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Analytical Expressions for the Exact Curved Surface Area and Volume of an Elliptic Paraboloid via Mellin-Barnes Type Contour Integration

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ABSTRACT: Our present investigation is motivated essentially by several interesting applications of generalized hypergeometric functions. The hypergeometric functions are potentially useful and have widespread applications related to the problems in the mathematical, physical, engineering and statistical sciences. In this article, we aim at obtaining the analytical expressions (not previously found and not recorded in the literature) for the exact curved surface area of an elliptic paraboloid in terms of Appell's double hypergeometric function of second kind. The derivation is based on Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and analytic continuation formula for Gauss function. Moreover, we also obtain the analytical expression for the volume of the right elliptic single cone. Some special cases related to right circular paraboloid are also discussed. The closed forms for the exact curved surface area and volume of an elliptic paraboloid are also verified numerically by using *Mathematica Program*.

Key Words: Appell's function of second kind, Mellin-Barnes type contour integral, Meijer's G-function, Elliptic paraboloid, Mathematica Program.

Contents

1. Introduction and preliminaries

For the definition of Pochhammer symbol, power series form of generalized hypergeometric function ${}_pF_q(z)$ and several related results, we refer the beautiful monographs [see e.g [2], [8], [15], [16], [28], [29] and [33]].

$$ {}_2F_1\left[\begin{array}{c} -\frac{1}{2},\ 1; \\ \\ 2\ ; \end{array} -z\right] = \frac{2}{3z}\left[(1+z)^{\frac{3}{2}}-1\right];\quad |z|<1. \tag{1.1}$$

The above formula can be derived simply by expanding the ${}_2F_1$ series.

Analytic continuation formula [8, p.63, Eq.(2.1.4(17)), [15], p.249, Eq.(9.5.9), [29], p.36, Eq.(1.8.1.11)]:

When $|z|>1$, then

$$ {}_2F_1\left[\begin{array}{c} a,\ b; \\ \\ c; \end{array} z\right] = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}(-z)^{-a}\ {}_2F_1\left[\begin{array}{c} a, 1+a-c; \\ \\ 1+a-b; \end{array} \frac{1}{z}\right]+ $$

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$$+\frac{\Gamma(c)\;\Gamma(a-b)}{\Gamma(a)\;\Gamma(c-b)}(-z)^{-b}\;{}_2F_1\left[\begin{array}{c}b,1+b-c;\\ \\ 1+b-a;\end{array}\;\frac{1}{z}\right], \tag{1.2}$$

where $|\arg(-z)|<\pi,\;|\arg(1-z)|<\pi$ and $a-b\neq 0,\pm1,\pm2,\pm3,....$

Mellin-Barnes type contour integral representation of Gauss' function [8, p.62, Eq.(15), [16], p.62, Eq.(28), [28], p.101, Eq.(9)]:

$${}_2F_1\left[\begin{array}{c}A,\;B;\\ \\ C;\end{array}\;z\right]=\frac{\Gamma(C)}{\Gamma(A)\Gamma(B)}\frac{1}{(2\pi i)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(A+s)\Gamma(B+s)\Gamma(-s)}{\Gamma(C+s)}(-z)^s\;ds, \tag{1.3}$$

where $z\neq 0, |\arg(-z)|<\pi, |z|<1$, and the parameters A,B,C are neither zero nor negative integers. Further $(B-A)$ is not an integer or zero.

Mellin-Barnes type contour integral representation of binomial function:

$$(1-z)^{-a}=\;{}_1F_0\left[\begin{array}{c}a;\\ \\ -;\end{array}\;z\right]=\frac{1}{(2\pi i)\;\Gamma(a)}\int_{-i\infty}^{+i\infty}\Gamma(a+s)\Gamma(-s)(-z)^s\;ds:\quad z\neq 0, \tag{1.4}$$

where $|\arg(-z)|<\pi, |z|<1, a\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $i=\sqrt{(-1)}$.
The equation (1.4) is the particular case of (1.3).

Appell's function of second kind [33, p.53, Eq.(5)] is defined as:

$$F_2\left[\;a;\;b,c;\;d,g;\;x,y\;\right]=F_{0:1;1}^{1:1;1}\left[\begin{array}{c}a:\;b;\;c;\\ \\ -:\;d;\;g;\end{array}\;x,y\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n\;x^m\;y^n}{(d)_m(g)_n\;m!\;n!}=\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\;y^n}{(g)_n\;n!}\;{}_2F_1\left[\begin{array}{c}a+n,\;b;\\ \\ d;\end{array}\;x\right]. \tag{1.5}$$

Convergence conditions of Appell's double series F_2:

(i) Appell's function F_2 is convergent when $|x|+|y|<1;\;a,b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

(ii) Appell's function F_2 is absolutely convergent when $|x|+|y|=1$;
$x\neq 0,\;y\neq 0;\;a,b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re(a+b+c-d-g)<0$.

