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Non-Superfluous Intersection Graph of Ideals of a Ring

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ABSTRACT: Let R be a commutative ring with unity. The non-superfluous intersection graph of ideals of R, denoted by $G(R)$, is the graph whose vertex set is the collection of non-trivial ideals of R and any two vertices, say I and J, are adjacent if and only if $I \cap J$ is non-superfluous. The connectedness of $G(R)$ is studied in this paper. The notions of clique, colourability, independence number, and domination number are also established. Eventually, we initiate the concept of traversability and planarity in $G(Z_n)$. The principal part of this paper is to point out the role of the non-superfluous intersection graph in Z_n.

Key Words: Non-superfluous intersection graph, ring, connected graph, Jacobson radical.

Contents

1. Introduction

For the last two decades, tremendous importance has been given to the study of algebraic structures by associating graphs, which has led to conclusive results and opened up new dimensions for further research. Istvan Beck [8], in 1988, initiated the notion of zero-divisor graph, which influenced many researchers to conduct studies in this area. This paper is motivated by the profound concept of the intersection graph of ideals of rings, which was discussed by Chakrabarty *et al*. [9]. Later, this concept was studied by Akbari *et al*. [1,2,3,4] and was further carried out by Rajkhowa and Saikia [15] and many other researchers, giving different insights into this notion [10,17].

In order to follow up, some basic definitions and notations are collected from already existing literature. Throughout this paper, all graphs G with a vertex set $V(G)$ are undirected, unless otherwise mentioned. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path, we define $d(x, y) = \infty$. The maximum distance among the distances between all pairs of vertices of G is termed as the diameter of G, denoted by $diam(G)$. A graph G is said to be complete if any two vertices are adjacent in G. A complete subgraph of G is called a clique. A maximum clique of G is a clique with the largest number of vertices, and the number of vertices of a maximum clique is called the clique number of G, and it is denoted by $\omega(G)$. The degree of a vertex v in a graph G is the number of edges incident with v, denoted by $deg(G)$. A walk in G is an alternating sequence of vertices and edges beginning and ending with vertices, in which each edge is incident with immediate preceding and succeeding vertices. A closed walk has the same first and last vertices. A path is a walk in which all the vertices are distinct. The graph G is connected if there is a path between every two distinct vertices and is disconnected if it is not connected. On the other hand, a totally disconnected graph G does not contain any edge. A circuit is a closed walk with all vertices distinct (except the first and last). The length of a circuit is nothing but the number of edges in the circuit. The length of the smallest circuit of G is called the girth of G, denoted by $girth(G)$. An independent set of G is a set of vertices of G which are not mutually adjacent. A maximum independent set of G is an independent set with the largest number of vertices, and the number of vertices of a maximum independent set is called the independence number

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of G, denoted by $\alpha(G)$. A subset S of V is called a dominating set if every vertex in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set in G. An Eulerian circuit is a closed walk that contains all the edges of G exactly once. A graph G is said to be Eulerian if it contains an Eulerian circuit. Any undefined terms are available in [7,13].

Now we define some already existing concepts related to ring theory [5,12,11,16]. Throughout this paper, R is a commutative ring with unity. The Jacobson radical of R, denoted by $J(R)$, is the intersection of all maximal ideals of R. An ideal I of R is said to be superfluous if for some ideal J of R, $I+J=R$ implies $J=R$. If the ideal is not superfluous, then it is non-superfluous. Let R be a ring and I, J be two ideals of R. If M is a maximal ideal of R, then $I \cap J \subseteq M$ implies $I \subseteq M$ or $J \subseteq M$. For R, we have $Rad(R) = \sum L$, L is superfluous in R.

In this paper, the concept of non-superfluous intersection graph of ideals of a ring is introduced. The non-superfluous intersection graph of R denoted by $G(R)$ is an undirected graph having vertex set as the collection of non-trivial ideals of R and any two vertices, say I and J, are adjacent if $I \cap J$ is non-superfluous. In the study of $G(R)$, Jacobson radical and maximal ideals play vital role to set up the relationship between graph theoretic and ring theoretic properties. The important area of this study is based on the ring Z_n and further generalisation. In Section 2, an attempt has been made to study the features of connectedness in the non-superfluous intersection graph $G(Z_n)$ and then pursue the same for finite and arbitrary ring. In that context, the results on diameter and girth are also obtained. In Section 3, some results on clique, colorability, independence number and domination number of $G(Z_n)$ are established. Finally, in Section 4, we conclude our results by giving insight into traversability and planarity.

2. Connectedness in $G(R)$

In this section, the characteristics of connectedness of non-superfluous intersection graphs of non-trivial ideals of R are discussed. We basically point out the role of the non-superfluous intersection graph of $G(Z_n)$. Since a field contains exactly two trivial ideals, the non-superfluous intersection graph $G(R)$ is a null graph if R is considered as a field. Thus, we conclude that R is not a field. An observation is made that the non-superfluous intersection graph $G(Z_n)$ is totally disconnected if $n = 6, 8, 10, 15, 16, \cdots$, and connected if $n = 4, 30, 210, \cdots$. This urges the following theorem.

Theorem 2.1 *$G(Z_n)$ is totally disconnected if and only if $n = pq$ or $n = p^m, m > 2$.*

Proof: Let $n = pq$ or $n = p^m$, $m > 2$.
Case I: If $n = pq$ where p and q are distinct primes. Then Z_n has non-trivial ideals (p) and (q). Therefore, $(p) \cap (q) = (0)$, which implies that (p) and (q) are not adjacent. Thus, $G(Z_n)$ is totally disconnected.

