigma}$. The theorem is established for $1 < \alpha < 2$, a recurrence on $[\alpha]$ gives the final result. □

Remark 3.1 Further research in the context of different types of Besov spaces related to the different integral transform is needed.

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References

1. Pathak, R.S., *Integral Transforms of Generalized Functions and Their Applications*, Gordon and Breach Science Publishers, UK, (1997).
2. Pathak, R. S.and Dixit, M. M., *Continuous and discrete Bessel wavelet transforms*, J. Comput. Appl. Math. 160 (1-2), 241-250, (2003).
3. Pathak, R. S., Upadhyay, S. K. and Pandey, R. S., *The Bessel wavelet convolution product*, Rend. Semin. Mat. Univ. Politec. Torino 69 (3), 267-279, (2011).
4. Pathak, R. S.and Upadhyay, S. K. *L^μ_p-boundedness of the pseudo-differential operator associated with the Bessel operator*, J. Math. Anal. Appl. 257 (1), 141-153, (2001).
5. Pathak, Ashish and Pandey, Shrish,*Besov-type spaces associated with Lebedev-Skalskaya wavelet transform*, Math. Methods Appl. Sci. 46 , no. 14, 15626–15640, (2023).
6. Pathak, Ashish and Pandey, Shrish, *Kontorovich-Lebedev wavelet transform on Besov type spaces*, Integral Transforms Spec. Funct. 34, no. 9, 659–674, (2023)
7. Pathak, Ashish and Pandey, Shrish, *Besov-type spaces for the k-Hankel wavelet transform on the real line* , Concr. Oper. 8, no. 1, 114–124, (2021).
8. Perrier, Valérie; Basdevant, Claude, *Besov norms in terms of the continuous wavelet transform. Application to structure functions*, Math. Models Methods Appl. Sci. 6 (5), 649-664, (1996).
9. Upadhyay, S. K. and Singh, Reshma, *Integrability of the continuum Bessel wavelet kernel*, Int. J. Wavelets Multiresolut. Inf. Process. 13 (5), 1550032, (2015).

10. Hirschman, I. I., Jr. *Variation diminishing Hankel transforms*, J. Analyse Math. 8, 307-336, (1960).

11. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, (1958).

12. Betancor, Jorge J., Rodríguez-Mesa, L., *On the Besov-Hankel spaces*, J. Math. Soc. Japan 50 (3), 781-788, (1998).

13. Haimo, D.T. , *Integral equations associated with Hankel convolutions*, Trans. Amer. Math. Soc., 116, 330-375, (1965).

14. Trime'che, K., *Generalized Wavelet and Hypergroup Gordon and Breach*, Amsterdam,(1997).

15. Stein, E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, (1993).

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