(iii) When a is a negative integer, then Appell's series F_2 will be a polynomial, $b,c,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

(iv) When b,c are negative integers, then Appell's series F_2 will be a polynomial, $a,d,g\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

For absolutely and conditionally convergence of Appell's double series F_2, we refer a beautiful paper of Hài *et al.* [10].

Mellin-Barnes type double integral representation of Appell's function of second kind [8, p.232, Eq.(5.8.3(11)), [5]]:

$$F_2\left[\;a;\;b,c;\;d,g;\;x,y\;\right]=\frac{\Gamma(d)\Gamma(g)}{\Gamma(a)\Gamma(b)\Gamma(c)(2\pi i)^2}\int_{t=-i\infty}^{+i\infty}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(a+t+s)\Gamma(b+t)\Gamma(c+s)}{\Gamma(d+t)\Gamma(g+s)}\times$$

$$\times\Gamma(-t)\Gamma(-s)(-x)^t(-y)^s\;dt\;ds;\quad x\neq 0,\;y\neq 0. \tag{1.6}$$

Mellin-Barnes type contour integral representation of Meijer's G-function ([33, p.45, Eq.(1)], see also [8,16]):
When $p \leq q$ and $1 \leq m \leq q,\ 0 \leq n \leq p$, then

$$G^{m,n}_{p,q}\left(z\,\middle|\, \begin{array}{l} \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n; \alpha_{n+1}, ..., \alpha_p \\ \beta_1, \beta_2, \beta_3, ..., \beta_m; \beta_{m+1}, ..., \beta_q \end{array}\right) =$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^{m}\Gamma(\beta_j - s)\prod_{j=1}^{n}\Gamma(1-\alpha_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+s)\prod_{j=n+1}^{p}\Gamma(\alpha_j - s)}(z)^s\, ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\beta_1 - s)...\Gamma(\beta_m - s)\Gamma(1-\alpha_1+s)...\Gamma(1-\alpha_n+s)}{\Gamma(1-\beta_{m+1}+s)...\Gamma(1-\beta_q+s)\Gamma(\alpha_{n+1}-s)...\Gamma(\alpha_p - s)}(z)^s\, ds, \tag{1.7}$$

where $z \neq 0, (\alpha_i - \beta_j) \neq$ positive integers, $i = 1, 2, 3, ..., n;\ j = 1, 2, 3, ..., m$, and For details of three contours, see [8, p.207, [16], p.144].

Convergence conditions of Meijer's G-function:
When $\Lambda = m + n - \left(\frac{p+q}{2}\right),\ \nu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$, then

(i) The integral (1.7) is convergent when $|arg(z)| < \Lambda\pi$ and $\Lambda > 0$.

(ii) If $|arg(z)| = \Lambda\pi$ and $\Lambda \geq 0$, then the integral (1.7) is absolutely convergent when $p = q$ and $\Re(\nu) < -1$.

(iii) If $|arg(z)| = \Lambda\pi$ and $\Lambda \geq 0$, then the integral (1.7) is also absolutely convergent, when $p \neq q,\ (q-p)\sigma > \Re(\nu) + 1 - \left(\frac{q-p}{2}\right)$ and $s = \sigma + ik$, where σ and k are real. σ is chosen so that for $k \to \pm\infty$.

For other two types of contours, following will be convergence conditions of the integral (1.7)

(iv) The integral (1.7) is convergent if $q \geq 1$ and either $p < q, 0 < |z| < \infty$ or $p = q,\ 0 < |z| < 1$.

(v) The integral (1.7) is convergent if $p \geq 1$ and either $p > q, 0 < |z| < \infty$ or $p = q,\ |z| > 1$.