Case II: If $n = p^m, m > 2$. Then the non-trivial ideals of Z_n are (p^{m-1}), (p^{m-2}),..., (p^2), (p). We have $(p^{m-1}) \subset (p^{m-2}) \subset ... \subset (p^{i+1}) \subset (p^i) \ldots \subset (p^2) \subset (p)$. Taking any arbitrary ideal $A = (p^i)$ and $B = (p^{i+1})$, then $A \cap B = B$ and it is easy to see that $A + B = (p^i, p^{i+1}) = (p^i) \neq (1) = Z_n$. So, A is not adjacent to B. Thus, $G(Z_n)$ is totally disconnected.

Conversely, we assume that $G(Z_n)$ is totally disconnected. Let $n = p_1 p_2 \ldots p_i,\ i > 2$ and p_i's be all primes, but not necessarily distinct. Let $I = (p_1)$, $J = (m)$, where $m|n$ and $m < n$. So, if $p_1|m$ then $(m) \subset (p_1)$, and this implies $(m) \cap (p_1) = (m)$. Then there exists an ideal (q) such that $(m, q) = 1$ so, $(m) + (q) = (1) = Z_n$. Thus, I and J are adjacent. Since we get at least one path between two vertices. Hence the graph is not totally disconnected. Next, for $i = 2$, $p_1 = p_2 = p$ the graph $G(Z_{p^2})$ contains a single vertex, and so connected. So, the only options left here are for $i > 2$ with $p_1 = p_2 = \cdots = p_i$, that is, $n = p^i, i > 2$ and $n = pq, p \neq q$.

Theorem 2.2 *$G(Z_n)$ is connected if and only if either $n = p_1 p_2 \ldots p_i$ where p_i's are all distinct primes, $i > 2$ or $n = p^2$.*

Proof: Let $n = p_1 p_2 \dots p_i$ where p_i's are distinct primes, $i > 2$. Let $I = (p_1)$ and J be any vertex of $G(Z_n)$ which is different from I. Then, we get $J = (m), m|n$ and $m < n$.

Case I: If $p_1|m$ then $(m) \subset (p_1)$. So $(m) \cap (p_1) = (m)$. It is easy to see that there exists an ideal $k = (q)$ such that $(m, q) = 1$ which implies $(m) + (q) = (m, q) = (1) = Z_n$. Thus, I and J are adjacent.

Case II: If $p_1 \nmid m$ then for any prime factor q of m, $(p_1, q) = 1$. Let $K = (q)$ then $(q) \cap (p_1) = (p_1 q)$. Again, it is easy to see that there exists an ideal (r) such that $(p_1 q, r) = 1$. This implies $(p_1 q) + (r) = (p_1 q, r) = (1) = Z_n$. Thus, I and K are adjacent. But $q|m$ and so $(m) \subset (q)$. So $(m) \cap (q) = (m)$ which implies that there exists an ideal (s) such that $(m, s) = 1$. Therefore, $(m) + (s) = (m, s) = (1) = Z_n$. Thus, $I - K - J$ is a path from I to J. Therefore, if we consider two vertices J_1 and J_2 different from I then from the above discussion we see both J_1 and J_2 are connected to I. Again, if $n = p^2$ then only non-trivial ideal is (p). Since vertex set contains only a single vertex, so the graph is connected.

Conversely, let $G(Z_n)$ be connected. By Theorem 2.1, for $n = p^m, m > 2$ and $n = pq, p \neq q$, the graph is totally disconnected. Let $n = p_1 p_2 \dots p_i$ where p_i's are primes but not necessarily distinct. Let $I = (p_1)$ and $J = (m)$ be any ideal where $m|n, m < n$. Let $p_1|m \Rightarrow m = p_1 q$ where $q = p_2 p_3 \dots p_i$. If $q \neq p_1$ then $m = p_1 q$. Now $(p_1 q) \cap (p_1) = (p_1 q)$. Then for any ideal (r) of $Z_n, (p_1 q, r) \neq 1$. Thus $(p_1 q) + (r) = gcd(p_1 q, r) \neq (1) = Z_n$. Hence I is not adjacent to J. So, the graph is disconnected. Thus, the only options left for connectedness are $n = p^2$ and $n = p_1 p_2 \dots p_i$ where p_i's are distinct primes.

Lemma 2.1 *Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_i^{k_i}$. Then $(p_1^{l_1} p_2^{l_2} p_3^{l_3} \dots p_s^{l_s})$, where at least one $l_i < k_i$, is an isolated vertex.*

Proof: The result trivially follows.

