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Multiplicity of weak solutions to a $p(\cdot)$-Laplacian problem with discontinuous Steklov boundary conditions

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ABSTRACT: The focus of this paper is to investigate the existence and multiplicity of weak solutions for an elliptic equation including discontinuous Steklov boundary conditions. The approach taken in this study involves the utilization of variational methods.

Key Words: $p(\cdot)$-Laplacian equation, discontinuous Steklov boundary conditions, variational methods.

Contents

1. Introduction

Though the article Ω is a bounded simply connected domain in $\mathbb{R}^N$, where $N \geq 2$, with a smooth boundary $\partial\Omega = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are compact simply connected smooth surfaces in $\mathbb{R}^{N-1}$, such that

$$\partial\Sigma_1 = \partial\Sigma_2 = \Sigma_1 \cap \Sigma_2 = \Gamma,$$

where Γ is a compact $(N-2)$-dimensional smooth surface without boundary. We consider the following $p(\cdot)$-Laplacian problem

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + \mathcal{R}(x)|u|^{p(x)-2}u = f(x,u) & x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda g(x) & x \in \Sigma_1, \\ u = 0 & x \in \Sigma_2, \end{cases} \tag{1.1}$$

where $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ is the outward unit normal to the smooth surface Σ_1 at x. We assume $\mathcal{R} \in L^\infty(\Omega)$ with $ess\inf_\Omega \mathcal{R} > 0$ and

$$f : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function that satisfies in the following conditions:

$(f1)$ There exists some $\rho > 0$ such that the following growth condition holds

$$|f(x,t)| \leq \rho |t|^{\theta(x)-1}, \qquad (x,t) \in \Omega \times \mathbb{R}$$

for continuous function θ with $1 \leq \theta(x) < p(x)$ a.e. in Ω.

$(f2)$ $t\, f(x,t) \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$;

The function $g : \Sigma_1 \to \mathbb{R}$ such that

$$g \in L^{(p^-)'}(\Sigma_1), \tag{g}$$

where $\frac{1}{p^-} + \frac{1}{(p^-)'} = 1$. We prove the existence and multiplicity of (weak) solutions to Problem (1.1).

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Several case of classical Steklov problems have been studied in the large literature. For instance, in 2017, Y. Karagiorgos and N. Yannakakis [12] considered the following Neumann problem with non-homogeneous boundary conditions

$$\begin{cases} -\Delta_{p(x)}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded smooth domain and

$$g \in L^\infty(\partial\Omega) \qquad \& \qquad \int_{\partial\Omega} g d\mathcal{H}^{N-1} = 0.$$

They verified some results on the existence and multiplicity of weak nontrivial solutions to the problem. Using the min-max method and Ekeland's variational principle, A. Khaled Ben [13] has demonstrated the existence of weak nontrivial solutions to the following non-linear Steklov boundary value problem

$$\begin{cases} -\Delta_{p(x)}u = a(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded smooth domain, λ is a positive parameter, $a \in L^\infty(\Omega)$ with $ess \inf_\Omega a > 0$, ν is the outer unit normal to $\partial\Omega$. Recently, A. Khaleghi and A. Razani [14] have been probed the existence and multiplicity of solutions to the $p(x)$-Laplacian problem

$$\begin{cases} -\Delta_{p(x)}u + c(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ is outer unit normal derivative. $p \in C(\overline{\Omega})$ with $p^- > N$ and the functions c, f, g are assumed to satisfy suitable assymptions. At the same time, R. Chammem et al. [7] have been probed the existence of multiple solutions to the $p(x)$-Laplacian problem

$$\begin{cases} (-\Delta)_{p(x)}u + a(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)}\frac{\partial u}{\partial \nu} + b(x)|u|^{q(x)-2}u = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ is outer unit normal derivative. The functions a, b, p, q, g and f hold in some hypotheses. We point out the authors have proved the existence of solutions to the problems in special cases with continuous boundary conditions on the Heisenberg groups [1,15,17,18,19,20,21,22,23,24,25,26,27,28].

The interest of discontinuous Steklov problems comes from the second order self-adjoint elliptic equations which arise in water wave problems [6].

Outline of this paper: In Section 2, we give some preliminary results which compose the tools that are needed for the next Section. In Section 3, we prove the existence and multiplicity of solutions for Problem (1.1).

