Bol. Soc. Paran. Mat. (3s.) v. 2025 (43) : 1–10.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.63913

On nilpotent homoderivations in semi-prime rings

Lahcen Taoufiq and Said Belkadi*

ABSTRACT: Let R be an associative ring and let $s \geq 1$ be a fixed integer. An additive map h on R is called a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ holds for all $x, y \in R$. In [4,5,6], Chung and Luh proved several results about the nilpotency of derivations in semi-prime rings. Similarly, the main objective of this paper is to provide a complete study about the nilpotency of homoderivations with nilpotency 's' in semi-prime rings.

Key Words: Homoderivation, nilpotent homoderivation, Leibniz formula, iterates of homoderivations, prime and semi-prime ring.

Contents

1. Introduction

Throughout the present paper, R always denotes an associative ring. R is said to be prime if for any $a, b \in R$, $aRb = 0$ implies that $a = 0$ or $b = 0$ and it is semi-prime if for any $a \in R$, $aRa = 0$ implies that $a = 0$. Let p be a positive integer, R is said to be p-torsion free if $px = 0$ (where $x \in R$) implies $x = 0$. For any $x, y \in R$. An additive mapping d of R into itself is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A classical example of such mapping is the inner derivation $d_a(y) = [a, y]$, where a is a fixed element in R. An additive mapping $f : R \longrightarrow R$ is called a homomorphism of R if $f(xy) = f(x)f(y)$ holds for all $x, y \in R$. Combining the notions of homomorphisms and derivations, El Sofy [9], introduced the concept of homoderivations in rings. Precisely, he gave the following definition. Let R be an associative ring. An additive mapping h on R is called a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ holds for all $x, y \in R$. An example of such mapping is the map $h : R \longrightarrow R$ such that $h(x) = f(x) - x$ for all $x \in R$, where f is an endomorphism on R.

Let s be a fixed positive integer. We say that a mapping ϕ is nilpotent on R with index of nilpotency s, if s is the least positive integer such that $\phi^s(x) = 0$ for all $x \in R$. Posner [14] started this research direction by showing that if d_1 and d_2 are two derivations of a prime ring R with characteristic not 2, then the composite $d_1 \circ d_2 = 0$ implies either $d_1 = 0$ or $d_2 = 0$. In [12], Herstein proved that, if x is an element of a prime ring R and there exists a positive integer n such that $[x, y]^n = 0$ for all $y \in R$, then x must be central in R. Also in [11], Herstein proved that if R is a prime ring of characteristic different from two which admits a non-zero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, Daif [8] showed that if a two torsion free semi-prime ring R admits a derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Motivated by the above mentioned result, Ashraf and Rehman [1] established that if R is a two torsion free prime ring admitting a nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. Keeping in mind that a commutator is simply the image of an element under an inner derivation, Chung and Luh [4] extended the previous work to semi-prime rings for any derivation d of R. Accurately, Their result is the following, given a 2-torsion free semi-prime ring with a derivation d satisfying $d^{2n}(x) = 0$ for all $x \in R$ then $d^{2n-1}(x) = 0$ for all

* Corresponding author

Submitted June 09, 2022. Published July 05, 2023
2010 *Mathematics Subject Classification:* 16N60.

$x \in R$. In other words, the index of nilpotency of derivations in a 2-torsion free semi-prime ring is always an odd number. Moreover, This conclusion sharpened in [5] by replacing the ring R by a non-zero one sided ideal I of R, yielding that $d^s(I) = \{0\}$ implies that $d^s(R) = \{0\}$. After that, the same authors [6], studied nilpotent derivations in a semi-prime ring with characteristic 2, in this case they proved that the index of nilpotency of a derivation is a power of 2. Finally, with Kobayashi, they finished their study by showing in [7] that for a general semi-prime ring with no torsion condition, the index of nilpotency of a derivation must be odd or a power of 2. At the same time, Bergen [3] demonstrated that if R is a semi-prime ring admitting an invertible or nilpotent derivation, then R is either a division ring or a 2×2 matrices over a division ring.

Inspired by these findings, the main objective of the present paper is to provide a study on nilpotent homoderiavations in semi-prime rings. Precisely, we prove the following results:

Theorem A: Let R be a 2-torsion free semi-prime ring and h be a homoderivation of R. If $h^{2n}(R) = \{0\}$ then $h^{2n-1}(R) = \{0\}$.

Theorem B: Let R be a semi-prime ring of characteristic 2. If h is a nilpotent homoderivation of R, then the index of nilpotency of h is a power of 2.