Relation between Meijer's G- function and ${}_2F_1(z)$ [19, p.61, [36], p.77, Eq.(1)]:

$$G^{2\ 2}_{2\ 2}\left(z\,\middle|\, \begin{array}{l} 1-a, 1-b; - \\ 0, c-a-b; - \end{array}\right) = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left[\begin{array}{ll} a,\ b; & \\ & 1-z \\ c; & \end{array}\right],$$

where $|1-z| < 1$ and $c-a, c-b \neq 0, -1, -2, ...$

$$G^{2\ 2}_{2\ 2}\left(z\,\middle|\, \begin{array}{l} a_1, a_2; - \\ b_1, b_2; - \end{array}\right) = \frac{\Gamma(1-a_1+b_1)\Gamma(1-a_1+b_2)\Gamma(1-a_2+b_1)\Gamma(1-a_2+b_2)z^{b_1}}{\Gamma(2-a_1-a_2+b_1+b_2)}\times$$

$$\times {}_2F_1\left[\begin{array}{ll} 1-a_1+b_1, 1-a_2+b_1; & \\ & 1-z \\ 2-a_1-a_2+b_1+b_2; & \end{array}\right];\quad |1-z| < 1. \tag{1.8}$$

Note: The equation of an elliptic paraboloid is given by $z = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right);\ c > 0;\ a \geq b > 0$, whose axis is z-axis and vertex is origin.
When $b = a$ and $c = h$ in the above elliptic paraboloid, we get the equation of a right circular paraboloid given by $z = \frac{h}{a^2}(x^2 + y^2)$, whose axis is z-axis, vertex is origin, "h" is the vertical height, radius of the circular base is "a" and its intersection with the plane $z = h$ will be a circle lying in the plane $z = h$ as

well as on the surface of $z = \frac{h}{a^2}\left(x^2+y^2\right)$.
Total area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be πab.

Suppose $\phi(x,y)=0$ is the projection of the curved surface of three dimensional figure $z=f(x,y)$ over the x-y plane, then curved surface area is given by

$$\hat{S} = \underbrace{\int\int}_{\substack{\text{over the area}\\ \phi(x,y)=0}} \sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}\, dx\, dy. \tag{1.9}$$

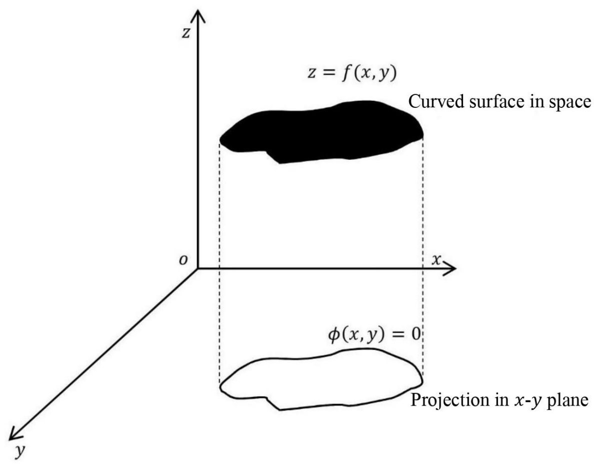

Figure 1: Projection of curved surface in x-y plane.

A definite integral:

$$\int_{\theta=0}^{\frac{\pi}{2}} \sin^{\alpha}\theta\cos^{\beta}\theta\, d\theta = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{2\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}, \tag{1.10}$$

where $\Re(\alpha) > -1,\ \Re(\beta) > -1$.

Motivated by the work of Abramowitz *et al.* [1], Andrews [3,4], Burchnall *et al.* [6] and others [7,9,11, 12,13,14,18,20,21,22,23,17,24,25,26,27,30,31,32,34,35,37,38], we evaluated an important definite integral $\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^s d\theta$ with suitable convergence conditions in **section 2**, by using Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and series manipulation technique. The integral is useful and helps in the derivation of closed form for the exact curved surface area of an elliptic paraboloid. In **section 3**, we derive the closed form for obtaining the exact curved surface area of an elliptic paraboloid by using Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and analytic continuation formula for Gauss function in terms of Appell's double hypergeometric function of second kind. In **section 4**, we obtain the formula for the volume of an elliptic paraboloid. In **section 5**, we derive some special cases related to the curved surface area and volume of right circular paraboloid.

2. Evaluation of some useful definite integrals

The following definite integrals hold true associated with suitable convergence conditions:

Theorem 2.1 $$\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^s d\theta=\frac{2\pi b}{a^{1+2s}}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ 1;\end{array} 1-\frac{b^2}{a^2}\right], \tag{2.1}$$

where $a \geq b > 0$ and it is obvious that $0 \leq (1-\frac{b^2}{a^2}) < 1$.

Theorem 2.2 $$\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^s d\theta=\frac{2\pi a}{b^{1+2s}}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ 1;\end{array} 1-\frac{a^2}{b^2}\right], \tag{2.2}$$

where $b \geq a > 0$ and it is obvious that $0 \leq (1-\frac{a^2}{b^2}) < 1$.

Remark: The above formulas are also verified numerically using Mathematica program.

