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Approximate Controllability of Non-autonomous Evolution System with Infinite Delay

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ABSTRACT: This article deals with the existence and approximate controllability results for a class of non-autonomous second-order evolution systems with infinite delay. To establish sufficient conditions for the proposed control problem the theory of evolution operator with Schauder's fixed point theorem is used. Further, we extend the approximate controllability results to the integro-impulsive differential system. Finally, to emphasize our theoretical concepts, an example is provided.

Key Words: Non-autonomous, Evolution operator, Approximate controllability, Infinite Delay.

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1. Introduction

In many real-world phenomena, there are many processes in which the rate of variation in the system state depends on past states. This feature is known as a delay or a time delay. Such systems can be described by differential equations so-called delay differential equations [23,7,32]. Delay can be easily seen in the networked control system, between the devices which are related through the network, during data exchanging to data congestion. Also in a biological system, the time necessary for red blood cells to mature in the bone marrow and there are plenty of real-life applications in which delay occurs. In recent years, many real-word applications have been investigated by authors to get the useful results for delay systems of various kinds [25,24,26,29,27]. On the other hand, Controllability plays a significant role in development of mathematical control theory and became an active area of research nowadays. Controllability is expressed as qualitative property of dynamical control systems introduced by Kalman, in 1960. It is widely used in various fields of science and engineering such as ecology, economics and biology etc. [11,9]. Controllability results for various dynamical systems have been extensively studied by research community [15,16,2,12] in past few years.

Approximate controllability results for a class of second order systems with infinite delay have been studied by Sakthivel et al. [17] with the help of Schauder's fixed point theorem. Existence results for non-autonomous second order systems have been established by Henríquez and Pozo [22]. Kumar et al. [28] investigated the existence of solution of non-autonomous fractional differential equations with integral impulse condition via the measure of non-compactness and k-set contraction. Ravikumar et al. [31] established the approximate controllability results for semi-linear differential system in Banach spaces using linear evolution theory with the resolvent operator and Schauder's fixed point theorem. Recently, Kumar et al. [29] investigated the approximate controllability results for the non-autonomous systems with finite delay in Banach space. We point out that, usually, researchers work on a time-independent system and less work have been reported in the literature on non-autonomous systems. For more information about non-autonomous (see [19,21,13,20,33]). To the best of the authors knowledge, the topic of non-autonomous second order evolution systems with infinite delay is not well developed.

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Motivated by the above facts, in this paper we consider the following non-autonomous second order system with infinite delay:

$$\begin{aligned} &\mathfrak{X}''(\varsigma) = \mathscr{A}(\varsigma)\mathfrak{X}(\varsigma) + \mathfrak{C}u(\varsigma) + \mathscr{F}(\varsigma, \mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)}), \quad \varsigma \in \mathfrak{J} = [0, \mathfrak{T}], \\ &\mathfrak{X}(0) = \varnothing(\varsigma) \in \mathscr{P}, \quad \mathfrak{X}'(0) = \xi_0, \quad \varsigma \in (-\infty, 0], \end{aligned} \tag{1.1}$$

where state function $\mathfrak{X}(\cdot)$ takes the values from Banach space $\mathcal{W}$ to $\mathcal{W}$ and linear operator $\mathscr{A}$ generates a strongly continuous cosine family of bounded linear operators in a Banach space $\mathcal{W}$. Let U is a Hilbert space and $u(\cdot) \in L^2([0, \mathfrak{T}]; U)$ is a control function. $\mathfrak{C} : U \to \mathcal{W}$ is a linear bounded operator, the functions $\mathfrak{X}_\varpi : (-\infty, 0] \to \mathcal{W}$, $\mathfrak{X}_\varpi(\theta) = \mathfrak{X}(\varpi + \theta)$ correspond to abstract phase space $\mathscr{P}$. $\mathscr{F} : \mathfrak{J} \times \mathscr{P} \to \mathcal{W}$ and $\rho : J \times \mathscr{P} \to (-\infty, \mathfrak{T}]$ are suitable non-linear functions to be stated subsequently.

2. Preliminaries

Consider the following non-autonomous differential system

$$\begin{aligned} \mathfrak{X}''(\varsigma) &= \mathscr{A}(\varsigma)\mathfrak{X}(\varsigma) + \mathscr{G}(\varsigma), \quad 0 \le \varsigma, \varpi \le \mathfrak{T}, \\ \mathfrak{X}(\varpi) &= \mathfrak{X}_0, \quad \mathfrak{X}'(\varpi) = \xi_0, \end{aligned} \tag{2.1}$$

where $\mathscr{A}(\varsigma) : \mathscr{D}(\mathscr{A}(\varsigma)) \subset \mathcal{W} \to \mathcal{W}$, is a closed operator dense in $\mathcal{W}$ and $\mathscr{G} : \mathfrak{J} \to \mathcal{W}$ is a suitable function. The existence of solutions for system (2.1) is associated to the existence of an evolution operator $\Psi(\varsigma, \varpi)$ for the homogeneous equation

$$\mathfrak{X}''(\varsigma) = \mathscr{A}(\varsigma)\mathfrak{X}(\varsigma), \quad 0 \le \varsigma, \varpi \le \mathfrak{T}. \tag{2.2}$$

Let us take that the domain of $\mathscr{A}(\varsigma)$ is a subspace $\mathscr{D}$ which is dense in $\mathcal{W}$ and independent of ς and for each $\mathfrak{X} \in \mathscr{D}$, the function$\varsigma \to \mathscr{A}(\varsigma)\mathfrak{X}$ is continuous. We refer to [8], for the fundamental solution of system (2.2). We will use the following concept of evolution operator for the development of our results.

Definition 2.1. *A family $\{\Psi(\varsigma, \varpi)\}$ of bounded linear operators from $\mathfrak{J} \times \mathfrak{J} \to \mathcal{L}(\mathcal{W})$ is called an evolution operator for (2.2) if the following conditions are holds:*

(S₁) $(\varsigma, \varpi) \in [0, \mathfrak{T}] \times [0, \mathfrak{T}] \to \Psi(\varsigma, \varpi)\mathfrak{X} \in \mathcal{W}$ *is of class* C^1, $\forall\ \mathfrak{X} \in \mathcal{W}$, *and*

(i) $\Psi(\varsigma, \varsigma) = 0, \ \forall\ \varsigma \in [0, \mathfrak{T}]$.