Theorem C: Let R be a semi-prime ring with a prime characteristic p. If h is a nilpotent homoderivation of R, then the index of nilpotency s of h is of the form: $s = \sum_{k=n}^{m} \delta_k p^k$ where $0 \leq n \leq m$, δ_k are integers such that $0 \leq \delta_k < p$, δ_n is odd and δ_k is even for all $k \neq n$.

In the next section, we state the relevant lemmas which we then apply, in Section 3, to prove our results.

2. Preliminary Lemmas

It is well known that in a ring R, an arbitrary derivation $d : R \to R$ satisfies the Leibniz formula

$$d^n(xy) = \sum_{j=0}^{n} \binom{n}{j} d^j(x) d^{n-j}(y) \quad \text{for all} \quad x, y \in R.$$

In [2], the authors along with S.Ali provided a similar formula for homoderivations in rings which will be frequently used in what follows.

Lemma 2.1 *[Proposition 2.1, [2]] Let h be a homoderivation in a ring R and $n \geq 1$ be an integer, then the following relation holds:*

$$h^n(xy) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} h^{n-k+j}(x) h^{n-j}(y) \quad \textit{for all} \quad x, y \in R. \tag{2.1}$$

As a consequence, we obtained:

Corollary 2.1 *[Corollary 2.2, [2]] Let R be a ring of characteristic p, where p is a prime number. If $h : R \to R$ is a homoderivation then h^{p^n} is also a homoderivation for any positive integer n.*

In [(3.23), [10]] Gould provided the following result

Lemma 2.2 *Let r be an integer ≥ 1, then*

$$\sum_{k=0}^{r} (-1)^k \binom{s}{k} \binom{s+r-k-1}{r-k} = 0.$$

Using the previous lemma, Chung and Luh demonstrated

Lemma 2.3 *[Lemma 1, [4]] Let s be an integer > 2, the determinant of the $j \times j$ matrix :*

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \binom{s}{0} & \binom{s}{1} \\ 0 & 0 & \cdots & \binom{s}{0} & \binom{s}{1} & \binom{s}{2} \\ . & . & \cdots & . & . & . \\ \binom{s}{0} & \binom{s}{1} & \cdots & \binom{s}{j-3} & \binom{s}{j-2} & \binom{s}{j-1} \\ \binom{s}{1} & \binom{s}{2} & \cdots & \binom{s}{j-2} & \binom{s}{j-1} & \binom{s}{j} \end{pmatrix}$$

is

$$\Delta_{s,j} = (-1)^{\frac{j(j-1)}{2}} \binom{s+j-1}{j}.$$

According to E. Lucas, we have the following:

Lemma 2.4 *[pp: 417-420, [13]] Let p be a prime number, if a and b are two positive integers such that:*

$$a = \sum a_i p^i \quad \text{and} \quad b = \sum b_i p^i \quad \text{with} \quad 0 \le a_i, b_i < p \quad \text{for all} \quad i$$

then $\binom{a}{b} \equiv \prod \binom{a_i}{b_i} \pmod{p}$.

Using this we certainly obtain the result below:

Lemma 2.5 *[Lemma 3, [4]] Let n be an integer such that $n = \sum_{i=0}^{N} \omega_i p^i$, where $\omega_0, \omega_N \neq 0$ and $0 \le \omega_i < \frac{p-1}{2}$ for $i = 0, 1, \cdots, N$. Then $\binom{2n}{n} \not\equiv 0 \pmod{p}$.*

3. Main Result

Throughout the rest of this paper, we denote $I_p = \{x \in R / \ px = 0\}$ where p is a prime number.

Lemma 3.1 *Let R be a semi-prime ring and h a homoderivation of R such that $h^{2n}(R) = \{0\}$ but $h^{2n-1}(R) \neq \{0\}$ for some given positive integer n. Then I_p is a semi-prime ideal of R and $h(I_p) \subset I_p$. Furthermore, if $h^{2n-1}(I_p) \neq \{0\}$ then $n = \sum_{i=0}^{N} \omega_i p^i$, where $\omega_0, \omega_N \neq 0$ and $0 \le \omega_i < \frac{p-1}{2}$ for $i = 0, 1, \cdots, N$.*

Proof: We will prove only the last part of this lemma since the first half is obvious. If $p > 2n$, it is sufficient to choose $N = 0$ and $\omega_0 = n$.

If $p < 2n$, there exist positive integers $a_0, \cdots, a_N$ such that $2n = \sum_{i=0}^{N} a_i p^i$ where $a_N \neq 0$ and $0 \le a_i \le p-1$.

Claim: for all $i \in \{0, 1, \cdots, N\}$ a_i is an even number.