Independent demonstration of the assertions (2.1) and (2.2)

$$\text{Suppose } I=\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^s d\theta.$$

$$=\frac{4}{b^{2s}}\int_{\theta=0}^{\frac{\pi}{2}}(\sin^2\theta)^s\left\{1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right\}^s d\theta$$

$$=\frac{4}{b^{2s}}\int_{\theta=0}^{\frac{\pi}{2}}\sin^{2s}\theta\ {}_1F_0\left[\begin{array}{c}-s;\\ -\ ;\end{array} \frac{-b^2\cos^2\theta}{a^2\sin^2\theta}\right] d\theta. \tag{2.3}$$

Employing the contour integral (1.4) of ${}_1F_0(.)$, we get

$$I=\frac{2}{\pi i\Gamma(-s)\ b^{2s}}\int_{\theta=0}^{\frac{\pi}{2}}\sin^{2s}\theta\left\{\int_{\zeta=-i\infty}^{+i\infty}\Gamma(-\zeta)\Gamma(-s+\zeta)\left(\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)^{\zeta} d\zeta\right\} d\theta. \tag{2.4}$$

Interchanging the order of integration in double integral of (2.4) and using the integral formula (1.10), we get

$$I=\frac{1}{\pi i\Gamma(-s)\Gamma(1+s)\ b^{2s}}\int_{\zeta=-i\infty}^{+i\infty}\Gamma(0-\zeta)\Gamma(1-(s+1)+\zeta)\Gamma\left(\frac{1}{2}+s-\zeta\right)\Gamma\left(1-\frac{1}{2}+\zeta\right)\left(\frac{b^2}{a^2}\right)^{\zeta} d\zeta. \tag{2.5}$$

Applying the definition (1.7) of Meijer's G-function, we get

$$I=\frac{2}{\Gamma(-s)\Gamma(1+s)\ b^{2s}}\ G^{2\ 2}_{2\ 2}\left(\frac{b^2}{a^2}\ \middle|\ \begin{array}{l}s+1,\frac{1}{2};-\\ 0,\frac{1}{2}+s;-\end{array}\right). \tag{2.6}$$

Employing the conversion formula (1.8) in equation (2.6), and after further simplification, we arrive at the result (2.1).
The proof of the result (2.2) follows the same steps as in the proof of (2.1). So we omit the details here.

3. Closed forms for curved surface area of an elliptic paraboloid

Theorem 3.1 The curved surface area of an elliptic paraboloid $z = c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$; $c>0$ and $a \geq b > 0$, whose vertical height is h along z-axis is given by:

I. when $a \geq b > 0$; $c < \frac{a^2}{4h}$, then curved surface area of an elliptic paraboloid is given by

$$\hat{S} = \left(\frac{b^2 h\pi}{c}\right) \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{m!}\, {}_2F_1\left[\begin{array}{c} 1+m, \frac{-1}{2}; \\ 2; \end{array} \frac{-4ch}{a^2}\right], \tag{3.1}$$

where h is the vertical height of an elliptic paraboloid and ${}_2F_1(.)$ is the Gauss hypergeometric function.

II. when $a \geq b > 0$; $c > \frac{a^2}{4h}$, then curved surface area of an elliptic paraboloid is given by

$$\hat{S} = \left(\frac{4\pi b^2 h\sqrt{h}}{3a\sqrt{c}}\right) \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{3}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{(m!)^2}\, {}_2F_1\left[\begin{array}{c} \frac{-1}{2}, \frac{-3}{2}; \\ \frac{-1}{2}-m\,; \end{array} \frac{-a^2}{4ch}\right] - \frac{a^2 b^2 \pi}{6c^2}, \tag{3.2}$$

where h is the vertical height of an elliptic paraboloid and ${}_2F_1(.)$ is the Gauss hypergeometric function.

Remark: For application purpose, the above formulas (3.1) and (3.2) containing infinite sum of Gauss functions, are verified numerically via *Mathematica* software under the stated associated conditions on a, b and c.

III. For the total surface area of an elliptic paraboloid $z = c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$; $c>0$ and $a \geq b > 0$, we add the curved surface area given by (3.1) (or (3.2)) and area of the base of an elliptic paraboloid given by $\frac{\pi abh}{c}$.

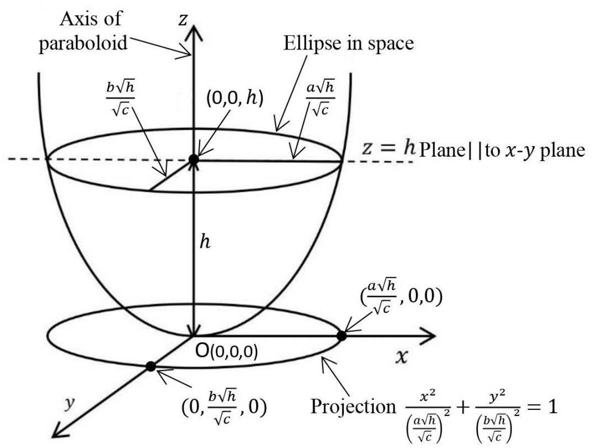

Figure 2: Elliptic paraboloid.