(ii) For all $\varsigma, \varpi \in [0, \mathfrak{T}]$, *and for each* $\mathfrak{X} \in \mathcal{W}$,

$$\frac{\partial}{\partial \varsigma}\Psi(\varsigma, \varpi)\mathfrak{X}\Big|_{\varsigma=\varpi} = \mathfrak{X}, \qquad \frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X}\Big|_{\varsigma=\varpi} = -\mathfrak{X}.$$

(S₂) *For all* $\varsigma, \varpi \in [0, \mathfrak{T}]$, *and if* $\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *then* $\Psi(\varsigma, \varpi)\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *the mapping* $(\varsigma, \varpi) \in [0, \mathfrak{T}] \times [0, \mathfrak{T}] \to \Psi(\varsigma, \varpi)\mathfrak{X} \in \mathcal{W}$ *is a class* C^2 *and*

(i) $\frac{\partial^2}{\partial \varsigma^2}\Psi(\varsigma, \varpi)\mathfrak{X} = \mathscr{A}(\varsigma)\Psi(\varsigma, \varpi)\mathfrak{X}$.

(ii) $\frac{\partial^2}{\partial \varpi^2}\Psi(\varsigma, \varpi)\mathfrak{X} = \Psi(\varsigma, \varpi)\mathscr{A}(\varpi)\mathfrak{X}$.

(iii) $\frac{\partial}{\partial \varpi}\frac{\partial}{\partial \varsigma}\Psi(\varsigma, \varpi)\mathfrak{X}\Big|_{\varsigma=\varpi} = 0$.

(S₃) *For all* $\varsigma, \varpi \in [0, \mathfrak{T}]$, *and if* $\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *then* $\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *as well as* $\frac{\partial^2}{\partial \varsigma^2}\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X}$, $\frac{\partial^2}{\partial \varpi^2}\frac{\partial}{\partial \varsigma}\Psi(\varsigma, \varpi)\mathfrak{X}$ *and*

(i) $\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *then* $\frac{\partial^2}{\partial \varsigma^2}\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X} = \mathscr{A}(\varsigma)\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X}$.

(ii) $\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X} \in \mathscr{D}(\mathscr{A})$, *then* $\frac{\partial^2}{\partial \varpi^2}\frac{\partial}{\partial \varsigma}\Psi(\varsigma, \varpi)\mathfrak{X} = \frac{\partial}{\partial \varsigma}\Psi(\varsigma, s)\mathscr{A}(\varpi)\mathfrak{X}$, *and the mapping* $(\varsigma, \varpi) \in [0, \mathfrak{T}] \times [0, \mathfrak{T}] \to \mathscr{A}(\varsigma)\frac{\partial}{\partial \varpi}\Psi(\varsigma, \varpi)\mathfrak{X}$ *is continuous.*

Suppose that $\Psi(\varsigma,\varpi)$ is an evolution operator associated with the operator $\mathscr{A}(\varsigma)$ and $\Phi(\varsigma,\varpi) = -\frac{\partial\Psi(\varsigma,\varpi)}{\partial\varpi}$. Furthermore, suppose that $\Phi(\varsigma,\varpi)$ and $\Psi(\varsigma,\varpi)$ is bounded by $\aleph$ and $\tilde{\aleph}$ respectively i.e $\sup_{0\leq\varsigma,\varpi\leq\mathfrak{T}}\|\Psi(\varsigma,\varpi)\| \leq \tilde{\aleph}$ and $\sup_{0\leq\varsigma,\varpi\leq\mathfrak{T}}\|\Phi(\varsigma,\varpi)\| \leq \aleph$. In addition, there is a constant $\aleph_1$ such that

$$\|\Psi(\varsigma+l,\varpi)-\Psi(\varsigma,\varpi)\| \leq \aleph_1|l|$$

for all $\varpi,\varsigma,\varsigma+l \in [0,\mathfrak{T}]$. The mild solution of the equation (2.1) is given by

$$\mathfrak{X}(\varsigma) = \Phi(\varsigma,\varpi)\mathfrak{X}_0 + \Psi(\varsigma,\varpi)\xi_0 + \int_{\varpi}^{\varsigma}\Psi(\varsigma,\tau)\mathscr{G}(\tau)d\tau.$$

provided that $\mathscr{G}:\mathfrak{J}\to\mathcal{W}$ is an integrable function.

In the literature, an abundance of techniques have been used to formulate the existence of the evolution operator $\Psi(\varsigma,\varpi)$. In particular, the quite well-known situation is that $\mathscr{A}(\varsigma)$ is the perturbation of operator $\mathscr{A}$ that generates a cosine family. Because of this, we briefly reviewing definition of the theory of cosine family and related terms.

Definition 2.2. *A one-parameter family $\{\Phi(\varsigma)\}_{\varsigma\in\mathbb{R}}$ of bounded linear operators mapping the Banach space $\mathcal{W}$ into $\mathcal{W}$ is called strongly continuous cosine family if and only if*
(i) $\Phi(\varpi+\varsigma)+\Phi(\varpi-\varsigma) = 2\Phi(\varpi)\Phi(\varsigma)$, $\forall\ \varpi,\varsigma\in\mathbb{R}$.
(ii) $\Phi(0)=\mathbf{I}$.
(iii) $\Phi(\varsigma)\mathfrak{X}$ *for each fixed point* $\mathfrak{X}\in\mathcal{W}$, *is continuous on* $\mathbb{R}$.

Let $\{\Phi(\varsigma)\}_{\varsigma\in\mathbb{R}}$ be a strongly continuous cosine family of bounded linear operators on Banach space $\mathcal{W}$ and have an infinitesimal generator $\mathscr{A}$ from $\mathscr{D}(\mathscr{A})$ to $\mathcal{W}$. We denote $\{\Psi(\varsigma)\}_{\varsigma\in\mathbb{R}}$ as the sine family associated with $\{\Phi(\varsigma)\}_{\varsigma\in\mathbb{R}}$ which is defined as follows:

$$\Psi(\varsigma)\mathfrak{X} = \int_0^{\varsigma}\Phi(\varpi)\mathfrak{X}\,d\varpi, \quad \mathfrak{X}\in\mathcal{W}, \quad \varsigma\in\mathbb{R}$$

The domain of the operator $\mathscr{A}$ is the Banach space and is defined as follows:

$$\mathscr{D}(\mathscr{A}) = \{\mathfrak{X}\in\mathcal{W} : \Phi(\varsigma)\mathfrak{X} \text{ is twice continuously differentiable in } \varsigma\}$$

endowed with norm

$$\|\mathfrak{X}\|_{\mathscr{A}} = \|\mathfrak{X}\| + \|\mathscr{A}\mathfrak{X}\|,\ \mathfrak{X}\in\mathscr{D}(\mathscr{A}).$$

Define $\tilde{\mathscr{D}} = \{\mathfrak{X}\in\mathcal{W} : \Phi(\varsigma)\mathfrak{X}$ is once continuously differentiable in $\varsigma\}$, is a Banach space endowed with norm

$$\|\mathfrak{X}\|_1 = \|\mathfrak{X}\| + \sup_{0\leq\varsigma\leq1}\|\mathscr{A}\Psi(\varsigma)\mathfrak{X}\|,\ \mathfrak{X}\in\tilde{\mathscr{D}}.$$