We will first deal with a_N, for this, we put: $h_1 = h^{p^N}$, due to Corollary 2.1, h_1 is also a homoderivation on the ring I_p. Since $0 \le a_i \le p-1$ then multiplying by p^i yields $0 \le a_i p^i \le p^{i+1} - p^i$. Using the summation: $0 \le \sum_{i=0}^{N-1} a_i p^i \le \sum_{i=0}^{N-1} p^{i+1} - p^i$, which means that $2n - a_N p^N \le p^N - 1$, this in turn gives $2n \le p^N(a_N + 1) - 1 < p^N(a_N + 1)$. Thus if a_N is odd then $h_1^{a_N+1}(I_p) = h^{p^N(a_N+1)}(I_p) = \{0\}$. which means h_1 is a nilpotent homoderivation on the semi-prime ring I_p with index of nilpotency $< 2n$. But $2n$ is the

least possible such even index. So $h_1^{a_N}(I_p) = \{0\}$, and hence, $h^{2n-1}(I_p) = \{0\}$, which contradicts our hypothesis. Thus a_N must be even.
Now, suppose $a_N, a_{N-1,\dots,}a_i$ are all even where $i > 1$ and let us prove that a_{i-1} is also even. Suppose to the contrary that a_{i-1} is odd, by the same manner as above one can show that $(a_{i-1}+1)\,p^{i-1} + a_i p^i + a_{i+1}p^{i+1} + \cdots + a_N p^N > 2n$ and then $h_2 = h^{p^{i-1}}$ is a nilpotent homoderivation on I_p with index $\leq$ the even number $a_{i-1} + 1 + a_i p + \cdots + a_N p^{N-i+1} < 2n$. Again we obtain a contradiction. Thus we conclude, by induction, that a_N, $a_{N-1}, \dots, a_1$ are even. and hence $a_0 = 2n - \sum_{i=1}^{N} a_i p^i$ is even too. □

Lemma 3.2 *Let R be a semi-prime ring and h a homoderivation of R such that $h^{2n}(R) = \{0\}$ but $h^{2n-1}(R) \neq \{0\}$ for some given positive integer n and let $a, b \in R$. If $h^n(x)ah^{2n-1}(b) = 0$ for all $x \in R$ then $ah^{2n-1}(b) = 0$.*

Proof: Replacing x by $xh^{n-1}(y)$ in the main equation and using Lemma 2.1, we get

$$\begin{aligned} h^n(xh^{n-1}(y))ah^{2n-1}(b) &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} h^{n-j}(x)h^{2n+j-k-1}(y)ah^{2n-1}(b) \\ &= h^n(x)h^{n-1}(y)ah^{2n-1}(b) = 0. \end{aligned}$$

Combining this identity with $h^n(h(x)h^{n-2}(y))ah^{2n-1}(b) = 0$ we obtain the identity

$$h^{n+1}(x)h^{n-2}(y)ah^{2n-1}(b) = 0.$$

Proceeding by the same argument we get $h^{2n-1}(x)yah^{2n-1}(b) = 0$ for all $x, y \in R$. In particular, $ah^{2n-1}(b)yah^{2n-1}(b) = 0$. But R is semi-prime, hence $ah^{2n-1}(b) = 0$. □

Lemma 3.3 *Let R be a semi-prime ring and h a homoderivation of R such that $h^{2n}(R) = \{0\}$ with $h^{2n-1}(R) \neq \{0\}$ and $h^{2n-1}(I_p) \neq \{0\}$ for some given positive integer n and let $n \leq s < 2n$.*

1. *If $t \in I_p$ and $h^s(x)t = 0$ for all $x \in R$ then $h^{s-1}(x)h^{s-1}(y)t = 0$ for all $x, y \in R$.*
2. *If R is $(2n)!$-torsion free, $t \in R$, and $h^s(x)t = 0$ for all $x \in R$ then $h^{s-1}(x)h^{s-1}(y)t = 0$ for all $x, y \in R$.*

Proof: Replacing x by xy and using Lemma 2.1 we have that

$$h^s(xy)t = \sum_{k=0}^{s} \binom{s}{k} \sum_{j=0}^{k} \binom{k}{j} h^{s-j}(x)h^{s-k+j}(y)t = 0 \quad \text{for all } \ x, y \in R. \tag{3.1}$$

For each $k \in \{0,\ \dots,\ s-1\}$ let $i_k = s-k-1$ and $i_s \in \{0,\ \dots,\ s-2\}$ and substitute $h^{k+i_k-s+1}(x)$ for x and $h^{s-i_k-1}(y)$ for y in (3.1) to obtain :

$$\sum_{k=0}^{s} \binom{s}{k} \sum_{j=0}^{k} \binom{k}{j} h^{k+i_k-j+1}(x)h^{2s-k-i_k+j-1}(y)t = 0 \quad \text{for all } \ x, y \in R. \tag{3.2}$$