Remark 2.1 *Every isolated vertex in $G(Z_n)$ is superfluous ideal and conversely. This can be extended to $G(R)$.*

Theorem 2.3 *$G(Z_n)$ is a disjoint union of complete subgraphs if and only if $n = p^r q^m$.*

Proof: Let $n = p^r q^m$.
The number of vertices of $G(Z_n)$ is $(r+1)(m+1) - 2 = (r+m) + (rm-1)$. Here,

$$(p^r) \subseteq (p^{r-1}) \subseteq \dots \subseteq (p). \tag{2.1}$$

For any arbitrary ideal (p^i) its intersection with any ideal in the sequence (2.1) is a subset of (p) say (p^k) and we can easily find an ideal $(q^j),\ j = 1, 2, \dots, m$ such that $(p^k) + (q^j) = Z_n$. Hence, any two ideals in sequence (2.1) are adjacent. Thus, we get a complete graph of order r. Similarly, we can show that the ideals in the sequence $(q^m) \subseteq (q^{m-1}) \subseteq \dots \subseteq (q)$ are mutually adjacent, and so we get a complete graph of order m. Again, since the rest of $rm - 1$ number of non-trivial ideals are of the form $(p^{l_1} q^{l_2})$ where at least one $l_i < r$ or $m, i \in \{1, 2\}$, so by Lemma 2.3, these are isolated vertices. Hence, $G(Z_n)$ is a disjoint union of complete subgraphs.

Conversely, we assume that $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_s^{k_s}$.

Case I: If $n = p_1^{k_1}$ then from Theorem 2.1 for $k_1 = 2$ we get one isolated vertex and for $k_1 > 2$ the graph is totally disconnected.

Case II: If $n = p_1 p_2 \dots p_s$ from Theorem 2.2, we get the graph is connected but not complete and hence a contradiction.

Case III: If $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_s^{k_s},\ s > 2$, then any arbitrary ideal $(p_1^{l_1} p_2^{l_2} p_3^{l_3} \dots p_s^{l_s})$, where at least one $l_i < k_i$, is an isolated vertex by Lemma 2.3. For any arbitrary ideal $(p_1^{k_1})$ we have $(p_1^{k_1})$ is not adjacent to $(p_2^{k_2} p_3^{k_3} \dots p_s^{k_s})$ as its intersection gives us $(p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_s^{k_s})$. So the graph $G(Z_n)$ does not have any complete disjoint components. Hence, only option left is $n = p^r q^m$.

Remark 2.2 *If we consider $n = p^2qr$, then we get a disconnected graph with two components, where one component includes (pqr) as an isolated vertex, and the other component is not complete.*

Theorem 2.4 *Let I and J be two ideals of a finite ring $R \cong F_{p_1} \times F_{p_2} \times F_{p_3} \times \ldots F_{p_m}$. Then $I \cap J$ is superfluous if and only if $I \cap J = (0, 0, \ldots, 0)$.*

Proof: Let $I \cap J$ be superfluous. Then, either $I \cap J = (0, 0, \ldots, 0)$ or $I \cap J \neq (0, 0, \ldots, 0)$. If $I \cap J = (0, 0, \ldots, 0)$, then we are done. But if $I \cap J \neq (0, 0, \ldots, 0)$, then as the non-trivial ideals are a collection of $m-$tuples, $m > 2$, so there always exists another $m-$tuple such that its sum gives the whole ring. Hence, we get a non-superfluous ideal, a contradiction. The converse case trivially follows.

Theorem 2.5 *For a finite ring R, $G(R)$ is connected if and only if $R \cong F_{p_1} \times F_{p_2} \times F_{p_3} \times \ldots F_{p_m}$, where $m > 2$.*

Proof: Let $G(R)$ be connected. Let $R \ncong F_{p_1} \times F_{p_2} \times F_{p_3} \times \ldots F_{p_m}$, where $m > 2$. Consider that $R \cong F_{p_1} \times F_{p_2}$. Here the non-trivial ideals are of the form $A = (F_{p_1}, 0)$, $B = (0, F_{p_2})$. The number of vertices of $G(R)$ is $2^m - 2 = 2^2 - 2 = 2$. Now, $(F_{p_1}, 0) \cap (0, F_{p_2}) = (0, 0)$, which is superfluous. Hence, A and B are not adjacent. So, the graph is totally disconnected, a contradiction. Hence, $R \cong F_{p_1} \times F_{p_2} \times F_{p_3} \times \ldots F_{p_m}$.

Conversely, let $R \cong F_{p_1} \times F_{p_2} \times F_{p_3} \times \ldots F_{p_m}$. Let I and J be two non-trivial ideals, such that I and J are not adjacent. So $I \cap J$ is superfluous and by Theorem 2.7, $I \cap J = (0, 0, \ldots, 0)$. Let K be any non-zero ideal. If $K \subseteq I$ then $K \cap I = K \neq (0, 0, \ldots, 0)$. Thus $K \cap I$ is non-superfluous. Therefore K and I are not adjacent. If $K \nsubseteq I$ then $K \cap I = P$, say. If $P \neq (0, 0, \ldots, 0)$ then K and I are adjacent. If $P = (0, 0, \ldots, 0)$ then $K \cap I$ is superfluous. But the ideals are a collection of m-tuples, so it is obvious that there always exists another ideal, say A, such that $A \nsubseteq I$ and $A \cap I \neq (0, 0, \ldots, 0)$. Thus, K and I are adjacent. Similarly, K and J are adjacent. Therefore, $I - K - J$ is a path. Hence, $G(R)$ is connected.

Theorem 2.6 *There exists an edge in $G(Z_n)$ if and only if there exist three non-trivial ideals in $G(Z_n)$ such that $|Max(Z_n)| \geq 2$.*

Proof: Let there exists an edge in $G(Z_n$. If there is only one non-trivial ideal, then it is an isolated vertex, and so we cannot get an edge. Next, for exactly two non-trivial ideals, we have the two cases for $n = pq$ and $n = p^3$. In either case from Theorem 2.1, we get $G(Z_n)$ is totally disconnected. So there must exist at least three non-trivial ideals in Z_n. Let $|Max(Z_n)| = 1$. Then either $n = p$ or $n = p^m$. The case $n = p$ is absurd as we have no vertex set. Next by Theorem 2.1, the graph is totally disconnected for $n = p^m$. So $|Max(Z_n)| \geq 2$.