2. Preliminaries and main tools

Let Ω be a bounded simply connected domain with the hypotheses of the previous section. We set

$$p^- = \inf_{x\in\overline{\Omega}} p(x) \quad \& \quad p^+ = \sup_{x\in\overline{\Omega}} p(x),$$

where $p \in C_+(\bar{\Omega}) = \{g \in C(\bar{\Omega}) : g^- > 1\}$. The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions u on Ω for which $\int_\Omega |u(x)|^{p(x)}dx < +\infty$ and has the norm

$$|u|_{p(\cdot)} = inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}.$$

Notice that if $q(\cdot)\equiv q,\ 1\le q<+\infty$, this norm is equal to the standard norm on $L^q(\Omega)$ that we denote it by $|.|_q$. Also, we put

$$|u|_{q,\partial} := \Big(\int_{\partial\Omega}|u|^q d\sigma\Big)^{\frac{1}{q}},$$

and

$$|u|_{q,\Sigma_1} := \Big(\int_{\Sigma_1}|u|^q d\sigma\Big)^{\frac{1}{q}}.$$

For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, where $L^{p'(\cdot)}(\Omega)$ is the conjugate space of $L^{p(\cdot)}(\Omega)$, the Hölder type inequality

$$\Big|\int_\Omega uv dx\Big| \le \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)},$$

holds true. The following theorem is in [16, Theorem 2.8].

Theorem 2.1 *Assume that $\Omega\subset\mathbb{R}^N$ is a bounded domain and $p,q\in C_+(\bar\Omega)$. Then*

$$L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$$

if and only if $q(x)\le p(x)$ a.e. $x\in\Omega$.

Following the authors of paper [17], for any $\iota>0$, we put

$$\iota^{\check r} := \begin{cases} \iota^{r^+} & \iota<1,\\ \iota^{r^-} & \iota\ge 1;\end{cases}$$

and

$$\iota^{\hat r} := \begin{cases} \iota^{r^-} & \iota<1,\\ \iota^{r^+} & \iota\ge 1;\end{cases}$$

for $r\in C_+(\Omega)$. Then the well-known proposition 2.7 of [11] will be rewritten as follows.

Proposition 2.1 *For each $u\in L^{p(\cdot)}(\Omega)$, we have*

$$|u|_{p(\cdot)}^{\check p} \le \int_\Omega |u(x)|^{p(x)}dx \le |u|_{p(\cdot)}^{\hat p}.$$

Normally, the Sobolev space associated with $L^{p(\cdot)}(\Omega)$ is defined as follows

$$W^{1,p(\cdot)}(\Omega)=\{u:\Omega\to\mathbb{R}:\ u,|\nabla u|\in L^{p(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|u\| := |\nabla u|_{p(\cdot)}+|u|_{p(\cdot)},$$

where $\nabla u = (\frac{\partial u}{\partial x_1}(x),\cdots,\frac{\partial u}{\partial x_N}(x))$ is the gradient of u at $x=(x_1,\cdots,x_N)$ and as usual $|\nabla u| = \big(\sum_{i=1}^N|\frac{\partial u}{\partial x_i}|^2\big)^{\frac{1}{2}}$. If $p(\cdot)\equiv p$, for $1\le p<+\infty$, we put

$$\|u\|_p = |\nabla u|_p + |u|_p,$$

which is equal to the norm of the Sobolev space $W^{1,p}(\Omega)$.

Remark 2.1 *As a consequence of Theorem 2.1 we have*

$$W^{1,p(\cdot)}(\Omega)\hookrightarrow W^{1,q(\cdot)}(\Omega),$$

if $q(x)\le p(x)$ a.e. $x\in\Omega$; moreover, there exists $c_q>0$ such that

$$\|u\|_{q(\cdot)}\le c_q\|u\|,$$

for each $u \in W^{1,p(\cdot)}(\Omega)$*. In a special case, one has*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p^-}(\Omega);$$

furthermore, there exists $\bar{c} > 0$ *such that*

$$\|u\|_{p^-} \leq \bar{c}\|u\|$$

for each $u \in W^{1,p(\cdot)}(\Omega)$.

The following is Sobolev embedding established in [8, Theorem 8.2.4] and also [10].