The results related with the existence of solutions for the second-order abstract Cauchy problem

$$\begin{aligned} \mathfrak{X}''(\varsigma) &= \mathscr{A}\mathfrak{X}(\varsigma)+\mathscr{K}(\varsigma), \quad \varpi\leq\varsigma\leq\mathfrak{T} \\ \mathfrak{X}(\varpi) &= \mathfrak{X}_0, \qquad \mathfrak{X}'(\varpi) = \xi_0, \end{aligned} \tag{2.3}$$

where $\mathscr{K}:[0,\mathfrak{T}]\to\mathcal{W}$ is an integrable function, the existence of solution for (2.3) is given in [1]. The existence of the solutions of semilinear second order abstract Cauchy problem has been discussed in [4]. The mild solution $\mathfrak{X}(\cdot)$ of the equation (2.3) is given by

$$\mathfrak{X}(\varsigma) = \Phi(\varsigma-\varpi)\mathfrak{X}_0 + \Psi(\varsigma-\varpi)\xi_0 + \int_{\varpi}^{\varsigma}\Psi(\varsigma-\tau)\mathscr{K}(\tau)d\tau, \quad 0\leq\varsigma\leq\mathfrak{T} \tag{2.4}$$

and when $\mathfrak{X}_0\in\tilde{\mathscr{D}}$, $\mathfrak{X}(\cdot)$ is continuously differentiable then

$$\mathfrak{X}'(\varsigma) = \mathscr{A}\Psi(\varsigma-\varpi)\mathfrak{X}_0 + \Phi(\varsigma-\varpi)\xi_0 + \int_{\varpi}^{\varsigma}\Psi(\varsigma-\tau)\mathscr{K}(\tau)d\tau, \quad 0\leq\varsigma\leq\mathfrak{T}.$$

In addition, if $\mathfrak{X}_0 \in \mathscr{D}(\mathscr{A}),\ \xi_0 \in \tilde{\mathscr{D}}$ and $\mathscr{G}$ is continuously differentiable function, then the function $\mathfrak{X}(\cdot)$ is a solution of the initial value problem (2.3). Let us take that $\mathscr{A}(\varsigma) = \mathscr{A} + \tilde{\mathscr{A}}(\varsigma)$ where $\tilde{\mathscr{A}}(\cdot) : \mathbb{R} \to \mathcal{L}(\tilde{\mathscr{D}}, \mathcal{W})$ is a map such that the function $\varsigma \to \tilde{\mathscr{A}}(\varsigma)\mathfrak{X}$ is a continuously differentiable in $\mathcal{W}$ for each $\mathfrak{X} \in \tilde{\mathscr{D}}$. For more details see [6], for each $(\mathfrak{X}_0, \xi_0) \in \mathscr{D}(\mathscr{A}) \times \tilde{\mathscr{D}}$ the non-autonomous Cauchy problem given below

$$\mathfrak{X}''(\varsigma) = (\mathscr{A} + \tilde{\mathscr{A}}(\varsigma))\mathfrak{X}(\varsigma), \quad \varsigma \in \mathbb{R} \tag{2.5}$$
$$\mathfrak{X}(0) = \mathfrak{X}_0, \qquad \mathfrak{X}'(0) = \xi_0 \tag{2.6}$$

has a unique solution $\mathfrak{X}(\cdot)$ such that the function $\varsigma \to \mathfrak{X}(\varsigma)$ is continuously differentiable in $\tilde{\mathscr{D}}$. Following similar argument, one can conclude that equation (2.5) with the initial condition of (2.3) has a unique solution $\mathfrak{X}(\cdot, \varpi)$ such that the function $\varsigma \to \mathfrak{X}(\varsigma, \varpi)$ is continuously differentiable in $\tilde{\mathscr{D}}$. It follows from (2.4) that

$$\mathfrak{X}(\varsigma, \varpi) = \Phi(\varsigma - \varpi)\mathfrak{X}_0 + \Psi(\varsigma - \varpi)\xi_0 + \int_{\varpi}^{\varsigma} \Psi(\varsigma - \tau)\tilde{\mathscr{A}}(\tau)\mathfrak{X}(\tau, \varpi)d\tau.$$

In particular, for $\mathfrak{X}_0 = 0$ we have

$$\mathfrak{X}(\varsigma, \varpi) = \Psi(\varsigma - \varpi)\xi_0 + \int_{\varpi}^{\varsigma} \Psi(\varsigma - \tau)\tilde{\mathscr{A}}(\tau)\mathfrak{X}(\tau, \varpi)d\tau.$$

Consequently,

$$\|\mathfrak{X}(\varsigma, \varpi)\|_1 \leq \|\Psi(\varsigma - \varpi)\|_{\mathcal{L}(\mathcal{W}, \tilde{\mathscr{D}})}\|\xi_0\| + \int_{\varpi}^{\varsigma} \|\Psi(\varsigma - \tau)\|_{\mathcal{L}(\mathcal{W}, \tilde{\mathscr{D}})}\|\tilde{\mathscr{A}}(\tau)\|_{\mathcal{L}(\mathcal{W}, \tilde{\mathscr{D}})}\|\mathfrak{X}(\tau, \varpi)\|_1 d\tau.$$

Gronwall's inequality implies that

$$\|\mathfrak{X}(\varsigma, \varpi)\|_1 \leq \tilde{\mathfrak{G}}\|\xi_0\|, \qquad \forall\, \varpi, \varsigma \in \mathfrak{J},$$

where $\tilde{\mathfrak{G}} = \|\Psi(\varsigma - \varpi)\| \exp[\|\Psi(\varsigma - \tau)\|\|\tilde{\mathscr{A}}(\tau)\|(\varsigma - \varpi)]$. Let us define the operator $\Psi(\varsigma, \varpi)\xi_0 = \mathfrak{X}(\varsigma, \varpi)$. By previous results it is concluded that $\Psi(\varsigma, \varpi)$ is a bounded linear map on $\tilde{\mathscr{D}}$. As $\tilde{\mathscr{D}}$ is dense in $\mathcal{W}$ so we can extend $\Psi(\varsigma, \varpi)$ to $\mathcal{W}$. For extension of $\Psi(\varsigma, \varpi)$, the notation $\Psi(\varsigma, \varpi)$ has been used. It is very well known fact that the cosine family $\Phi(\varsigma)$ can not be compact unless the $\dim(\mathcal{W}) < \infty$. By contrast, for the cosine family that arise in specific applications, the sine family $\Psi(\varsigma)$ is very often a compact operator for all $\varsigma \in \mathbb{R}$.