For $i_s = 0$, (3.2) reads :

$$\binom{s}{0} h^s(x)h^{s-2}(y)t + \binom{s}{1} h^{s-1}(x)h^{s-1}(y)t = 0 \quad \text{for all } \ x, y \in R. \tag{3.3}$$

For $i_s = 1$, (3.2) reads :

$$\begin{aligned} &\binom{s}{0} h^{s+1}(x)h^{s-3}(y)t + \binom{s}{1} h^s(x)h^{s-2}(y)t \\ &+ \binom{s}{2} h^{s-1}(x)h^{s-1}(y)t = 0 \quad \text{for all } \ x, y \in R. \end{aligned} \tag{3.4}$$

Proceeding in this way by giving i_s the values from 1 to $s-2$ we get a system of $s-1$ equations that can be expressed in matrix form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \binom{s}{0} & \binom{s}{1} \\ 0 & 0 & \cdots & \binom{s}{0} & \binom{s}{1} & \binom{s}{2} \\ . & . & \cdots & . & . & . \\ 0 & 0 & \cdots & \binom{s}{2n-s-1} & \binom{s}{2n-s} & \binom{s}{2n-s+1} \\ . & . & \cdots & . & . & . \\ \binom{s}{0} & \binom{s}{1} & \cdots & \binom{s}{j-3} & \binom{s}{j-2} & \binom{s}{j-1} \\ \binom{s}{1} & \binom{s}{2} & \cdots & \binom{s}{j-2} & \binom{s}{j-1} & \binom{s}{j} \end{pmatrix}$$

$$\begin{pmatrix} h^{2s-3}(x)h(y)t \\ h^{2s-4}(x)h^2(y)t \\ \vdots \\ h^{2n}(x)h^{2s-2n-2}(y)t \\ h^{2n-1}(x)h^{2s-2n-1}(y)t \\ \vdots \\ h^s(x)h^{s-2}(y)t \\ h^{s-1}(x)h^{s-1}(y)t \end{pmatrix} = 0$$

1. Since $h^{2n}(R) = \{0\}$ we have

   $$h^{2s-2}(x)yt = h^{2s-3}(x)h(y)t = \cdots = h^{2n}(x)h^{2s-2n-2}(y)t = 0 \text{ for all } x, y \in R.$$

   Hence, to prove our result we just need to show that there exists $j \in \{2n-s+1, \cdots, s-1\}$ such that $\Delta_{s,j} \not\equiv 0 \pmod{p}$.
   Proceeding by induction on s and taking $s = n+1$ and $j = n$ we get by Lemma 2.5 and Lemma 3.1 $\Delta_{s,n} = (-1)^{\frac{n(n-1)}{2}} \binom{2n}{n} \not\equiv 0 \pmod{p}$. Now we assume $n < k < 2n$ and $\Delta_{k,j} \not\equiv 0 \pmod{p}$ for some j where $2n-k+1 \leq j \leq k-1$. Denote that $\Delta_{k,j} = \Delta_{k+1,j} + (-1)^j \Delta_{k+1,j-1}$ at least one of $\Delta_{k+1,j}$ and $\Delta_{k+1,j-1}$ is not congruent to zero modulo p where $2n-k < j-1 < j < k$. Therefore, $\Delta_{s,j} \not\equiv 0 \pmod{p}$ for some j as we desired.

2. can be seen analogously by noticing that $(2n)!$ is divided by $\Delta_{s,2n-s+1}$.

□

Lemma 3.4 *Let R be a $(2n)!$-torsion free semi-prime ring and h a homoderivation of R such that $h^{2n}(R) = \{0\}$ but $h^{2n-1}(R) \neq \{0\}$ and for some given positive integer n and let $a, b \in R$ and $n < s \leq 2n$. Suppose either $h^{2n-1}(a) \in I_p$ and $h^{2n-1}(I_p) \neq \{0\}$ where p is a prime number or $h^{2n-1}(a) \in R$. If $h^s(x)bh^{2n-1}(a) = 0$ for all $x \in R$ then $bh^{2n-1}(a) = 0$.*

Proof: We proceed by induction on s. It is true for $s = n$ by Lemma 3.2. Now we assume that the induction hypothesis holds for $s = k$, where $n \leq k < 2n-1$. If $h^{k+1}(x)bh^{2n-1}(a) = 0$ for all $x \in R$, then by Lemma 3.3 $h^k(x)h^k(y)bh^{2n-1}(a) = 0$ for all $x, y \in R$. Using the induction hypothesis twice we obtain $bh^{2n-1}(a) = 0$. □