Proposition 2.2 *Assume that* Ω *is a bounded and smooth domain in* $\mathbb{R}^N$ *and* $p, q \in C_+(\bar{\Omega})$*. If* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$*, the embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact and continuous, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) \geq N. \end{cases}$$

Moreover, there exists $\kappa_q > 0$ *such that*

$$|u|_{q(\cdot)} \leq \kappa_q \|u\|.$$

Remark 2.2 *For* $u \in W^{1,p(x)}(\Omega)$*, there exist* $\mu, M > 0$ *such that*

$$\mu\|u\|^{\check{p}} \leq \int_\Omega (|\nabla u|^{p(x)} + \mathcal{R}(x)|u|^{p(x)})dx \leq M\|u\|^{\hat{p}}.$$

Proof: Since $ess\inf_\Omega \mathcal{R} > 0$, there exists $0 < \delta < 1$ such that $\delta < \mathcal{R}(x)$ a.e. in Ω. Using Proposition 2.1, Hölder inequality and hypothesis $\mathcal{R} \in L^\infty(\Omega)$, we gain

$$\delta|u|^{\check{p}}_{p(\cdot)} \leq \int_\Omega \mathcal{R}(x)|u(x)|^{p(x)}dx \leq |\mathcal{R}|_\infty |u|^{\hat{p}}_{p(\cdot)},$$

and

$$\delta|\nabla u|^{\check{p}}_{p(\cdot)} \leq |\nabla u|^{\check{p}}_{p(x)} \leq \int_\Omega |\nabla u(x)|^{p(x)}dx \leq |\nabla u|^{\hat{p}}_{p(\cdot)}.$$

Bearing in mind the following elementry inequality due to J.A. Clarkson: for all $\gamma > 0$, there exists $C_\gamma > 0$ such that

$$|\alpha + \beta|^\gamma \leq C_\gamma(|\alpha|^\gamma + |\beta|^\gamma),$$

for all $\alpha, \beta \in \mathbb{R}$. Then we deduce

$$\frac{\delta}{C_{\check{p}}}\|u\|^{\check{p}} \leq \int_\Omega (|\nabla u|^{p(x)} + \mathcal{R}(x)|u|^{p(x)})dx \leq (1 + |\mathcal{R}|_\infty)\|u\|^{\hat{p}}.$$

So, the proof is complete. It is enough to put $\mu = \frac{\delta}{C_{\check{p}}}, M = 1 + |\mathcal{R}|_\infty$.

□

The next is trace theorem [9, Chapter 5, Theorem 1] that plays a key role to prove our claims.

Theorem 2.2 *Assume that* $1 \leq p < +\infty$ *and* Ω *is bounded domain in* $\mathbb{R}^N, N > 1$*, and* $\partial\Omega$ *is* C^1*. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) *$Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;*

(ii)
$$|u|_{p,\partial} \leq c\|u\|_p$$
for each $u \in W^{1,p}(\Omega)$, with constant c depending only on p and Ω.

Remark 2.3 *For each $u \in W^{1,p(\cdot)}(\overline{\Omega})$, there exists c^* such that*
$$|u|_{p^-,\Sigma_1} \leq c^*\|u\|.$$

Proof: Let $u \in W^{1,p(\cdot)}(\overline{\Omega})$. Combining part (ii) of Theorem 2.2 and embedding inequality of Remark 2.1, we deduce that
$$|u|_{p^-,\Sigma_1} \leq |u|_{p^-,\partial} \leq c\|u\|_{p^-} \leq c\bar{c}\|u\|,$$
then it is enough to set $c^* = c\bar{c}$. □

Here, we recall recent critical points results obtained by Bonanno et al. where we make use of them to prove our claims in the next section.

Definition 2.1 *($(PS)^{[r]}$ condition) Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; Put*
$$I = \Phi - \Psi,$$
and fixed some $r \in [-\infty, +\infty]$. We say that I satisfies the Palais-Smale condition cut off upper at r (in short the $(PS)^{[r]}$), if any sequence $\{u_k\}_{k\in\mathbb{N}}$ in X such that

(i) *$\{I(u_k)\}$ is bounded;*

(ii) *$I'(u_k) \to 0$;*

(iii) *$\Phi(u_k) < r$ for all $k \in \mathbb{N}$;*

admits a convergent subsequence.

The next is a particular case of Bonanno [2, Theorem 5.1] which is one of the main tools of this note (See also [5]).