Conversely, we assume that there exist at least three non-trivial ideals such that $|Max(Z_n)| \geq 2$. Let $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}\ldots p_s^{k_s}$. Clearly, $I = (p_1)$, $J = (p_2)$, $\ldots$,$K = (p_k) \in Max(Z_n)$. Let $A = (p_1^{k_1}p_2^{k_2}p_3^{k_3}\ldots p_r^{k_r})$, $r \leq k$, $1 \leq l_i \leq n_i$ and $B = (p_1^{l_1}p_2^{l_2}p_3^{l_3}\ldots p_s^{l_s})$, $s \leq k$, $1 \leq k_i \leq n_i$. So $I \cap A = A$ and since we have at least two maximal ideals, so there exist a maximal ideal $Q \in Max(Z_n), Q \neq I$ such that $A + Q = Z_n$. Hence I and A are adjacent. Thus, there exists at least one edge in $G(Z_n)$.

Theorem 2.7 *There exists an edge in $G(R)$ if and only if there exist at least three non-trivial ideals in R such that $|Max(R)| \geq 2$.*

Proof: Let there exist an edge in $G(R)$. Suppose that there exist at most two non-trivial ideals in R.

Case I: If there is only one non-trivial ideal then it is an isolated vertex and hence there does not exist any edge, which contradicts the given statement.

Case II: If there are only two non-trivial ideals, say I and J, then either of I and J is contained in the other, say $I \subseteq J$ or $I \cap J = (0)$. If $I \subseteq J$ then $I \cap J = I$ and we do not have any ideal other than J and so $I + J = J \subsetneq R$, which implies that I is superfluous. Hence, I and J are not adjacent. Again, if

$I \cap J = (0)$ then I and J are not adjacent, a contradiction. Therefore, there must exist at least three non-trivial ideals in R. Now, let $|Max(R)| < 2$. Assume that $|Max(R)| = 1$. Let $A \in Max(R)$. Then for any arbitrary ideals, say I, J of R, we have $I \subseteq A$, $J \subseteq A$ and so $I \cap J \subseteq A$. This implies that $I \cap A$, $J \cap A$, $I \cap J$ are superfluous. Thus, there does not exist any edge in $G(R)$.

Conversely, let there exist three non-trivial ideals such that $|Max(R)| \geq 2$. Let $M_1, M_2 \in Max(R)$. Then there exist at least two non-trivial ideals I and J with $I \subseteq M_1$ and $J \subseteq M_2$. Now, $I \cap M_1 \nsubseteq Rad(R)$. This implies that $I \cap M_1$ is non-superfluous. Thus, I and M_1 are adjacent. Similarly, J and M_2 are adjacent. Hence, the result.

Theorem 2.8 *$G(R)$ is connected if and only if $|Max(R)| \geq 3$ and $J(R) = 0$ or R has exactly one non-trivial ideal.*

Proof: Suppose $G(R)$ is connected so there exists an edge and hence from Theorem 2.10, there must exist at least three non-trivial ideals such that $|Max(R)| \geq 2$. For $|Max(R)| = 2$, let $M_1, M_2 \in Max(R)$ and I and J are two non-trivial ideals such that $I \subseteq M_1,\ J \subseteq M_2$, respectively. So, $I \cap J \subseteq M_1 \cap\ M_2$. This implies that $I \cap J$ is superfluous and thus I and J are not adjacent. Moreover, $M_1 \cap\ M_2$ is also superfluous and so M_1 and M_2 are not adjacent. Thus, in any way I is never adjacent to J. Hence, $G(R)$ is disconnected, a contradiction. Therefore, $|Max(R)| \geq 3$. Next, let $J(R) \neq 0$. Let $M_1, M_2, M_3 \in Max(R)$. So $J(R) = M_1 \cap\ M_2 \cap M_3 = (k)$, say. Thus, (k) is superfluous and also (k) an isolated vertex. Hence $G(R)$ disconnected, a contradiction. Thus, $J(R) = 0$.

Conversely, if R has exactly one non-trivial ideal, then clearly $G(R)$ is connected. Let $|Max(R)| \geq 3$, and $J(R) = 0$. Let $M_1, M_2, M_3 \in Max(R)$. Let I and J be two non-trivial ideals such that $I \subseteq M_1$, $J \subseteq M_2$ and I and J are not adjacent. Now, $I \subseteq M_1$ and this implies $I \cap M_1 = I$. Thus, I and M_1 are adjacent. Similarly J and M_2 are adjacent. Again, $M_1 \cap\ M_2$ is non-superfluous. Thus, M_1 and M_2 are adjacent. So $I - M_1 - M_2 - J$ is a path. Hence, $G(R)$ is connected.