Lemma 3.5 *Let R be a semi-prime ring and h a homoderivation of R such that $h^{2n}(R) = \{0\}$ but $h^{2n-1}(R) \neq \{0\}$ for some given positive integer n then R is not torsion free and $h^{2n-1}(I_p) = \{0\}$ for any prime number p.*

Proof: Let J $= \sum I_p$ then J is an ideal of R. Suppose by contradiction that R is torsion free, which means that J $= \{0\}$, hence by using the second part of Lemma 3.3 and Lemma 3.4 $h^{2n-1}(R) = \{0\}$, a contradiction. Now if $h^{2n-1}(I_p) \neq \{0\}$, since J $= \sum I_q$ and $I_p I_q = \{0\}$ for $p \neq q$, then by the first part of Lemma 3.4 we obtain $h^{2n-1}(R)h^{2n-1}(I_p)\mathrm{J} = \{0\}$, which implies by the semi-primness of R that $h^{2n-1}(R)h^{2n-1}(I_p) = \{0\}$, and again by Lemma 3.4 $h^{2n-1}(R) = \{0\}$, a contradiction. □

Now we can prove our first main result of this paper.

Theorem 3.1 *Let R be a 2-torsion free semi-prime ring and h be a homoderivation of R. If $h^{2n}(R) = \{0\}$ then $h^{2n-1}(R) = \{0\}$.*

Proof: Suppose that there exists a least positive integer n with $h^{2n}(R) = \{0\}$ but $h^{2n-1}(R) \neq \{0\}$. We should note that J $= \sum I_p$ is a semi-prime ideal or R/J is a semi-prime ring. We also note that there exists a naturally induced homoderivation $\bar{h}$ by h on R/J verifying $\bar{h}^{2n} = 0$. Since R/J is torsion free, by Lemma 3.5 we get $\bar{h}^{2n}(\mathrm{J}) = \{0\}$ which is equivalent to $h^{2n-1}(R) \subset \mathrm{J}$, this in turn gives that $xh^{2n-1}(R) \subset \mathrm{J}$ for all $x \in R$. Using again Lemma 3.5 yields $h^{2n-1}\big(xh^{2n-1}(R)\big) = 0$ which means that $h^{2n-1}(x)h^{2n-1}(R) = 0$ for all $x \in R$. Hence $\big(h^{2n-1}(R)\big)^2 = \{0\}$. Assume $h^{2n-1}(R) \neq \{0\}$, then there exists a non zero element $\alpha \in R$ with $h^{2n-1}(\alpha) \in I_p$ for some prime p. But $h^{2n-1}(x)h^{2n-1}(\alpha) = 0$ for all $x \in R$, from Lemma 3.4 we obtain $h^{2n-1}(\alpha) = 0$ a contradiction. Thus $h^{2n-1}(R) = \{0\}$ another contradiction. This completes the proof. □

Lemma 3.6 *Let R be a semi-prime ring with char$(R) = 2$. If h is a homoderivation of R such that $h^3 = 0$. Then $h^2 = 0$.*

Proof: Using lemma 2.1 and the fact that $h^3 = 0$ we have

$$\begin{aligned} 0 = h^3(h(x)y) &= \sum_{k=0}^{3} \binom{3}{k} \sum_{j=0}^{k} \binom{k}{j} h^{3-k+j}(h(x))h^{3-j}(y) \\ &= 3h^2(x)h^2(y) \quad \text{for all} \;\; x, y \in R. \end{aligned} \tag{3.5}$$

Since the characteristic of R is 2 we get

$$h^2(x)h^2(y) = 0 \;\; \text{for all} \;\; x, y \in R. \tag{3.6}$$

According to Corollary 2.1, h^2 is also a homoderivation of R. Hence, replacing y by yx in (3.6) yields

$$\begin{aligned} h^2(x)h^2(yx) &= h^2(x)h^2(y)h^2(x) + h^2(x)h^2(y)x + h^2(x)yh^2(x) \\ &= h^2(x)yh^2(x) = 0 \quad \text{for all} \;\; x, y \in R. \end{aligned} \tag{3.7}$$

From the semi-primeness of R we deduce that $h^2(x) = 0$ for all $x \in R$. This completes the proof. □

Lemma 3.7 *Let R be a semi-prime ring with char$(R) = 2$. If h is a homoderivation of R such that $h^s = 0$ where $2^n < s < 2^{n+1} - 1$ for some positive integer n. Then $h^{2^n} = 0$.*

Proof: Proceeding by induction on n, for $n = 1$ we get $s = 3$. Thus, the result is true in view of Lemma 3.1 Now let $n > 1$ and suppose the result holds for $n - 1$. Since $2^n < s < 2^{n+1} - 1$ then $s = \sum_{k=0}^{n} \alpha_k 2^k$ where $\alpha_k \in \{0 \;,\; 1\}$ for all $k \in \{0 \;,\; \dots \;,\; n\}$ with at least one of the α_k's is zero and $\alpha_n = 1$. Put $i = \min\{k \;/\; \alpha_k = 0\}$.