Proof: Equation of an elliptic paraboloid is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z}{c}; \quad a > b > 0;\ c > 0. \tag{3.3}$$

Therefore,

$$z = c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right); \ c > 0, \tag{3.4}$$

Intersection of the surface (3.4) with the plane $z = h$, where $h > 0$ (parallel to x-y plane, lying above x-y plane) will be an ellipse (in space) lying in the plane $z = h$ as well as lying on the surface of an elliptic paraboloid. The projection of that ellipse in x-y plane will be

$$\frac{x^2}{(\frac{a\sqrt{h}}{\sqrt{c}})^2} + \frac{y^2}{(\frac{b\sqrt{h}}{\sqrt{c}})^2} - 1 = 0, \tag{3.5}$$

whose semi-major and semi-minor axes will be $\frac{a\sqrt{h}}{\sqrt{c}}$ and $\frac{b\sqrt{h}}{\sqrt{c}}$ respectively.

Now from equation (3.4), we have

$$\frac{\partial z}{\partial x} = \frac{2cx}{a^2}, \quad \frac{\partial z}{\partial y} = \frac{2cy}{b^2}. \tag{3.6}$$

Substitute the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in equation (1.9). Therefore the curved surface area of an elliptic paraboloid will be

$$\hat{S} = \underbrace{\int\int}_{\substack{\text{over the area of an ellipse}\\ \frac{x^2}{\left(a\sqrt{\frac{h}{c}}\right)^2}+\frac{y^2}{\left(b\sqrt{\frac{h}{c}}\right)^2}=1}} \sqrt{\left\{1 + \frac{4c^2x^2}{a^4} + \frac{4c^2y^2}{b^4}\right\}}\, dx\ dy. \tag{3.7}$$

Put $x = aX\sqrt{(\frac{h}{c})},\ y = bY\sqrt{(\frac{h}{c})}$, therefore

$$\hat{S} = \frac{abh}{c} \underbrace{\int\int}_{\substack{\text{over the area of a circle}\\ X^2+Y^2=1}} \sqrt{\left\{1 + 4ch\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2}\right)\right\}}\, dX\ dY. \tag{3.8}$$

When $X = r\cos\theta,\ Y = r\sin\theta$, then $dX\ dY = rdr\ d\theta$.

$$\text{Therefore } \hat{S} = \frac{abh}{c}\int_{\theta=-\pi}^{\pi}\int_{r=0}^{1}\left\{1 + 4chr^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\right\}^{\frac{1}{2}} rdr\ d\theta. \tag{3.9}$$

Remark: Since we have no standard formula of definite/indefinite integrals in the literature of integral calculus for the integration with respect to "r" and "θ" in double integral (3.9). Therefore we can solve such integrals exactly through hypergeometric function approach.

$$\text{Therefore } \hat{S} = \frac{abh}{c}\int_{\theta=-\pi}^{\pi}\int_{r=0}^{1} {}_1F_0\left[\begin{array}{c} \frac{-1}{2}; \\ \\ -; \end{array} -4chr^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\right] rdr\ d\theta. \tag{3.10}$$

Since there is uncertainty about the argument of ${}_1F_0$ in equation (3.10), because the argument of ${}_1F_0$ in equation (3.10) may be greater than 1. Therefore applying contour integral (1.4) of ${}_1F_0(.)$ in equation (3.10), we get

$$\hat{S} = \frac{abh}{c}\int_{\theta=-\pi}^{\pi}\int_{r=0}^{1}\left[\frac{1}{(2\pi i)\Gamma\left(\frac{-1}{2}\right)}\int_{s=-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)\left\{4chr^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\right\}^s ds\right] rdr\ d\theta, \tag{3.11}$$

where $|\arg\left\{4chr^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\right\}| < \pi$ and $i = \sqrt{(-1)}$.