Theorem 2.9 *$G(R)$ is totally disconnected if and only if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n,\ n \geq 3$ or every maximal ideal is minimal.*

Proof: Let every non-trivial ideal of R be maximal as well as minimal. Thus, for any two non-trivial ideals I and J, $I \cap J = (0)$ and so I and J are not adjacent. Hence, $G(R)$ is totally disconnected. Again if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n,\ n \geq 3$ then for any ideals say I_i and $I_j,\ i \neq j,\ i < j$, $I_i \cap I_j = I_i \subseteq I_n$. This implies that $I_i \cap I_j$ is superfluous. Hence I_i and I_j are not adjacent. As I_i and I_j are arbitrary ideals, so there is no edge between any pair of distinct ideals. Hence, $G(R)$ is totally disconnected.

Conversely, we assume that $G(R)$ is totally disconnected. Then each component in $G(R)$ is an isolated vertex. Let I and J be two non-trivial ideals and so $I \cap J = (0)$ or (k), where $k(\neq 0) \in R$. If $I \cap J = (0)$, then I and J are minimal ideals. Let I and J be not maximal ideals then there are two maximal ideals M_1 and M_2 such that $I \subseteq M_1$ and $J \subseteq M_2$, respectively. Since $|Max(R)| = 2$ and we have at least three ideals in R, so by Theorem 2.10 there exists an edge, which contradicts the given statement. Hence I and J are maximal ideals. Let $I \cap J = (k)$, where (k) is superfluous. This implies $(k) \subseteq Rad(R)$. Let $M_1, M_2 \in Max(R)$ such that $I \subseteq M_1,\ J \subseteq M_2$, then by Theorem 2.10 there exist an edge in $G(R)$, a contradiction. Thus, $M_1 = M_2$. Therefore, $|Max(R)| = 1$. Let I and J be two arbitrary ideals such that $I, J \subseteq M$, M is the maximal ideal in R. Now $I \cap J \subseteq M$. We show that $I \subseteq J$ or $J \subseteq I$. Let $I \nsubseteq J$. Then $I \cap J \subseteq I \nsubseteq J$. This implies $I \cap J \nsubseteq J$, a contradiction. Thus, $I \subseteq J$. Hence, the result.

Theorem 2.10 *$G(R)$ is the disjoint union of complete subgraphs if and only if $|Max(R)| \leq 2$.*

Proof: Suppose $G(R)$ is the disjoint union of complete subgraphs. Let $|Max(R)| > 2$. Then we have two cases.

Case I: For $|Max(R)| > 2$ and $J(R) = 0$, we have from Theorem 2.11 the graph is connected, a contradiction.

Case II: For $|Max(R)| > 2$ and $J(R) \neq 0$, let $M_1, M_2, M_3 \in Max(R)$. Suppose $M_1 \cap M_2 \cap M_3 = (k)$. This implies that (k) is superfluous. Let I be an ideal such that $I \nsubseteq M_1$ and $I \subseteq M_2$, $I \subseteq M_3$ then $I \cap M_1 \subseteq M_1 \cap M_2 \cap M_3 = (k)$. This implies that $I \cap M_1$ is superfluous. Also it is easy to see that M_1 and M_2, M_2 and M_3, M_1 and M_3 are adjacent. Thus we get a contradiction to the given statement. Hence $|Max(R)| \leq 2$.

Conversely, we assume that $|Max(R)| \leq 2$. Let $|Max(R)| = 1$. So every vertex is isolated and hence $G(R)$ is the disjoint union of complete subgraphs. Let $|Max(R)| = 2$ and $M_1,\ M_2 \in Max(R)$. Then M_1 and M_2 are not adjacent. Hence, we have disjoint components in $G(R)$. Let $I, J \subset M_1$ such that $I, J \nsubseteq M_2$. Then $I \cap M_1 = I$. Let $I \subseteq M_1 \cap\ M_2$ but $I \nsubseteq M_2$. So $I \nsubseteq M_1 \cap\ M_2 = Rad(R)$. Thus, I is non-superfluous. Therefore, I and M_1 are adjacent. Similarly, J is non-superfluous, and so J and M_1 are adjacent. Again, $I \cap J = (k) \subseteq M_1$, which is again non-superfluous and so, I and J are adjacent. Hence $I - J - M_1 - I$ is a circuit. Similarly we have $P - Q - M_2 - P$ is a circuit and so is complete. Next, if $I \subseteq M_1,\ I \subseteq M_2$ then $I \subseteq M_1 \cap\ M_2$ and this implies that I is an isolated vertex. Hence, we get that $G(R)$ is the disjoint union of complete subgraphs.

Next, we find the diameter of $G(Z_n)$.

Theorem 2.11 *For Z_n, $diam(G(Z_n)) = 0, 2,$ or ∞.*

Proof: If $n = p^2$ then $diam(G(Z_n) = 0$. Again, if $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_s^{k_s}$ where p_i's are not necessarily distinct, then $diam(G(Z_n)) = \infty$. Let $n = p_1 p_2 \dots p_i$ where p_i's are distinct primes. If (l) and (m) are two vertices such that $(l) \cap (m) = (0)$, then (l) and (m) are not adjacent. In this case, we find a divisor r of m such that $(r) \cap (l) = (q)(\neq 0)$. Thus, we get a path $(l) - (q) - (m)$. Therefore, $diam(G(Z_n)) = 2$.

Now, we find the diameter of non-superfluous intersection graph of a commutative ring with unity.