Theorem 2.3 *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*
$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$
Assume that there exists $r > 0$ and $\overline{x} \in X$, with $0 < \Phi(\overline{x}) < r$, such that

(i) *$\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;*

(ii) *for each $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r} \Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda_r$, there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that $I'_\lambda(x_{0,\lambda}) \equiv \Theta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \leq I_\lambda(x)$, for all $x \in \Phi^{-1}(]0, r[)$.

The other tool is the following abstract result which was proved in [4].

Theorem 2.4 *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*
$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the Palais-Smale condition and it is unbounded from below. Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

The last tool is achieved by G. Bonnano and S.A. Marano [3], that we recall in a convenient form.

Theorem 2.5 *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

- *(i)* $\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;
- *(ii) for each $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r} \Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

From now on we denote by X the reflexive Banach space $W^{1,p(\cdot)}(\Omega)$ endowed with the norm $\|u\|$. In the next section we state the main results of the note and their proofs.

3. Exsitence and mulitplicity of weak solution

Let Carathéodory function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ hold in conditions $(f1)$ and $(f2)$. Put

$$F(x,t) = \begin{cases} \int_0^t f(x,s)ds & t > 0, \\ 0 & t \le 0. \end{cases}$$

Integrating the growth condition $(f1)$, it gains that

$$0 \le F(x,t) \le \frac{\rho}{\theta^-}|t|^{\theta(x)} \tag{3.1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$. Now, define the functional $\Phi : X \to \mathbb{R}$ as follows

$$\Phi(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \mathcal{R}(x)|u|^{p(x)}\Big)dx - \int_\Omega F(x,u(x))dx.$$

Remark 3.1 *By simple calculations, from Remark 2.2, we have*

$$\frac{\mu}{p^+}\|u\|^{\check{p}} \le \Phi(u) \le \frac{1}{p^-}(M + \rho k_\theta)\|u\|^{\hat{p}}$$

for $u \in X$.

It is known that Φ is continuously Gâteaux differentiable functional; moreover,

$$\langle \Phi'(u), v\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + \mathcal{R}(x)|u|^{p(x)-2}uv + f(x,u)v\Big)dx$$

for $u, v \in X$.

Now, let $g : \partial\Omega \to \mathbb{R}$ be a function with the condition (g), then the functional $\Psi : X \to \mathbb{R}$ defined by

$$\Psi(u) := \int_{\Sigma_1} g(x)u(x)d\sigma,$$

is well-defined and continuously Gâteaux differentiable functional, with the compact derivative

$$\langle \Psi'(u), v\rangle = \int_{\Sigma_1} g(x)v(x)d\sigma,$$

for each u, v in X.

Definition 3.1 *(Solution) We say that $u \in X$ is a weak solution of the problem (1.1), if*

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda g(x) \quad \text{on } \Sigma_1 \setminus \Gamma \quad \& \quad u = 0 \quad \text{on } \Sigma_2$$

and the following integral equality is true

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + \mathcal{R}(x)|u|^{p(x)-2}uv + f(x,u)v\Big)dx = \lambda \int_{\Sigma_1} g(x)v(x)d\sigma$$

for any $v \in X$.

It is clear that the critical points of $I_\lambda = \Phi - \lambda\Psi$ are the weak solutions of the problem (1.1). Set

$$\delta(x) := \begin{cases} \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\} & \partial B(x,\delta)\cap\Sigma_1 \neq \emptyset, \\ 0 & \partial B(x,\delta)\cap \Sigma_1 = \emptyset. \end{cases}$$

And, define

$$R := \sup_{x\in\Omega} \delta(x).$$

Obviously, there exists $x_0 \in \Omega$ such that $R = \delta(x_0) > 0$:

$$B(x_0, R) \subseteq \Omega \qquad \& \qquad \partial B(x_0,R)\cap \Sigma_1 \neq \emptyset.$$

The next is one of the main results of this paper.

Theorem 3.1 *Assume that $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $(f1)$, $g : \Sigma_1 \to \mathbb{R}$ is such that the condition (g) holds true and*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Sigma_1} g(x)}{t^{\hat{p}-1}} = +\infty. \tag{3.2}$$

Then for every $\lambda \in]0, \lambda^*[$ with

$$\lambda^* := \frac{\mu^{\frac{1}{\hat{p}}}}{c^*|g|_{(p^-)',\Sigma_1}\, p^{+\frac{1}{\hat{p}}}},$$

the problem (1.1) has at least one nontrivial weak solution.