Interchanging the order of integration in double integral of (3.11), we get

$$\hat{S} = \frac{abh}{-4c\pi\sqrt{\pi}\ i}\int_{s=-i\infty}^{+i\infty}\Gamma(-s)\Gamma(\frac{-1}{2}+s)(4ch)^{s}\left\{\int_{r=0}^{1} r^{2s+1}\,dr\right\}\times \tag{3.12}$$

$$\times\left\{\int_{\theta=-\pi}^{\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^{s}d\theta\right\}ds. \tag{3.13}$$

Since $a > b$ then employing the useful integral (2.1) in (3.13), we get

$$\hat{S} = \frac{-b^2h}{4c\sqrt{\pi}\ i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(\frac{-1}{2}+s)(4ch)^{s}}{(1+s)\ a^{2s}}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},1+s;\\ \\ 1;\end{array}\ 1-\frac{b^2}{a^2}\right]ds, \tag{3.14}$$

where $\left(1-\frac{b^2}{a^2}\right) < 1$.

Employing contour integral (1.3) of ${}_2F_1(z)$ in (3.14), we get

$$\hat{S} = \frac{-b^2h}{4c\sqrt{\pi}\ i}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma\left(\frac{-1}{2}+s\right)}{(1+s)}\left(\frac{4ch}{a^2}\right)^{s}\times \tag{3.15}$$

$$\times\left\{\frac{1}{\Gamma(1+s)\Gamma(\frac{1}{2})}\frac{1}{(2\pi i)}\int_{t=-i\infty}^{+i\infty}\frac{\Gamma(1+s+t)\Gamma(\frac{1}{2}+t)\Gamma(-t)}{\Gamma(1+t)}\left(\frac{b^2}{a^2}-1\right)^{t}dt\right\}ds, \tag{3.16}$$

where $|\arg\left(\frac{b^2}{a^2}-1\right)| < \pi$.

$$\text{Therefore } \hat{S} = \frac{b^2h}{8c\pi^2}\int_{t=-i\infty}^{+i\infty}\int_{s=-i\infty}^{+i\infty}\frac{\Gamma(1+t+s)\Gamma(\frac{1}{2}+t)\Gamma\left(\frac{-1}{2}+s\right)\Gamma(-t)\Gamma(-s)}{\Gamma(1+t)\Gamma(2+s)}\times$$

$$\times\left(\frac{b^2}{a^2}-1\right)^{t}\left(\frac{4ch}{a^2}\right)^{s}dt\ ds;\ \left(\frac{b^2}{a^2}-1\right)\neq 0,\ \left(\frac{4ch}{a^2}\right)\neq 0. \tag{3.17}$$

Expressing the Mellin-Barnes type double contour integral (3.17) in terms of Appell's double hypergeometric function F_2 defined by (1.6), we get

$$\hat{S} = \left(\frac{b^2h\pi}{c}\right)F_2\left[\ 1;\ \tfrac{1}{2},\ \tfrac{-1}{2};\ 1,\ 2;\ 1-\tfrac{b^2}{a^2},\tfrac{-4ch}{a^2}\ \right];\ \left(1-\frac{b^2}{a^2}\right)\neq 0,\left(\frac{-4ch}{a^2}\right)\neq 0 \tag{3.18}$$

Case I

When $a \geq b > 0,\ c < \frac{a^2}{4h}$, then $|1-\frac{b^2}{a^2}|+|\frac{4ch}{a^2}| < 1$ or > 1 is always possible, therefore on further simplification, we get

$$\hat{S} = \left(\frac{b^2h\pi}{c}\right)\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^m}{m!}\ {}_2F_1\left[\begin{array}{c}1+m,\frac{-1}{2};\\ \\ 2;\end{array}\ \frac{-4ch}{a^2}\right],\ \left|\frac{4ch}{a^2}\right| < 1 \tag{3.19}$$

and hence we arrive at the result (3.1), where the double series always converges and ${}_2F_1(.)$ is the Gauss hypergeometric function.

Case II

When $a \geq b > 0$ and $c > \frac{a^2}{4h}$ i.e, $\left|\frac{-4ch}{a^2}\right| > 1$, then $|1-\frac{b^2}{a^2}|+|\frac{4ch}{a^2}| > 1$ is always possible, therefore applying analytic continuation formula (1.2) in Gauss' function ${}_2F_1(.)$ of (3.19), we get

$$\hat{S} = \left(\frac{b^2 h\pi}{c}\right) \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{m!} \left\{ \frac{\Gamma(\frac{3}{2}+m)}{\Gamma(1+m)\Gamma(\frac{5}{2})} \left(\frac{4ch}{a^2}\right)^{\frac{1}{2}} {}_2F_1 \left[\begin{array}{c} \frac{-1}{2}, \frac{-3}{2}; \\ \frac{-1}{2}-m; \end{array} \frac{-a^2}{4ch} \right] + \right.$$