To establish the approximate controllability of the system (1.1), we assume the following condition: $(\mathbf{Y_1})$ $\alpha\mathscr{R}(\alpha, \Gamma_0^{\mathfrak{T}}) \to 0$ as $\alpha \to 0^+$ in the strong operator topology. The condition $(\mathbf{Y_1})$ holds if and only if the following second order linear control system

$$\begin{aligned} \mathfrak{X}''(\varsigma) &= \mathscr{A}\mathfrak{X}(\varsigma) + \mathfrak{C}u(\varsigma), \quad \varsigma \in \mathfrak{J} \\ \mathfrak{X}(0) &= \mathfrak{X}_0, \quad \mathfrak{X}'(0) = \xi_0, \end{aligned} \tag{2.7}$$

is approximately controllable on $\mathfrak{J}$, for instance see [10]. In [3], the phase space $\mathscr{P}$ be a linear space of functions

$$\mathscr{P} = \{\mathfrak{X} \mid \mathfrak{X} : (-\infty, 0] \to \mathcal{W}\}$$

endowed with a seminorm $\|.\|_{\mathscr{P}}$ and satifying the following axioms:

$(\mathbf{R_1})$ if $\mathfrak{X} : (-\infty, \sigma] \to \mathcal{W}$, $\sigma > 0$ is continuous on $[0, \sigma]$ and $\mathfrak{X}_0 \in \mathscr{P}$ then for every $\varsigma \in [0, \sigma]$ the following conditions holds:

(i) $\mathfrak{X}_\varsigma$ is in $\mathscr{P}$.

(ii) $\|\mathfrak{X}(\varsigma)\| \leq \tilde{\mathscr{H}}\|\mathfrak{X}_\varsigma\|_{\mathscr{P}}$.

(iii) $\|\mathfrak{X}_\varsigma\|_{\mathscr{P}} \leq \tilde{\mathscr{K}}(\varsigma)\|\mathfrak{X}_0\|_{\mathscr{P}} + \tilde{\mathscr{M}}(\varsigma) sup\{\|\mathfrak{X}(\varpi)\| : 0 \leq \varpi \leq \varsigma\}$ where $\tilde{\mathscr{H}} > 0$ is a constant ; $\tilde{\mathscr{K}}, \tilde{\mathscr{M}} : [0, \infty) \to [1, \infty)$, $\tilde{\mathscr{K}}$ is continuous, $\tilde{\mathscr{M}}$ is locally bounded and $\tilde{\mathscr{H}}, \tilde{\mathscr{K}}, \tilde{\mathscr{M}}$ are independent of $\mathfrak{X}(\cdot)$.

$(\mathbf{R_2})$ For the function $\mathfrak{X}(\cdot)$ in $(\mathbf{R_1})$, $\mathfrak{X}_\varsigma$ is a $\mathscr{P}$-valued continuous function on $[0,\sigma]$.

$(\mathbf{R_3})$ The space $\mathscr{P}$ is complete.

Definition 2.3. *A function $\mathfrak{X}:(-\infty,\mathfrak{T}]\to\mathcal{W}$ is said to be mild solution of the differential system (1.1) if $\mathfrak{X}(\cdot)\in C(\mathfrak{J},\mathcal{W})$, $\mathfrak{X}(\varsigma)=\varnothing(\varsigma)$ for $\varsigma\in(-\infty,0]$ and satisfies of the following integral equation*

$$\mathfrak{X}(\varsigma) = \Phi(\varsigma,0)\varnothing(0)+\Psi(\varsigma,0)\xi_0+\int_0^\varsigma \Psi(\varsigma,\varpi)[\mathcal{C}u(\varpi)+\mathscr{F}(\varsigma,\mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)})]d\varpi. \tag{2.8}$$

Definition 2.4. *The system (1.1) is called approximately controllable if $\overline{\tilde{\mathscr{D}}(\mathfrak{T})}=\mathcal{W}$ where*

$$\tilde{\mathscr{D}}(\mathfrak{T})=\{\mathfrak{X}(\mathfrak{T};u)\ :\ u(\cdot)\in L^2([0,\mathfrak{T}];U)\}$$

and $\mathfrak{X}(\varsigma,u)$ is a mild solution of (1.1).

Now define the operators

$$\Gamma_0^{\mathfrak{T}} = \int_0^{\mathfrak{T}} \Psi(\mathfrak{T},\varpi)\mathcal{C}\mathcal{C}^*\Psi^*(\mathfrak{T},\varpi)d\varpi$$
$$\mathscr{R}(\alpha,\Gamma_0^{\mathfrak{T}}) = (\alpha\mathbf{I}+\Gamma_0^T)^{-1},$$

where $\mathbf{I}$ is an identity operator and Γ_0^T is a linear operator.

In order to establish the controllability result of the system (1.1), we consider the following assumptions:

$(\mathbf{P_1})$ $\Psi(\varsigma),\varsigma>0$ is compact.

$(\mathbf{P_2})$ $\rho:\mathfrak{J}\times\mathscr{P}\to(-\infty,\mathfrak{T}]$ is continuous.

$(\mathbf{P_3})$ The function $\varsigma\to\varnothing_\varsigma$ is well defined and continuous from the set

$$\pounds(\rho^{-1})=\{\rho(\varpi,\xi):(\varpi,\xi)\in\mathfrak{J}\times\mathscr{P},\rho(\varpi,\xi)\le 0\}\text{ into }\mathscr{P},$$

and there exist a bounded continuous function

$$\mathscr{H}^{\varnothing}:\pounds(\rho^{-1})\to(0,\infty)\text{ such that }\|\varnothing_\varsigma\|_{\mathscr{P}}\le\mathscr{H}^{\varnothing}(\varsigma)\|\varnothing\|_{\mathscr{P}}\text{ for every }\varsigma\in\pounds(\rho^{-1}).$$

$(\mathbf{P_4})$ The function $\mathscr{F}:\mathfrak{J}\times\mathscr{P}\to\mathcal{W}$ satisfies the following conditions.

(i) $\mathscr{F}(\varsigma,\xi):\mathfrak{J}\times\mathcal{W}$ is strongly measurable, $\forall\ \xi\in\mathscr{P}$.

(ii) $\mathscr{F}(\varsigma,.):\mathscr{P}\to\mathcal{W}$ is continuous, $\forall\varsigma\in\mathfrak{J}$.