Case 1 *If $i = 0$. Then $s = 2s_0$ where $s_0 = \sum_{k=1}^{n} \alpha_k 2^{k-1}$ and $2^{n-1} < s_0 \leq 2^n - 1$. Set $\eta = h^2$. Then using Corollary 2.1, η is also a homoderivation with $\eta^{s_0} = 0$. By the induction hypothesis, $\eta^{2^{n-1}} = 0$, hence $h^{2^n} = 0$.*

Case 2 *If $i > 0$. Then $\alpha_0 = 1$ and $s + 1 = 2^i + \sum_{k=i+1}^{n} \alpha_k 2^k = 2^i s_{i-1}$ where $s_{i-1} = 1 + \sum_{k=i+1}^{n} \alpha_k 2^{k-i}$. Let $\beta = h^{2^i}$. Then again using corollary 2.1, β is a homoderivation of R and $2^{n-i} < s_{i-1} \leq 2^{n-i+1} - 1$. Once more by the induction hypothesis, $\beta^{2^{n-i}} = 0$, thus $h^{2^n} = 0$.*

□

Let $\mathbb{N}^*$ be the set of all non-zero positive integers.

Lemma 3.8 *Let R be a semi-prime ring of characteristic 2, h a homoderivation of R such that $h^{2^{n+1}-1} = 0$ but $h^{2^{n+1}-2} \neq 0$ for some integer $n \geq 1$, I the ideal generated by h^{2^n} and $\mathbb{W} = \{(p, q) \in \mathbb{N}^*/\exists r \in I$ such that $r \neq 0,\ h(r) = 0,\ h^p(R)r = \{0\} = rh^q(R)\}$. Then the next hold:*

1. *$\mathbb{W} \neq \emptyset$.*
2. *In respect to the following partial order relation : $(i_1, j_1) < (i_2, j_2)$ if and only if $i_1 < j_1$ and $i_2 < j_2$. If (i, j) is a minimal element in $\mathbb{W}$ then $i > 2^n$ and $j > 2^n$.*

Proof:

1. Since R of characteristic 2 and $h^{2^{n+1}-1} = 0$ we obtain

$$\begin{aligned} 0 &= h^{2^{n+1}-1}\big(h^{2^{n+1}-3}(x)(y)\big) \\ &= \sum_{k=0}^{2^{n+1}-1} \binom{2^{n+1}-1}{k} \sum_{p=0}^{k} \binom{k}{p} h^{2^{n+2}-4-k+p}(x) h^{2^{n+1}-1-p}(y) \\ &= h^{2^{n+1}-2}(x) h^{2^{n+1}-2}(y) \quad \text{for all } \ x, y \in R. \end{aligned} \tag{3.8}$$

   Using (3.8) along with the hypothesis $h^{2^{n+1}-2} \neq 0$ we derive the existence of $t \in R$ such that $w = h^{2^{n+1}-2}(t) \neq 0$, $h(w) = 0$ and $wh^{2^{n+1}-2}(y) = 0 = h^{2^{n+1}-2}(x)w$ for all $x, y \in R$. Hence, $(2^{n+1} - 2\ ,\ 2^{n+1} - 2) \in \mathbb{W}$ implying that $\mathbb{W} \neq \emptyset$.

2. Without loss of generality, assume that $i \leq 2^n$. Since $(i\ ,\ j) \in \mathbb{W}$. Then there exists $x_0 \in I \setminus \{0\}$ such that $h(x_0) = 0$, $h^i(R)x_0 = \{0\} = x_0 h^j(R)$. Then for all $a, b \in R$

$$0 = h^{2^n}(ab)x_0 = h^{2^n}(a)h^{2^n}(b)x_0 + h^{2^n}(a)bx_0 + ah^{2^n}(b)x_0 = h^{2^n}(a)bx_0.$$

   Therefore, $Ix_0 = \{0\}$. In view of the semi-primness of R we get $x_0 = 0$. This contradicts our choice of x_0. Thus $i > 2^n$ and $j > 2^n$.