$$\left. + \frac{\Gamma(\frac{-3}{2}-m)}{\Gamma(1-m)\Gamma(\frac{-1}{2})} \left(\frac{4ch}{a^2}\right)^{-(m+1)} {}_2F_1 \left[\begin{array}{c} 1+m,\ m; \\ m+\frac{5}{2}; \end{array} \frac{-a^2}{4ch} \right] \right\}$$

$$= \left(\frac{2b^2 h\pi}{3c}\right) \left\{ \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{3}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m \left(\frac{4ch}{a^2}\right)^{\frac{1}{2}}}{(1)_m \ m!} {}_2F_1 \left[\begin{array}{c} \frac{-1}{2}, \frac{-3}{2}; \\ \frac{-1}{2}-m; \end{array} \frac{-a^2}{4ch} \right] - \frac{a^2}{4ch} \right\}. \tag{3.20}$$

After further simplification, we arrive at the result (3.2). □

4. Volume of an elliptic paraboloid

The volume of an elliptic paraboloid having the vertical height h, α and β, the lengths of semi-major axis and semi-minor axis of elliptic base respectively, is given by

$$\mathrm{V} = \frac{\pi}{2}\alpha\beta h. \tag{4.1}$$

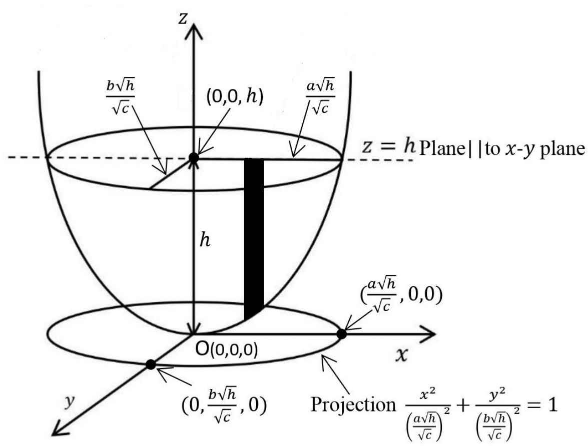

Figure 3: Elliptic paraboloid.

The equation of an elliptic paraboloid is given by

$$z = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right); \quad c > 0; a > b > 0. \tag{4.2}$$

$$\text{Now volume } = \underbrace{\iint}_{\substack{\text{over the projection of an elliptic} \\ \text{paraboloid bounded by } z=h}} \int_{z=c\lambda}^{z=h} dz\ dy\ dx; \quad \text{where } \lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{4.3}$$

$$= \underbrace{\int\int}_{\substack{\text{over the area of an ellipse}\\ \frac{x^2}{\left(a\sqrt{\frac{h}{c}}\right)^2}+\frac{y^2}{\left(b\sqrt{\frac{h}{c}}\right)^2}=1}} \left[h - c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right] dx\ dy. \tag{4.4}$$

Put $x = \frac{aX\sqrt{h}}{\sqrt{c}},\ y = \frac{bY\sqrt{h}}{\sqrt{c}}$, therefore,

$$\mathrm{V}\ = \frac{abh}{c} \underbrace{\int\int}_{\substack{\text{over the area of circle}\\ X^2+Y^2=1}} \left[h - c\left(\frac{h}{c}X^2 + \frac{h}{c}Y^2\right)\right]\ dX\ dY \tag{4.5}$$

$$= \frac{abh}{c} \underbrace{\int\int}_{\substack{\text{over the area of circle}\\ X^2+Y^2=1}} \left[h - h\left(X^2+Y^2\right)\right]\ dX\ dY. \tag{4.6}$$

When $X = r\cos\theta,\ Y = r\sin\theta$, then $dX\ dY = rdr\ d\theta$.

$$\text{Therefore V } = \frac{abh^2}{c}\int_{\theta=-\pi}^{\pi}\int_{r=0}^{1}\left\{1-r^2\right\}\ rdr\ d\theta \tag{4.7}$$

$$= \frac{abh^2}{c}\left(\int_{\theta=-\pi}^{\pi} d\theta\right)\left(\int_{r=0}^{1}\left\{r-r^3\right\}\ dr\right)$$

$$= \frac{2\pi abh^2}{c}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1. \tag{4.8}$$

On further simplification, we arrive at the volume of an elliptic paraboloid $z = c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right);\ c>0$ having vertical height h, is given by

$$\mathrm{V} = \frac{abh^2\pi}{2c} \tag{4.9}$$

$$= \frac{\pi}{2}\left(\frac{a\sqrt{h}}{\sqrt{c}}\right)\left(\frac{b\sqrt{h}}{\sqrt{c}}\right)h \tag{4.10}$$

$= \frac{\pi}{2}$ (length of semi-major axis of elliptic base)(length of semi-minor axis of elliptic base) (vertical height of an elliptic paraboloid).