(iii) For each $r>0$, there exist a function $\lambda_r\in L^1(\mathfrak{J},R^+)$ such that

$$\sup_{\|\xi\|\le r}\|\mathscr{F}(\varsigma,\xi)\|\le\lambda_r(\varsigma)\text{ for a.e }\varsigma\in\mathfrak{J},$$

and
$$\lim_{r\to\infty}\inf\int_0^{\mathfrak{T}}\frac{\lambda_r(\varsigma)}{r}d\varsigma=\delta<\infty.$$

$(\mathbf{P_5})$ There exist $L>0$ such that

$$\|\mathscr{F}(\varsigma,\xi)\|\le L\quad\text{for all }(\varsigma,\xi)\in\mathfrak{J}\times\mathscr{P}.$$

Lemma 2.5. *([14]) Let $\mathfrak{X}:(-\infty,\mathfrak{T}]\to\mathcal{W}$ be a function such that $\mathfrak{X}_0=\varnothing$, then*

$$\|\mathfrak{X}_\varpi\|_{\mathscr{P}}\le(\mathscr{H}_3+\mathscr{H})\|\varnothing\|_{\mathscr{P}}+\mathscr{H}_2\sup\{\|x(\theta)\|;\theta\in[0,max\{0,\varpi\}]\},\varpi\in\pounds(\rho^{-1})\cup\mathfrak{J}$$

where $\mathscr{H}=\sup_{\varsigma\in\pounds(\rho^{-1})}\mathscr{H}^{\varnothing}(\varsigma)$, $\mathscr{H}_3=\sup_{\varsigma\in\mathfrak{J}}\tilde{M}(\varsigma)$ and $\mathscr{H}_2=\sup_{\varsigma\in\mathfrak{J}}\tilde{K}(\varsigma)$.

3. Main results

In this segment, we prove the approximate controllability of second-order non-autonomous system with infinite delay by using the Lebesgue dominated convergence theorem and Schauder's fixed point theorem. We will obtain sufficient conditions ensuring the existence of mild solution of differential system (1.1). To prove the results, we introduce the following notations.

$$M_{\mathcal{C}} = \|\mathcal{C}\| \;\; K = \|\mathfrak{X}_{\mathcal{T}}\| + \tilde{\mathscr{H}}\aleph\|\varnothing\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\|$$

Let $\mathscr{Z} = \{\mathfrak{X} \in C(\mathfrak{J}, \mathcal{W}); \mathfrak{X}(0) = \varnothing(0)\}$ be a Hilbert space. Consider a set $\mathscr{Q} = \{\mathfrak{X} \in \mathscr{Z}; \|\mathfrak{X}\| \leq \mathfrak{r}\}$ where $\mathfrak{r} > 0$ is a constant. It will be shown that system (1.1) is approximate controllable, if for all $\alpha > 0$ there exist a continuous function $\mathfrak{X}(\cdot) \in \mathscr{Z}$ such that

$$\begin{aligned} \mathfrak{X}(\varsigma) &= \Phi(\varsigma, 0)\varnothing(0) + \Psi(\varsigma, 0)\xi_0 + \int_0^{\varsigma} \Psi(\varsigma, \varpi)[\mathcal{C}u(\varpi) + \mathscr{F}(\varsigma, \mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)})]d\varpi \\ u(\varsigma) &= \mathcal{C}^*\Psi^*(T, \varsigma)\mathscr{R}(\alpha, \Gamma_0^{\mathcal{T}})\mathfrak{q}(\mathfrak{X}(\cdot)) \\ \mathfrak{q}(\mathfrak{X}(\cdot)) &= \mathfrak{X}_{\mathcal{T}} - \Phi(\mathcal{T}, 0)\varnothing(0) - \Psi(\mathcal{T}, 0)\xi_0 - \int_0^{\mathcal{T}} \Psi(\mathcal{T}, \varpi)\Big[\mathscr{F}(\varsigma, \mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)})\Big]d\varpi. \end{aligned}$$

Theorem 3.1. *Assume that condition* $(\mathbf{P_1}) - (\mathbf{P_4})$ *are holds and suppose that for all* $\alpha > 0$

$$(1 + \frac{1}{\alpha}\aleph^2 M_{\mathcal{C}}^2 \mathcal{T})\tilde{\aleph}\mathscr{H}_2\delta < 1,$$

then system (1.1) has solution on $\mathfrak{J}$.

Proof. Define the operator $\mathbb{F}_\alpha : \mathscr{Z} \to \mathscr{Z}$, as follows

$$\mathbb{F}_\alpha\mathfrak{X}(\varsigma) \;= \Phi(\varsigma, 0)\varnothing(0) + \Psi(\varsigma, 0)\xi_0 + \int_0^{\varsigma} \Psi(\varsigma, \varpi)[\mathcal{C}v(\varpi) + \mathscr{F}(\varsigma, \bar{\mathfrak{X}}_{\rho(\varsigma,\bar{\mathfrak{X}}_\varsigma)})]d\varpi, \;\; \alpha > 0$$

where

$$\begin{aligned} v(\varsigma) &= \mathcal{C}^*\Psi^*(\mathcal{T}, \varsigma)\mathscr{R}(\alpha, \Gamma_0^{\mathcal{T}})\mathfrak{q}(\mathfrak{X}(\cdot)) \\ \mathfrak{q}(\mathfrak{X}(\cdot)) &= \mathfrak{X}_{\mathcal{T}} - \Phi(\mathcal{T}, 0)\varnothing(0) - \Psi(\mathcal{T}, 0)\xi_0 - \int_0^{\mathcal{T}} \Psi(\mathcal{T}, \varpi)[\mathscr{F}(\varsigma, \bar{\mathfrak{X}}_{\rho(\varsigma,\bar{\mathfrak{X}}_\varsigma)})]d\varpi. \end{aligned}$$

and $\bar{\mathfrak{X}} : (-\infty, \mathcal{T}] \to \mathcal{W}$ such that $\bar{\mathfrak{X}}_0 = \varnothing$ and $\bar{\mathfrak{X}} = \mathfrak{X}$ on $[0, \mathcal{T}]$, the proof of this theorem is divided into three steps.

Step 1. We have to prove that $\mathbb{F}_\alpha$ is self map. Let $\bar{\varnothing} : (-\infty, \mathcal{T}] \to \mathcal{W}$ be the extension of $\varnothing$ to $(-\infty, \mathcal{T}]$ such that $\bar{\varnothing}(\theta) = \varnothing(0)$ on $\mathfrak{J}$. Our aim, is to show that the operator $\mathbb{F}_\alpha : \mathscr{Z} \to \mathscr{Z}$ has a fixed point. We assume that there exist $\mathfrak{r} > 0$ such that $\mathbb{F}_\alpha(\mathscr{Q}) \subset \mathscr{Q}$. Suppose that our assumption is false, then there exist $\alpha > 0$ such that for all $\mathfrak{r} > 0$, there exist $\mathfrak{X}' \in \mathscr{Q}$ and $\varsigma' \in \mathfrak{J}$ such that $\mathfrak{r} < \|\mathbb{F}_\alpha\mathfrak{X}'(\varsigma')\|$. We get

$$\begin{aligned} \mathfrak{r} &< \|\mathbb{F}_\alpha\mathfrak{X}'(\varsigma')\| \\ &\leq \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \tilde{\aleph}M_{\mathcal{C}}\int_0^{\varsigma} \|v(\varpi)\|d\varpi + \tilde{\aleph}\int_0^{\varsigma} \|[\mathscr{F}(\varpi, \bar{\mathfrak{X}}'_{\rho(\varsigma,\bar{\mathfrak{X}}'_\varpi)})]\|d\varpi, \end{aligned}$$

for any $\mathfrak{X} \in \mathscr{Q}$, it follows from above Lemma (2.5)

$$\|\bar{\mathfrak{X}}'_{\rho(\varsigma,\bar{\mathfrak{X}}'_\varpi)}\| \leq (\mathscr{H}_3 + \mathscr{H})\|\varnothing\|_{\mathscr{P}} + \mathscr{H}_2\mathfrak{r} = \mathfrak{r}^*, \quad \mathfrak{r}^* > 0. \tag{3.1}$$