□

Lemma 3.9 *Let R be a semi-prime ring of characteristic 2. If h is a homoderivation of R such that $h^{2^{n+1}-1} = 0$ for some positive integer n. Then $h^{2^n} = 0$.*

Proof: Proceeding by induction on n, for $n = 1$ we get $s = 3$. Thus, the result is true in view of Lemma 3.6. Now let $n > 1$ and suppose the result holds for $n - 1$. Due to Lemma 3.7 it will be sufficient to prove that $h^{2^{n+1}-2} = 0$. Let us assume by contradiction that $h^{2^{n+1}-2} \neq 0$. Using Lemma 3.8 and keeping the same notations, let (i, j) be a minimal element of $\mathbb{W}$. Then there exists $z \in I \setminus \{0\}$ such that $h(z) =$

0, $h^i(R)z = zh^j(R) = \{0\}$. Consequently

$$\begin{aligned}
0 &= h^{2^{n+1}-1}\big(h^{i-2^n-1}(x)zh^{j-2^n}(y)\big) \\
&= h^{2^n-1}\big(h^{2^n}\big(h^{i-2^n-1}(x)zh^{j-2^n}(y)\big)\big) \\
&= h^{2^n-1}\big(h^{i-1}(x)zh^{j-2^n}(y) + h^{i-2^n-1}(x)zh^{j}(y)\big) \\
&= h^{2^n-1}\big(h^{i-1}(x)zh^{j-2^n}(y)\big) \\
&= \sum_{k=0}^{2^n-1}\binom{2^n-1}{k}\sum_{p=0}^{k}\binom{k}{p}h^{2^n-1-k+p}\big(h^{i-1}(x)z\big)h^{2^n-1-p}\big(h^{j-2^n}(y)\big) \\
&= \sum_{k=0}^{2^n-1}\binom{2^n-1}{k}\sum_{p=0}^{k}\binom{k}{p}h^{2^n-1-k+p}\big(h^{i-1}(x)z\big)h^{j-1-p}(y) \\
&= h^{i-1}(x)zh^{j-1}(y) \quad \text{for all} \quad x, y \in R.
\end{aligned} \tag{3.9}$$

If $zh^{j-1}(y) = 0$ for all $y \in R$, then $(i, j-1) \in \mathbb{W}$ which contradicts the minimality of (i, j) in $\mathbb{W}$. Thus there exists $y_0 \in R$ such that $z_0 = zh^{j-1}(y_0) \neq 0$. Since $z_0 \in I$ and $h(z_0) = h\big(zh^{j-1}(y_0)\big) = h(z)h^j(y_0) + h(z)h^{j-1}(y_0) + zh^j(y_0) = 0$ as well as

$$\begin{aligned}
0 = zh^j\big(h^{j-1}(y_0)y\big) &= \sum_{k=0}^{j}\binom{j}{k}\sum_{p=0}^{k}\binom{k}{p}zh^{2j-1-k+p}(y_0)h^{j-p}(y) \\
&= zh^{j-1}(y_0)h^j(y) = z_0h^j(y) \quad \text{for all} \quad y \in R.
\end{aligned}$$

Then $(i-1, j) \in \mathbb{W}$ which leads again to a contradiction with the minimality of (i, j) in $\mathbb{W}$. Therefore $h^{2^{n+1}-2} = 0$ and the lemma follows from this. □

Combining Lemma 3.7 and Lemma 3.9 we deduce our second main conclusion of this paper

Theorem 3.2 *Let R be a semi-prime ring of characteristic 2. If h is a nilpotent homoderivation of R, then the index of nilpotency of h is a power of 2.*

Next, we will provide a result in case of general characteristic restrictions

Theorem 3.3 *Let R be a semi-prime ring with a prime characteristic p. If h is a nilpotent homoderivation of R, then the index of nilpotency s of h is of the form: $s = \sum_{k=n}^{m}\delta_k p^k$ where $0 \leq n \leq m$, δ_k are integers such that $0 \leq \delta_k < p$, δ_n is odd and δ_k is even for all $k \neq n$.*

Proof: If $p = 2$, then by Theorem 3.2, $s = 2^{p^n}$. Hence the result is true in this case. Now suppose $p \geq 3$, then there exists an integer $m \geq 0$ such that for all $k \in \{0, \dots, m\}$ there exist $0 \leq \delta_k < p$ with $s = \sum_{k=0}^{m}\delta_k p^k$ with $\delta_m \neq 0$. Put $i = \max\{k \,/\, \delta_k \text{ is odd}\}$ and let $\eta = h^{p^i}$. According to Corollary 2.1 η is also a homoderivation of R and considering the fact that $p^i \geq \sum_{k=0}^{i-1}\delta_k p^k$ we get $\eta^N = 0$ where $N = 1 + \sum_{k=i}^{m}\delta_k p^{k-i}$. Since p is odd. Then R is 2-torsion free and as N is even, hence using Lemma 3.7 we conclude that $\eta^{N-1} = h^{p^i(N-1)} = 0$. Therefore, $s = p^i(N-1) = \sum_{k=i}^{m}\delta_k p^k$ because s is the index of nilpotency of h. Moreover δ_i is odd and the other δ_k are even. □