Theorem 2.12 *If $G(R)$ is connected, then $diam(G(R)) = 0, 2$.*

Proof: Let I and J be two non-trivial ideals of R such that I and J are not adjacent. Thus, $I \cap J$ is superfluous. Let $I \subseteq M_1,\ J \subseteq M_2$ where $M_1, M_2 \in Max(R)$. If $I \cap M_2$ is non-superfluous then $I - M_2 - J$ is a path and $d(I, J) = 2$. Similarly if $J \cap M_1$ is non-superfluous, then $d(I, J) = 2$. Suppose that $I \cap M_2$ and $J \cap M_1$ are superfluous. Since $G(R)$ is connected, so $|Max(R)| \geq 3$, and $J(R) = 0$, by Theorem 2.11. Again $I \cap J$ is superfluous, and so $I \cap J \subseteq J(R) \subseteq M_3$. Thus, $I \subseteq M_3$ or $J \subseteq M_3$. Let $I \subseteq M_3$. We need to show that $J \cap M_3$ is non-superfluous. Let $J \cap M_3$ be superfluous. Therefore, $J \cap M_3 \subseteq J(R) \subseteq M_1$. This implies $J \subseteq M_1$. So $J = J \cap M_1 = (0)$, as $J \cap M_1 \subseteq J(R) = 0$. Thus, $J = (0)$, a contradiction. Hence $J \cap M_3$ is non-superfluous and $I - M_3 - J$ is a path. So, $d(I, J) = 2$.

Next, we obtain the girth of $G(Z_n)$. Then we generalise the result for $G(R)$.

Theorem 2.13 *If $G(Z_n)$ contains a circuit, then $girth(G(Z_n) = 3$.*

Proof: Observe that n is not equal to $p^s,\ p^2q^2,\ pq,\ p^2q$. Let $n = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ be the prime factorization of n. For $s = 2, k_i \geq 3,\ i \in \{1, 2\}$, we have from Theorem 2.5 that the graph is the disjoint union of complete components, hence $girth(G(Z_n)) = 3$. Again, for $n = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s},\ s \geq 3,\ k_i \geq 1,\ i \in \{1, 2, \dots, s\}$, we have, for any arbitrary ideals $(p_i), (p_j),\ (p_i) \cap (p_j) = (p_i p_j)$. Then there exists at least one (p_k) such that $(p_i p_j) + (p_k) = Z_n$. Hence (p_i) and (p_j) are adjacent. Again, $(p_j) \cap (p_i p_j) = (p_i p_j)$. So, $(p_i p_j) + (p_k) = Z_n$. Hence, $(p_i p_j)$ and (p_j) are adjacent. Similarly, $(p_i p_j)$ and (p_i) are adjacent. Thus, $(p_i) - (p_j) - (p_i p_j) - (p_i)$ forms a circuit. Hence $girth(G(Z_n) = 3$.

Theorem 2.14 *If $G(R)$ contains a circuit, then $girth(G(R)) = 3$.*

Proof: We know from Theorem 2.10 that there exists an edge in G(R) if and only if there exist at least three non-trivial ideals such that $|Max(R)| \geq 2$. If $|Max(R)| = 2$, then by Theorem 2.10, $G(R)$ is the disjoint union of complete graphs. So $girth(G(R)) = 3$ if $G(R)$ contains a circuit. If $|Max(R)| \geq 3$, then for $M_1, M_2, M_3 \in Max(R)$, $M_1 - M_2 - M_3 - M_1$ is a circuit. Hence $girth(G(R)) = 3$.

3. Clique, colourability, independence number and domination number

In this section the concept of clique, independence number, domination number of $G(Z_n)$ are discussed. Here, it is necessary to mention that in ring Z_n, $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ is the prime factorization of n for the following sequel.

Theorem 3.1 *The clique number of* $G(Z_n)$ *is* $(k_1+1)(k_2+1)\ldots(k_i+1)\ldots(k_{j-1}+1)(k_{j+1}+1)\ldots(k_r+1)-1$ *where* k_i *is the maximum among all* $k_j, 1 \leq j \leq r$.

Proof: The vertex set of $G(Z_n)$ has $T_1 = (k_1+1)(k_2+1)\ldots(k_i+1)\ldots(k_j+1)\ldots(k_r+1)-2$ elements. Consider the element $(p_i^{k_i})$, where $1 \leq i \leq r$. Then $(p_i^{k_i})$ is adjacent to all vertices generated by the divisors of $p_1^{k_1} p_2^{k_2} \ldots p_i^{k_i} \ldots p_{j-1}^{k_{j-1}} p_{j+1}^{k_{j+1}} \ldots p_r^{k_r}$. If C is the set of vertices generated by the divisors of $p_1^{k_1} p_2^{k_2} \ldots p_i^{k_i} \ldots p_{j-1}^{k_{j-1}} p_{j+1}^{k_{j+1}} \ldots p_r^{k_r}$, then the number of elements in C equals $(k_1+1)(k_2+1)\ldots(k_i+1)\ldots(k_{j-1}+1)(k_{j+1}+1)\ldots(k_r+1)-1$. The number of elements in C is maximum for the maximum value of k_i, $i = 1, 2, \ldots r$. Thus, the clique number of $G(Z_n)$ is $(k_1+1)(k_2+1)\ldots(k_i+1)\ldots(k_{j-1}+1)(k_{j+1}+1)\ldots(k_r+1)-1$.

Corollary 3.1 *For* $G(Z_n)$, $\omega(G(Z_n)) = \chi(G(Z_n))$.

Theorem 3.2 *For the connected graph* $G(Z_n)$, *the independence number is* $r_{C_{r-1}}$.