Hence, we get

$$\mathfrak{r} \leq \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \tilde{\aleph}M_{\mathfrak{C}}\mathfrak{T}\|v(\varpi)\| + \tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{r^*}(\varpi)d\varpi$$

$$\mathfrak{r} \leq \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \tilde{\aleph}M_{\mathfrak{C}}\mathfrak{T}\Big\{\mathfrak{C}^*\Psi^*(\mathfrak{T},\varsigma)\mathscr{R}(\alpha,\Gamma_0^{\mathfrak{T}})\big[\mathfrak{X}_{\mathfrak{T}} - \Phi(\mathfrak{T},0)\varnothing(0) - \Psi(\mathfrak{T},0)\xi_0$$

$$- \int_0^{\mathfrak{T}} \Psi(\mathfrak{T},\varpi)[\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}_{\varpi}})})]d\varpi\big]\Big\} + \tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{r^*}(\varpi)d\varpi$$

$$\mathfrak{r} \leq \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \tilde{\aleph}M_{\mathfrak{C}}\mathfrak{T}\Big\{\frac{1}{\alpha}\tilde{\aleph}M_{\mathfrak{C}}[K + \tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{r^*}(\varpi)d\varpi]\Big\} + \tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{r^*}(\varpi)d\varpi$$

$$\mathfrak{r} \leq \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \frac{1}{\alpha}\tilde{\aleph^2}M_{\mathfrak{C}}^2K\mathfrak{T} + (1 + \frac{1}{\alpha}\tilde{\aleph^2}M_{\mathfrak{C}}^2\mathfrak{T})\tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{\mathfrak{r}^*}(\varpi)d\varpi$$

$$\mathfrak{r} \leq K^* + (1 + \frac{1}{\alpha}\tilde{\aleph^2}M_{\mathfrak{C}}^2\mathfrak{T})\tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{\mathfrak{r}^*}(\varpi)d\varpi,$$

where $K^* = \tilde{\mathscr{H}}\aleph\|\varnothing(0)\|_{\mathscr{P}} + \tilde{\aleph}\|\xi_0\| + \frac{1}{\alpha}\tilde{\aleph^2}M_{\mathfrak{C}}^2K\mathfrak{T}$, we note that K^* is independent of $\mathfrak{r}$ and $\mathfrak{r}^* \to \infty$ as $\mathfrak{r} \to \infty$. Now

$$\lim_{\mathfrak{r}\to\infty} \inf \int_0^{\mathfrak{T}} \frac{\lambda_{\mathfrak{r}^*}(\varpi)}{\mathfrak{r}}d\varpi = \lim_{\mathfrak{r}\to\infty} \inf \int_0^{\mathfrak{T}} \frac{\lambda_{\mathfrak{r}^*}(\varpi)\mathfrak{r}^*}{\mathfrak{r}^*\mathfrak{r}}dt \ = \delta\mathscr{H}_2.$$

Hence for $\alpha > 0$, we obtain

$$(1 + \frac{1}{\alpha}\tilde{\aleph^2}M_{\mathfrak{C}}^2\mathfrak{T})\tilde{\aleph}\mathscr{H}_2\delta > 1,$$

which contradict our hypothesis, therefore, for $\alpha > 0$, $\exists\ \mathfrak{r} > 0$ such that $\mathbb{F}_\alpha(\mathscr{Q}) \subset \mathscr{Q}$.

Step 2. Next we prove that the operator $\mathbb{F}_\alpha$ is compact. Consider $\Pi(\varsigma) = \{\mathbb{F}_\alpha\mathfrak{X}(\varsigma) : \mathfrak{X} \in \mathscr{Q}\}$. Define

$$(\mathbb{F}_\alpha^\varepsilon\mathfrak{X})(\varsigma) = \Phi(\varsigma,0)\varnothing(0) + \Psi(\varsigma,0)\xi_0 + \int_0^{\varsigma-\varepsilon} \Psi(\varsigma,\varpi)[\mathfrak{C}v(\varpi) + \mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}_{\varpi}})})]d\varpi.$$

Since $\Psi(\varsigma,\varpi)$ is compact, the set $\Pi_\varepsilon(\varsigma) = \{\mathbb{F}_\alpha^\varepsilon\mathfrak{X}(\varsigma) : \mathfrak{X} \in \mathscr{Q}\}$ is relatively compact in $\mathcal{W}$, That is, a finite set $\{\varrho_i, 1 \leq i \leq n\}$ in $\mathcal{W}$ exists such that $\Pi_\varepsilon(\varsigma) \subset \bigcup_{i=1}^{\infty} \tilde{\aleph}(\varrho_i,\frac{\tau}{2})$, where $\bigcup_{i=1}^{\infty} \tilde{\aleph}(\varrho_i,\frac{\tau}{2})$ is an open ball in $\mathcal{W}$ with center at ϱ_i and radius $\frac{\tau}{2}$. On the other hand,

$$\begin{aligned}\|(\mathbb{F}_\alpha\mathfrak{X})(\varsigma) - (\mathbb{F}_\alpha^\varepsilon\mathfrak{X})(\varsigma)\| &\leq \tilde{\aleph}M_{\mathfrak{C}}\int_{\varsigma-\varepsilon}^{\varsigma} \|v(\varpi)\|d\varpi \\ &\quad + \tilde{\aleph}\int_{\varsigma-\varepsilon}^{\varsigma} \|\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}_{\varpi}})})\|d\varpi \\ &\leq \frac{\tau}{2}.\end{aligned}$$