Proof: We apply Theorem 2.3 in the case $r = 1$.

Let X, Φ and Ψ be as above and fix $\lambda \in]0, \lambda^*[$. By (3.2) there exists

$$0 < \delta_\lambda < \min\left\{1, \Big(\frac{p^-}{(M+\rho k_\theta)(\frac{1}{R})^{\hat{p}} m R^N}\Big)^{\frac{1}{\hat{p}}}\right\},$$

such that

$$\frac{p^- \inf_{x\in\Sigma_1} g(x)\epsilon}{(M+\rho\kappa_\theta)(\frac{\delta_\lambda}{R})^{\hat{p}-1}(2^N-1)} > \frac{1}{N\lambda}, \tag{3.3}$$

where M is defined as in Remark 3.1 and $\epsilon \in (0,1]$ appears in

$$HV(\partial B(x_0,R)\cap\Sigma_1) = \epsilon\, HV(\partial B(x_0,R)), \tag{3.4}$$

in which $HV(D)$ is hypervolum of $D \subset \mathbb{R}^{N-1}$. We define $\bar{u}_\lambda \in X$ as follows

$$\bar{u}_\lambda(x) := \begin{cases} 0 & x \in \Omega \backslash \overline{B(x_0,R)}, \\ \frac{\delta_\lambda}{R}(R-|x-x_0|) & x \in \overline{B(x_0,R)} \backslash \Sigma_1 \\ \delta_\lambda & x \in \partial B(x_0,R) \cap \Sigma_1. \end{cases}$$

By Remark 3.1, we have

$$\begin{aligned} \Phi(\bar{u}_\lambda) &\leq \frac{1}{p^-}(M+\rho\kappa_\theta)\|\bar{u}_\lambda\|^{\hat{p}} \\ &\leq \frac{1}{p^-}(M+\rho\kappa_\theta)(\frac{\delta_\lambda}{R})^{\hat{p}} m R^N, \end{aligned}$$

where $m = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ is the measure of the unit ball in $\mathbb{R}^N$. So, clearly $0 < \Phi(\bar{u}_\lambda) < 1$. Moreover,

$$\Psi(\bar{u}_\lambda) \geq \int_{\partial B(x_0,R)\cap\Sigma_1} g(x)\bar{u}_\lambda(x)d\sigma \geq \inf_{x\in\Sigma_1} g(x)\delta_\lambda \mathfrak{m} R^{N-1}\epsilon, \tag{3.5}$$

in which $\mathfrak{m} = \frac{N\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ is the hypervolume of the $(N-1)$-dimensional unit sphere and ϵ is as in (3.4). So thanks to (3.3), one has

$$\frac{\Psi(\bar{u}_\lambda)}{\Phi(\bar{u}_\lambda)} \geq \frac{N \inf_{x\in\Sigma_1} g(x)\epsilon}{\frac{1}{p^-}(M+\rho\kappa_\theta)(\frac{\delta_\lambda}{R})^{\hat{p}-1}} > \frac{1}{\lambda}.$$

On the other hand, using Remark 3.1 for each $u \in \Phi^{-1}(]-\infty,1[)$, we have

$$\|u\| \leq (\frac{p^+}{\mu}\Phi(u))^{\frac{1}{\hat{p}}} < (\frac{p^+}{\mu})^{\frac{1}{\hat{p}}}.$$

From Hölder inequality and Remark 2.3, we have

$$\begin{aligned} \Psi(u) &= \int_{\Sigma_1} g(x)u(x)d\sigma \\ &\leq |g|_{(p^-)',\Sigma_1}|u|_{p^-,\Sigma_1} \\ &\leq c^*|g|_{(p^-)',\Sigma_1}\|u\| \\ &\leq c^*|g|_{(p^-)',\Sigma_1}\,(\frac{p^+}{\mu})^{\frac{1}{\hat{p}}}, \end{aligned} \tag{3.6}$$

for $u \in \Phi^{-1}(]-\infty,1[)$. So, we deduce that

$$\sup_{\Phi(u)<1} \Psi(u) \leq c^*|g|_{(p^-)',\Sigma_1}\,(\frac{p^+}{\mu})^{\frac{1}{\hat{p}}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}.$$

Then

$$\sup_{\Phi(u)<1} \Psi(u) < \frac{1}{\lambda} < \frac{\Psi(\bar{u}_\lambda)}{\Phi(\bar{u}_\lambda)}.$$

Therefore, Theorem 2.3 guarantees the existence of local minimum point u_0 for I_λ and the result would be achieved. □

Subsequently, we prove multiplicity of weak solutions for our problem.