Proof: Since the graph is connected, so $n = p_1 p_2 \ldots p_r$. An independence set contains the ideals of Z_n generated by the combinations of $r-1$ primes over r primes. We take the independent set $I = \{(p_1 p_2 \ldots p_{r-1}), (p_1 p_2 \ldots p_{r-2} p_r), \ldots\}$ where no two vertices of I are adjacent. If we insert any other vertex, say (q), then (q) is adjacent to at least one of the vertices in I. So I is the maximum independent set. Thus, the independence number of $G(Z_n)$ is $r_{C_{r-1}}$.

Lemma 3.1 *If* $G(Z_n)$ *is disconnected, then the number of isolated vertices in* $G(Z_n)$ *is* $k_1 k_2 \ldots k_r - 1$, $r \geq 2$.

Proof: Since $G(Z_n)$ is disconnected and $r \geq 2$, so $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}$, $r \geq 2$ and not all k_i's are equal to one. Here, the number of isolated vertices is equal to the number of superfluous ideals. Hence, the number of isolated vertices of $G(Z_n)$ is $k_1 k_2 \ldots k_r - 1$, $r \geq 2$

Theorem 3.3 *If* $G(Z_n)$ *is disconnected, then the independence number of* $G(Z_n)$ *is* $r_{C_{r-1}} + (k_1 k_2 \ldots k_r - 1)$, $r \geq 2$.

Proof: Since $G(Z_n)$ is disconnected and so for $r = 1$, we get $n = p^m$. In this case, $G(Z_n)$ is totally disconnected, and clearly the independence number is equal to the number of isolated vertices of $G(Z_n)$. Let $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, $r \geq 2$ and not all k_i's are equal to 1. Since $G(Z_n)$ is disconnected, so the number of isolated vertices in $G(Z_n)$ is $k_1 k_2 \ldots k_r - 1$. Clearly, all the isolated vertices must belong to any independence set. Along with the isolated vertices, the maximum independence set also contains those ideals which are generated by the combinations of $r-1$ primes over r primes $(p_1), (p_2), \ldots, (p_r)$. Thus, the independence number of $G(Z_n)$ is $r_{C_{r-1}} + (k_1 k_2 \ldots k_r - 1)$, $r \geq 2$.

Theorem 3.4 *If* $G(Z_n)$ *is a totally disconnected graph, then the domination number of* $G(Z_n)$ *is* $k_1 k_2 \ldots k_r + 1$.

Proof: Since the graph is not totally disconnected, so $n \neq p^m$ and $n \neq pq$. Thus $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, where $r \geq 2$ and at least one of $k_i \geq 2$ for $i = 1, 2, \ldots, r$. The number of isolated vertices in the disconnected graph $G(Z_n)$ is $k_1 k_2 \ldots k_r - 1$, and, by using the same formula, we get that the number of isolated vertices is zero for connected graph. So, it does not violate the connectedness of the graph. Now it is obvious that all the isolated vertices belong to dominating set, otherwise there are no vertices which are adjacent to these isolated vertices. Again, we have for any prime, say p_1, in $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots p_r^{k_r}$, (p_1) is adjacent to all vertices which are generated by except the multiples of $p_2 p_3 \ldots p_r$. So, our dominating set must contain at least two such primes along with the isolated vertices. Thus, the minimum dominating set contains all the isolated vertices and two primes. So, $\gamma(G(Z_n)) = (k_1 k_2 \ldots k_r - 1) + 2 = k_1 k_2 \ldots k_r + 1$.

4. Traversability and planarity

Firstly this section focuses on how to count the degree of vertices in $G(Z_n)$. Let $n = p_1^{k_1} p_2^{k_2} \dots p_i^{k_i} p_{i+1}^{k_{i+1}} \cdots p_r^{k_r}$, where p_i's are distinct primes and $k_1, k_2, \dots k_r \in N$. Then the number of vertices in $G(Z_n)$ is $T = (k_1+1)(k_2+1)\dots(k_r+1)-2$. Let $I = (p_1^{k_1} p_2^{k_2} \dots p_i^{k_i})$ be any vertex. Now, $(p_1^{k_1} p_2^{k_2} \dots p_i^{k_i})$ is adjacent to those vertices which are not a multiple of $p_{i+1}p_{i+2}\dots p_r$. Now, T_1 = number of vertices which are multilpes of $p_{i+1}p_{i+2}\dots p_r = (k_1+1)(k_2+1)\dots(k_i+1)k_{i+1}k_{i+2}\dots k_r - 1$. Therefore, $deg(I) = T - T_1 - 1$. The number of vertices of $deg(I)$ equals $k_1 k_2 \dots k_i$. Then an observation is made on how to find the degree and the number of vertices having such degree for connected and disconnected graph of Z_n. Let the graph be disconnected. Then we have two cases:

Case I: Let $n = p_1^{k_1} p_2^{k_2} \dots p_i^{k_i} \dots p_r^{k_r}$, $r > 2$, k_i's not all be equal to 1. Then the number of vertices $T = (k_1+1)(k_2+1)\dots(k_r+1)-2$. Now $deg(p_i^{l_i}) = T - (k_i+1)k_1k_2\dots k_{i-1}k_{i+1}\dots k_r$, $l_i \leq k_i$. The number of vertices that have such a degree is equal to k_i. Also, $deg(p_1^{l_1} p_2^{l_2} \dots p_i^{l_i}) = T - (k_1+1)(k_2+1)\dots(k_i+1)k_{i+1}k_{i+2}\dots k_r$, where $i < r$, $l_i \leq k_i$. The number of vertices having this degree is equal to $k_1 k_2 \dots k_i$
Again, $deg(p_1^{l_1} p_2^{l_2} \dots p_r^{l_r}) = 0$, where at least one $l_i < k_i$. The number of vertices that have such a degree is equal to $k_1 k_2 \dots k_r - 1$.