Consequently, $\Pi(\varsigma) \subset \bigcup_{i=1}^{\infty} \tilde{\aleph}(\varrho_i,\tau)$. Hence, there exist relatively compact set arbitrarily close to $\Pi(\varsigma) = \{\mathbb{F}_\alpha\mathfrak{X}(\varsigma) : \mathfrak{X}(\varsigma) \in \mathscr{Q}\}$ as $\varepsilon \to 0$. Since it is compact at $\varsigma = 0$, hence set $\Pi(\varsigma)$ is relatively compact in $\mathcal{W}$, $\ \forall\, \varsigma \in [0,\mathfrak{T}]$. Now we prove that $\Pi(\varsigma) = \{\mathbb{F}_\alpha\mathfrak{X}(\cdot) : \mathfrak{X}(\cdot) \in \mathscr{Q}\}$ is equicontinuous on $[0,T]$ for $0 < \varsigma_1 < \varsigma_2 < T$,

$$\begin{aligned}\|(\mathbb{F}_\alpha\mathfrak{X})(\varsigma_2) - (\mathbb{F}_\alpha\mathfrak{X})(\varsigma_1)\| \ &\leq \ \|\Phi(\varsigma_2,0) - \Phi(\varsigma_1,0)\|\|\varnothing(0)\|_{\mathscr{P}}\tilde{\mathscr{H}} + \|\Psi(\varsigma_2,0) - \Psi(\varsigma_1,0)\|\|\xi_0\| \\ &+ \ \tilde{N}M_{\mathfrak{C}}\int_{\varsigma_1}^{\varsigma_2} \|u(\varpi)\|d\varpi + M_{\mathfrak{C}}\int_0^{\varsigma_1} \|\Psi(\varsigma_2,\varpi) - \Psi(\varsigma_1,\varpi)\|\|u(\varpi)\|d\varpi \\ &+ \ \tilde{\aleph}\int_{\varsigma_1}^{\varsigma_2} \lambda_{\mathfrak{r}^*}(\varpi)d\varpi + \int_0^{\varsigma_1} \|\Psi(\varsigma_2,\varpi) - \Psi(\varsigma_1,\varpi)\|\lambda_{\mathfrak{r}^*}(\varpi)d\varpi.\end{aligned}$$

$$
\begin{aligned}
&\|(\mathbb{F}_\alpha\mathfrak{X})(\varsigma_2)-(\mathbb{F}_\alpha\mathfrak{X})(\varsigma_1)\|\\
&\leq \|\Phi(\varsigma_2,0)-\Phi(\varsigma_1,0)\|\|\varnothing(0)\|_{\mathscr{P}\tilde{\mathscr{H}}}+\|\Psi(\varsigma_2,0)-\Psi(\varsigma_1,0)\|\|\xi_0\|\\
&+\frac{\tilde{\aleph}^2M_{\mathcal{C}}^2}{\alpha}\int_{\varsigma_1}^{\varsigma_2}\Big(\|\mathfrak{X}_{\mathcal{T}}\|+\aleph\|\varnothing(0)\|_{\mathscr{P}\tilde{\mathscr{H}}}+\tilde{\aleph}\|\xi_0\|+\tilde{\aleph}\int_0^{\mathcal{T}}\lambda_{\mathfrak{r}^*}(\varpi)d\varpi\Big)d\eta\\
&+\frac{\tilde{\aleph}M_{\mathcal{C}}^2}{\alpha}\int_0^{\varsigma_1}\|\Psi(\varsigma_2,\varpi)-\Psi(\varsigma_1,\varpi)\|\Big(\|\mathfrak{X}_{\mathcal{T}}\|+\aleph\|\varnothing(0)\|_{\mathscr{P}\tilde{\mathscr{H}}}+\tilde{\aleph}\|\xi_0\|\\
&+\tilde{\aleph}\int_0^{\mathcal{T}}\lambda_{r^*}(\varpi)d\varpi\Big)d\eta\\
&+\tilde{\aleph}\int_{\varsigma_1}^{\varsigma_2}\lambda_{r^*}(\varpi)d\varpi+\int_0^{\varsigma_1}\|\Psi(\varsigma_2,\varpi)-\Psi(\varsigma_1,\varpi)\|\lambda_{\mathfrak{r}^*}(\varpi)d\varpi.
\end{aligned}
$$

Here, it can be seen that $\|(\mathbb{F}_\alpha\mathfrak{X})(\varsigma_2)-(\mathbb{F}_\alpha\mathfrak{X})(\varsigma_1)\|\to 0$ as $(\varsigma_1-\varsigma_2)\to 0$. Also, the compactness of evolution operator $\Psi(\varsigma,\varpi)$ implies the continuity in the uniform operator topology. Thus, the set $\Pi(\varsigma)=\{\mathbb{F}_\alpha\mathfrak{X}(\varsigma):\mathfrak{X}\in\mathscr{Q}\}$ is equicontinuous on $[0,\mathcal{T}]$.

Step 3. Let $\{\mathfrak{X}^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{Q}$ and $\mathfrak{X}\in\mathscr{Q}$ such that $\mathfrak{X}^n\to\mathfrak{X}$ in $\mathscr{Z}$. From axioms $(\mathbf{R_1})$, we find that $\bar{\mathfrak{X}}^n_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)}\to\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}_\varpi)}$ as $n\to\infty\ \forall\ \varpi\in J$. Now from the inequality,

$$
\begin{aligned}
\|\mathscr{F}(\varpi,\bar{\mathfrak{X}}^n_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)}) &- \mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}_\varpi)})\|\\
&\leq \|\mathscr{F}(\varpi,\bar{\mathfrak{X}}^n_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)})-\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)})\|\\
&+ \|\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)})-\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}_\varpi)})\|.
\end{aligned}
$$

We infer that

$$
\mathscr{F}(\varpi,\bar{\mathfrak{X}}^n_{\rho(\varpi,\bar{\mathfrak{X}}^n_\varpi)})\to\mathscr{F}(\varpi,\bar{\mathfrak{X}}_{\rho(\varpi,\bar{\mathfrak{X}}_\varpi)})\ \ as\ \ n\to\infty\ \ \forall\ \ \varpi\in\mathfrak{J}.
$$

By the assumption $(\mathbf{R_2})$, and Lebesgue dominated convergence theorem, we concludes that $\mathbb{F}_\alpha\mathfrak{X}^n\to\mathbb{F}_\alpha\mathfrak{X}$ in $\mathscr{Z}$. Hence, $\mathbb{F}_\alpha(\cdot)$ is continuous on $\mathscr{Q}$. Thus in view of Schauder's fixed point theorem, the operator $\mathbb{F}_\alpha$ has a fixed point and the delay non-autonomous system (1.1) has a solution on $\mathfrak{J}$. □

Theorem 3.1. *If the assumptions* $(\mathbf{P_1})-(\mathbf{P_5})$ *are holds and linear system* (2.7) *is approximate controllable on* $\mathfrak{J}$*, then the nonlinear delay non-autonomous system* (1.1) *is approximate controllable.*

Proof. Let $\mathfrak{X}^\alpha(\cdot)$ be a fixed point of $\mathbb{F}_\alpha$ in $\mathscr{Q}$. Any fixed point of $\mathbb{F}_\alpha$ is a mild solution of the system (1.1) under the control

$$
u^\alpha(\varsigma)=\mathcal{C}^*\Psi^*(\mathcal{T},\varsigma)\mathscr{R}(\alpha,\Gamma_0^{\mathcal{T}})\mathfrak{q}(\mathfrak{X}^\alpha),
$$

where

$$
\mathfrak{q}(\mathfrak{X}^\alpha)=\mathfrak{X}_{\mathcal{T}}-\Phi(\mathcal{T},0)\varnothing(0)-\Psi(\mathcal{T},0)y_0-\int_0^{\mathcal{T}}\Psi(\mathcal{T},\varpi)\Big[\mathscr{F}(\varpi,\mathfrak{X}_{\rho(\varpi,\mathfrak{X}_\varpi)})\Big]d\varpi,
$$

and satisfies the inequality

$$
\mathfrak{X}^\alpha(\mathcal{T})=\mathfrak{X}_{\mathcal{T}}+\alpha\mathscr{R}(\alpha,\Gamma_0^{\mathcal{T}})\mathfrak{q}(\mathfrak{X}^\alpha).
$$