Corollary 3.1 *Let R be a semi-prime ring with char$(R) = q = \prod_{j=1}^{r} p_j$ with p_j are distinct prime numbers for all $j \in \{1, ..., r\}$. If h is a nilpotent homoderivation of R, then the index of nilpotency s of h is of the form: $s = \sum_{k=n}^{m} \delta_k p^k$ where $0 \le n \le m$, δ_k are integers such that $0 \le \delta_k < p$, δ_n is odd and δ_k is even for all $k \ne n$, for some prime divisor p of q.*

Proof: Since $q = \prod_{j=1}^{r} p_k$ then we can write R as the direct sum $R = \bigoplus_{j=1}^{r} R_j$ with $R_j = \{x \in R \,/\, p_j x = 0\}$. For all $j \in \{1, \ldots, r\}$ R_j is a sub-ring of R with prime characteristic p_j such that $h(R_j) \subset R_j$. Let s_j be the index of nilpotency of $h_{/R_j}$ the restriction of h to R_j. Then, by Theorem 3.3, $s_j = \sum_{k=n}^{m} \delta_k p_j^k$ where $0 \le n \le m$, δ_k are integers such that $0 \le \delta_k < p_j$, δ_n is odd and the other δ_k are even. It follows that the index of nilpotency of h is $s = \max\{s_k \,/\, 1 \le k \le r\}$. □

The following example proves the necessity of our assumptions in Theorem 3.3.

Example 3.1 *Let $\mathbb{F}_2$ be the Galois field of order 2 and put $R = \mathcal{M}_3(\mathbb{F}_2)$. Then R is a non semi-prime ring with char$(R) = 2$. Set the invertible matrix A in R as follows:*

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = Id_R + B$$

For all $X \in R$ define h by

$$h(X) = AXA^{-1} - X = \big(AX - XA\big)A^{-1} = \big(BX - XB\big)A^{-1} = d_B(X)A^{-1}$$

Since $X \mapsto AXA^{-1}$ is an inner automorphism of R, then h is a homoderivation on R. Furthermore, as $B^3 = 0$ but $B^2 \ne 0$ we deduce that $h^3 = 0$ however $h^2 \ne 0$. Thus the index of nilpotency of h is 3 which is not a power of 2.

References

1. M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results. Math. 42, 3-8, (2002).
2. S. Belkadi, S. Ali and L. Taoufiq, *On nilpotent homoderivations in prime rings*, Comm. Algebra. 51(9), 4044-4053, (2023).
3. J. Bergen, *Lie ideals with regular and nilpotent elements and a result on derivations*, Rend. Circ. Math. Palermo(2) 99-108, (1984).
4. L. O. Chung and J. Luh, *Nilpotency of derivations*, Canad. Math. Bull. 26, 341-346, (1983).
5. L. O. Chung and J. Luh, *Nilpotency of derivations on an ideal*, Proc. Amer. Math. Soc. 90(2), 211-214, (1984).
6. L. O. Chung and J. Luh, *Nilpotency of derivations II*, Proc. Amer. Math. Soc. 91(3), 357-358, (1984).
7. L. O. Chung, Y. Kobayashi and J. Luh, *Remark on nilpotency of derivations*, Proc. Japan. Acad. 60, 329-330, (1984).
8. M. N. *Commutativity results for semi-prime rings for derivations*, Internat. J. Math. Math. Sci. 21(3), 259-263, (1998).
9. M. M. El-Sofy, *Rings With Some Kinds of Mappings*, Ph.D. Thesis, Cairo University, Branch of Fayoum, Cairo, Egypt (2000).
10. H. W. Gould, *Combinatorial Identities*, revised éd., Morgantown, W. Va. (1972).
11. I. N. Herstein, *A note on derivations*, Math. Bull. 21(13), 369-370, (1978).
12. I. N. Herstein, *Sui commutatori degli anelli semlici*, Rend. Semin. Milano. 33, 3-9, (1963).
13. E. Lucas, *Théorie des Nombres*, Paris, (1891).
14. E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8, 1093-1100, (1957).

Lahcen Taoufiq,
Mathematics Computer Sciences and Applications Laboratory, E.N.S.A,
IBN ZOHR University,
MOROCCO.
E-mail address: l.taoufiq@uiz.ac.ma

and

Said Belkadi,
Mathematics Computer Sciences and Applications Laboratory, E.N.S.A,
IBN ZOHR University,
MOROCCO.
E-mail address: said.belkadi@edu.uiz.ac.ma