Theorem 3.2 *Let $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition $(f1)$ and $(f2)$. Besides, assume that there exist $\vartheta > p^+$ and $r > 0$ such that*

$$0 < \vartheta F(x,t) \leq tf(x,t), \tag{3.7}$$

for each $x \in \Omega$ and $|t| \geq r$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem (1.1) admits at least two distinct weak solutions where

$$\lambda^* := \frac{1}{c^* |g|_{(p^-)', \Sigma_1} \ (\frac{p^+}{\mu})^{\frac{1}{\check{p}}}}.$$

Proof: We employ Theorem 2.4 in the case $r = 1$.

Let X, Φ, Ψ be as before. Integrating condition (3.7), there exist α, β such that

$$F(x,t) \geq \alpha |t|^{\vartheta} - \beta \tag{3.8}$$

for each $(x,t) \in \Omega \times \mathbb{R}$. Fix $\omega \in X \backslash \{\Theta_X\}$. From definitions of Φ and Ψ, Remark 3.1 and (3.8), one has

$$\begin{aligned} I_\lambda(t\omega) &= \Phi(t\omega) - \lambda \Psi(t\omega) \\ &\leq \frac{1}{p^-}(M + \rho\kappa_\theta) t^{p^+} \|\omega\|^{\hat{p}} - \lambda \alpha t^{\vartheta} \int_\Omega |\omega|^{\vartheta} dx + \lambda \beta |\Omega| - t \int_{\partial\Omega} g(x)|\omega| d\sigma \end{aligned}$$

for each $t > 1$. Since $\vartheta > p^+ > 1$, I_λ is unbounded from below. Now, let $\{u_k\} \subset X$ be a sequence such that $I(u_k) < +\infty$ and $I'(u_k) \to 0$. On the other hand, since

$$\begin{aligned} |\int_\Omega f(x,u) u dx| &\leq \int_\Omega |f(x,u)u| dx \\ &\leq \rho \int_\Omega |u|^{\theta(x)} dx \leq \rho\kappa_\theta \|u\|^{\hat{\theta}}, \end{aligned} \tag{3.9}$$

for each $u \in X$, we have the following estimate

$$\begin{aligned} \langle I'(u_k), u_k \rangle &= \langle \Phi'(u_k), u_k \rangle - \lambda \langle \Psi'(u_k), u_k \rangle \\ &\geq \mu \|u_k\|^{\check{p}} - \rho\kappa_\theta \|u_k\|^{\hat{\theta}} - \lambda c^* |g|_{(p^-)', \Sigma_1} \|u_k\|, \end{aligned}$$

then, for k large enough, one has

$$\|u_k\|^{\check{p}} \leq \rho\kappa_\theta \|u_k\|^{\hat{\theta}} + \lambda c^* |g|_{(p^-)', \Sigma_1} \|u_k\|,$$

which means $\{u_k\}$ is a bounded sequence in reflexive Banach space X, and so $\{u_k\}$ has a convergence subsequence. Then I_λ verifies Palais-Smale compactness condition. Therefore, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical point that are weak solutions of our problem. □

The next is the last result of this paper.

Theorem 3.3 *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition $(f1)$ and $(f2)$. And, there exist $r > 0$, $\delta > 0$ with*

$$r < \frac{\mu}{p^+} (\frac{\delta}{R})^{\check{p}} m R^N,$$

such that

$$\eta_r := c^* |g|_{(p^-)', \Sigma_1} \left(\frac{p^+}{\mu} r\right)^{\frac{1}{\check{p}}} < \frac{p^- N \inf_{x \in \Sigma_1} g(x) \epsilon}{(M + \rho\kappa_\theta)(\frac{\delta}{R})^{\hat{p}-1}}.$$

Then, for every λ where

$$\lambda \in \Lambda_{r,\delta} :=]\frac{(M + \rho\kappa_\theta)(\frac{\delta}{R})^{\hat{p}-1}}{p^- N \inf_{x \in \Sigma_1} g(x) \epsilon}, \frac{1}{\eta_r}[,$$

the problem (1.1) admits at least three weak solutions.