Case II: Let $n = p_1^{k_1} p_2^{k_2}$ Then $deg(p_1^{r_1}) = T - (k_1+1)k_2$, where $r_1 \leq k_1$. The number of vertices of this degree is equal to k_1. Also $deg(p_1^{r_1} p_2^{r_2}) = 0$, where at least one $r_i < k_i$, $i \in \{1,2\}$. The number of vertices of this degree is equal to $k_1 k_2 - 1$. Let the graph be connected. Then $n = p_1 p_2 \dots p_r$, and $deg(p_1 p_2 \dots p_i) = T - 2^i$, $i \leq r$. The number of vertices of this degree is equal to r_{C_i}, $i \leq r$. Thus, the total degree of T number of vertices is $2e = (T-2)r_{C_1} + (T-2^2)r_{C_2} + \dots + (T-2^{r-1})r_{C_{r-1}}$, where e is the number of edges in $G(Z_n)$. Thus, $e = 1/2[(T-2)r_{C_1} + (T-2^2)r_{C_2} + \dots + (T-2^{r-1})r_{C_{r-1}}]$. Now, we interpret the case of a connected graph with an example. Let $n = p_1 p_2 p_3 p_4$. Then, the number of vertices T in $G(Z_n)$ is $(1+1)(1+1)(1+1)(1+1) - 2 = 14$. Here $r = 4$. Thus, the number of edges in $G(Z_n)$ is $e = 1/2[(T-2)r_{C_1} + (T-2^2)r_{C_2} + \dots + (T-2^{r-1})r_{C_{r-1}}] = 1/2[(14-2)4_{C_1} + (14-4)4_{C_2} + (14-8)4_{C_3}]$ $= 66$.

Theorem 4.1 *$G(Z_n)$ is Eulerian if and only if $n = p_1 p_2 \dots p_i$ where p_l's are all distinct primes and $i > 2$*

Proof: Let $n = p_1 p_2 \dots p_t p_{t+1} \dots p_i$. The vertex set of $G(Z_n)$ contains $T_1 = (1+1)(1+1)\dots(1+1) - 2$ elements which are even. Consider a vertex $I = (p_1 p_2 \dots p_t)$ where $t \geq 1$. Then I is adjacent to the vertices that are generated by not a multiple of $p_{t+1}p_{t+2}\dots p_i$. Therefore, there are $T_2 = 2^t - 1$ numbers of vertices that are not adjacent to I, which is odd. Considering this, we conclude that I is an even degree vertices of degree $T_1 - T_2 - 1$ i.e. $T_1 - 2^t$. Hence $G(Z_n)$ is Eulerian.
Conversely, let $G(Z_n)$ be Eulerian and $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_j^{k_j}$, for all $k_j \neq 1, j > 1$ then the graph is disconnected and we cannot have an Eulerian circuit. For $n = p^2$ we have only one vertex and hence cannot form an Eularian circuit. So, the only option left is $n = p_1 p_2 \dots p_i$ where p_i's are all distinct primes and $i > 2$.

Theorem 4.2 *$G(Z_n)$ is planar if and only if $n = p^k$, $k > 1$ or $n = p_1 p_2 p_3$ or $n = p_1^{k_1} p_2^{k_2}$, where $k_1, k_2 \leq 4$.*

Proof: Let $G(Z_n)$ be a planar graph. Let us assume that n is not of the given form. We assume $n = p_1^{k_1} p_2^{k_2} \dots p_i^{k_i}$, where $i \geq 2$ and at least one $k_i \geq 5$. Then we see K_5 as a subgraph in $G(Z_n)$ with vertex set $V = \{(p_i), (p_i^2), (p_i^3), (p_i^4), (p_i^5)\}$ contradicting the planarity of the graph.

For the converse direction, we consider the following three cases:

Case I: Let $n = p_1^{k_1}, k_1 > 1$. When $k_1 = 2$, we get a connected graph with one vertex and no edge and for $k_1 > 2$ we get a totally disconnected graph, having K_1 as subgraph and hence a planar graph.

Case II: Let $n = p_1^{k_1} p_2^{k_2}$, where $k_1, k_2 \leq 4$. We see from Theorem 2.3 that the graph is disconnected with a disjoint union of complete components and $deg(p_i) = \{(k_1+1)(k_1+1)-2\} - (k_1+1)k_2 = (k_1-1) \leq 3$. Since it is the union of complete graphs, so every vertex is of $degree \leq 3$. Thus, we cannot have a subgraph $K_{3,3}$ or K_5 in $G(Z_n)$. Therefore, $G(Z_n)$ is planar graph.

Case III: Let $n = p_1p_2p_3$. Here $deg(p_i)$ is 4 and all other vertices $(p_1p_2), (p_2p_3)$, (p_1p_3) are of degree 2. Thus in this case also we do not have $K_{3,3}$ or K_5 as a subgraph of $G(Z_n)$. So $G(Z_n)$ is a planar graph.

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