5. Special cases of (3.1), (3.2) and (4.9)

I. From the closed form (3.1), we have

$$\hat{S} = \left(\frac{b^2h\pi}{c}\right)\left\{ {}_2F_1\left[\begin{array}{c} 1, \frac{-1}{2}; \\ 2; \end{array} \frac{-4ch}{a^2}\right] + \sum_{m=1}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(1-\frac{b^2}{a^2}\right)^m}{m!}\ {}_2F_1\left[\begin{array}{c} 1+m, \frac{-1}{2}; \\ 2; \end{array} \frac{-4ch}{a^2}\right]\right\}, \tag{5.1}$$

For curved surface area of a right circular paraboloid having vertical height h and radius of base a, put $b = a$ and $c = h$ $(h < a)$ in formula (5.1), we get

$$\hat{S} = a^2\pi\ {}_2F_1\left[\begin{array}{c} \frac{-1}{2}, 1; \\ 2; \end{array} -\frac{4h^2}{a^2}\right];\ \left|\frac{-4h^2}{a^2}\right| < 1. \tag{5.2}$$

Employing the result (1.1) in equation (5.2), we arrive at the curved surface area of a right circular paraboloid which comes out to be

$$\hat{S} = \frac{a\pi}{6h^2}\left[(a^2+4h^2)^{\frac{3}{2}} - a^3\right], \tag{5.3}$$

where a is the radius of circular base and h is the vertical height of right circular paraboloid.

II. From the closed form (3.2), we have

$$\hat{S} = \left(\frac{4\pi b^2 h\sqrt{h}}{3a\sqrt{c}}\right)\left\{ {}_1F_0\left[\begin{array}{c} \frac{-3}{2}; \\ -; \end{array} \frac{-a^2}{4ch}\right] + \sum_{m=1}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{3}{2}\right)_m \left(1-\frac{b^2}{a^2}\right)^m}{(m!)^2} \times \right.$$

$$\left. \times\ {}_2F_1\left[\begin{array}{c} \frac{-1}{2}, \frac{-3}{2}; \\ \frac{-1}{2}-m\ ; \end{array} \frac{-a^2}{4ch}\right]\right\} - \frac{a^2b^2\pi}{6c^2}, \tag{5.4}$$

For curved surface area of a right circular paraboloid having vertical height h and radius of base a, put $b = a$ and $c = h$ $(h > a)$ in formula (5.4), we get

$$\hat{S} = \left(\frac{4\pi ah}{3}\right)\ {}_1F_0\left[\begin{array}{c} \frac{-3}{2}; \\ -; \end{array} \frac{-a^2}{4h^2}\right] - \frac{a^4\pi}{6h^2};\ \left|\frac{-4h^2}{a^2}\right| > 1. \tag{5.5}$$

On using binomial theorem in equation (5.5), we get

$$\hat{S} = \left(\frac{4\pi ah}{3}\right)\left(1+\frac{a^2}{4h^2}\right)^{\frac{3}{2}} - \frac{a^4\pi}{6h^2}. \tag{5.6}$$

On further simplification, we arrive at the curved surface area of a right circular paraboloid

$$\hat{S} = \frac{a\pi}{6h^2}\left[(a^2+4h^2)^{\frac{3}{2}} - a^3\right], \tag{5.7}$$

where a is the radius of circular base and h is the vertical height of right circular paraboloid.

In both the cases, the formula for the curved surface area of a right circular paraboloid is same.

III. For right circular paraboloid, put $b = a$ and $c = h$ in formula (4.9), we get volume of right circular paraboloid

$$V = \frac{\pi a^2 h}{2}, \tag{5.8}$$

where a is the radius of circular base and h is the vertical height of right circular paraboloid.

6. Conclusion

In this paper, we obtained the closed form for exact curved surface area of an elliptic paraboloid $z = c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$, intercepted by the plane $z = h$, through hypergeometric function approach i.e, by using Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and analytic continuation formula for Gauss function in terms of Appell's double hypergeometric function of second kind. We also derived a closed form for obtaining the volume of an elliptic paraboloid. These formulas are neither available anywhere in the literature of mathematics nor found in any mathematical tables. So we believe that these formulas are new. For verification and justification, we have also given the solution of some numerical problems. Moreover, we also derived some special cases related to the right circular paraboloid. We conclude that many formulas for curved surface areas of other three dimensional figures can be derived in an analogous manner, using Mellin-Barnes contour integration. Moreover, the results deduced above (presumably new), have potential applications in the fields of applied mathematics, statistics and engineering sciences.

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