Proof: We are going to apply Theorem 2.5.

As we mentioned before, X, Φ, Ψ satisfy the regularity hypotheses requested in Theorem 2.5. Now, let $\underline{u}$ be defined as follows

$$\underline{u}(x) := \begin{cases} 0 & x \in \Omega \backslash \overline{B(x_0, R)}, \\ \frac{\delta}{R}(R - |x - x_0|) & x \in \overline{B(x_0, R)} \backslash \Sigma_1 \\ \delta & x \in \partial B(x_0, R) \cap \Sigma_1. \end{cases}$$

Then, we have

$$\begin{aligned} \frac{\mu}{p^+}(\frac{\delta}{R})^{\check{p}} m R^N &\leq \Phi(\underline{u}) \\ &\leq \frac{1}{p^-}(M + \rho\kappa_\theta)\|\underline{u}\|^{\hat{p}} \\ &\leq \frac{1}{p^-}(M + \rho\kappa_\theta)(\frac{\delta}{R})^{\hat{p}} m R^N. \end{aligned}$$

Using assumption $r < \frac{\mu}{p^+}(\frac{\delta}{R})^{\check{p}} m R^N$, one has $r < \Phi(\underline{u})$. Also

$$\Psi(\underline{u}) \geq \int_{\partial B(x_0,R)\cap\Sigma_1} g(x)\underline{u}(x)dx \geq \inf_{x\in\Sigma_1} g(x)\delta \mathfrak{m} R^{N-1}\epsilon,$$

where as above $\mathfrak{m}$ is the hypervolume of the $(N-1)$-dimensional unit sphere.

Then

$$\frac{\Psi(\underline{u})}{\Phi(\underline{u})} \geq \frac{N \inf_{x\in\Sigma_1} g(x)\epsilon}{\frac{1}{p^-}(M + \rho\kappa_\theta)(\frac{\delta}{R})^{\hat{p}-1}}. \tag{3.10}$$

For each $u \in \Phi^{-1}(]-\infty, r[)$, thanks to Remark 3.1, we gain the following relation

$$\|u\| \leq \Big(\frac{p^+}{\mu}\Phi(u)\Big)^{\frac{1}{\check{p}}} \leq \Big(\frac{p^+}{\mu} r\Big)^{\frac{1}{\check{p}}}. \tag{3.11}$$

From Hölder inequality and Remark 2.3, we have

$$\begin{aligned} \Psi(u) &= \int_{\Sigma_1} g(x)u(x)dx \\ &\leq |g|_{(p^-)',\Sigma_1}|u|_{(p^-),\Sigma_1} \\ &\leq c^*|g|_{(p^-)',\Sigma_1}\|u\|. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we gain

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) \leq c^*|g|_{(p^-)',\Sigma_1}\Big(\frac{p^+}{\mu} r\Big)^{\frac{1}{\check{p}}}.$$

Then, by our assumptions and (3.10), we have

$$\frac{1}{r}\sup_{\Phi(u)\leq r}\Psi(u) < \frac{\Psi(\underline{u})}{\Phi(\underline{u})},$$

which means condition (i) of Theorem 2.5 is verified. What remains is to prove that for each $\lambda > 0$,

$$I_\lambda = \Phi - \lambda\Psi$$

is coercive. But inequalities (3.9) and (3.12) lead to

$$I_\lambda(u) \geq \frac{\mu}{p^+}\|u\|^{\check{p}} - \rho\kappa_\theta\|u\|^{\hat{\theta}} - \lambda c^*|g|_{(p^-)',\Sigma_1}\|u\|,$$

and according to our hypothesis $1 \leq \theta(x) < p(x)$ a.e. in Ω, then coercivity of I_λ is obtained. If we consider

$$\Lambda_{r,\delta} \subseteq]\frac{\Phi(\underline{u})}{\Psi(\underline{u})}, \frac{r}{\sup_{\Phi(u)<r} \Psi(u)}[,$$

then Theorem 2.5 certifies that for each $\lambda \in \Lambda_{r,\delta}$, the functional I_λ admits at least three critical points in X that are weak solutions of our problem. $\square$

Remark 3.2 *We point out that, if $f(x,0) \neq 0$, then by Theorems 2.4 and 2.5, we obtain the existence of at least two and three non-zero weak solutions, respectively.*

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