By the assumption $(\mathbf{A_3})$

$$
\int_0^{\mathcal{T}}\|\mathscr{F}(\varpi,\bar{\mathfrak{X}^\alpha}_{\rho(\varpi,\bar{\mathfrak{X}^\alpha}_\varpi)})\|^2d\varpi\ \leq\ \mathcal{T}L^2.
$$

Consequently, the sequence $\{\mathscr{F}(\varpi, \bar{\mathfrak{X}^\alpha}_{\rho(\varpi,\bar{\mathfrak{X}^\alpha}_\varpi)})\}$ is bounded in $L^2(\mathfrak{J},\mathcal{W})$ and there exists a subsequence denoted by $\{\mathscr{F}(\varpi, \bar{\mathfrak{X}^\alpha}_{\rho(\varpi,\bar{\mathfrak{X}^\alpha}_\varpi)})\}$, that weakly converges to $\mathscr{F}(\varpi)$ in $L^2(\mathfrak{J},\mathcal{W})$. By using infinite dimensional version of the Ascoli-Arzela theorem, an operator $l(\cdot) \to \int_0^\cdot \Psi(\cdot,\varpi)l(\varpi)d\varpi \; : \; L^2(\mathfrak{J},\mathcal{W}) \to C(\mathfrak{J},\mathcal{W})$ is compact, We obtain

$$\begin{aligned} \|\mathfrak{q}(\mathfrak{X}^\alpha) - w\| &= \Big\| \int_0^{\mathfrak{T}} \Psi(\mathfrak{T},\varpi)\Big[\|\mathscr{F}(\varpi, \bar{\mathfrak{X}^\alpha}_{\rho(\varpi,\bar{\mathfrak{X}^\alpha}_\varpi)}) - \mathscr{F}(\varpi)\|\Big]d\varpi\Big\| \\ &\leq \sup_{\varsigma\in\mathfrak{J}} \Big\| \int_0^{\varsigma} \Psi(\mathfrak{T},\varpi)\Big[\mathscr{F}(\varpi, \bar{\mathfrak{X}^\alpha}_{\rho(\varpi,\bar{\mathfrak{X}^\alpha}_\varpi)}) - \mathscr{F}(\varpi)\Big]d\varpi\Big\| \to 0 \end{aligned}$$

as $\alpha \to 0^+$, where

$$w = \mathfrak{X}_{\mathfrak{T}} - \Phi(\mathfrak{T},0)\varnothing(0) - \Psi(\mathfrak{T},0)y_0 - \int_0^{\mathfrak{T}} \Psi(\mathfrak{T},\varpi)[\mathscr{F}(\varpi, \mathfrak{X}_{\rho(\varpi,\mathfrak{X}_\varpi)})]d\varpi.$$

Then

$$\begin{aligned} \|\mathfrak{X}^\alpha(T) - \mathfrak{X}_{\mathfrak{T}}\| &\leq \|\alpha\mathscr{R}(\alpha,\Gamma_0^T)(w)\| + \|\alpha\mathscr{R}(\alpha,\Gamma_0^{\mathfrak{T}})\|\|\mathfrak{q}(\mathfrak{X}^\alpha) - w\| \\ &\leq \|\alpha\mathscr{R}(\alpha,\Gamma_0^T)(w)\| + \|\mathfrak{q}(\mathfrak{X}^\alpha) - w\|. \end{aligned}$$

In view the hypothesis $(\mathbf{Y_1})$ and above inequality. We assert that $\|\mathfrak{X}^\alpha(\mathfrak{T}) - \mathfrak{X}_{\mathfrak{T}}\| \to 0$ as $\alpha \to 0^+$. Hence the delay non-autonomous (1.1) is approximate controllable. □

4. Integro and Impulsive System

In this section, we establish the approximate controllability of second order non-autonomous infinite delay integro differential system with non-instantaneous impulse.

$$\begin{cases} \mathfrak{X}''(\varsigma) = \mathscr{A}(\varsigma)\mathfrak{X}(\varsigma) + \mathfrak{C}u(\varsigma) + \mathscr{F}(\varsigma, \mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)}) + \int_0^\varsigma \mathscr{K}(\varsigma-\varpi)\mathscr{G}(\varpi,\mathfrak{X}(\varpi))d\varpi, \; \varsigma \in (\varpi_i, \varsigma_{i+1}], \; i = 0,1,\cdots,m. \\ \mathfrak{X}(\varsigma) = \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-)), \; \varsigma \in (\varsigma_i, \varpi_i], \; i = 1,2,\cdots,m. \\ \mathfrak{X}'(\varsigma) = \psi^2(\varsigma, \mathfrak{X}(\varsigma_i^-)), \; \varsigma \in (\varsigma_i, \varpi_i], \; i = 1,2,\cdots,m. \\ \mathfrak{X}(\varsigma) = \varnothing(\varsigma) \in \mathscr{P}, \quad \mathfrak{X}'(0) = \xi_0, \quad \varsigma \in (-\infty, 0] \end{cases} \tag{4.1}$$

where $\mathscr{A}, \mathscr{F}, \mathfrak{C}$ are defined in equation(1.1). $\mathfrak{X}(\varsigma)$ is a state function with time interval

$$0 = \varpi_0 = \varsigma_0 < \varsigma_1 < \varpi_1 < \varsigma_2, \cdots, \varsigma_m < \varpi_m < \varsigma_{m+1} = \mathfrak{T} < \infty.$$

Consider the state function $\mathfrak{X} \in C((\varsigma_i, \varsigma_{i+1}], X), \; i = 0,1,\cdots,m$ and there exist $\mathfrak{X}(\varsigma_i^-)$ and $\mathfrak{X}(\varsigma_i^+), \; i = 1,2,\cdots,m$ with $\mathfrak{X}(\varsigma_i^-) = \mathfrak{X}(\varsigma_i)$. The functions $\psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-))$ and $\psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-))$ represent non-instantaneous impulses during the intervals $(\varsigma_i, \varpi_i]$, $i = 1,2,\cdots,m$.
Let $PC([0,\mathfrak{T}],\mathcal{W})$ be the space of piecewise continuous functions $\mathfrak{X} : [0,\mathfrak{T}] \to \mathcal{W}$ a Banach space, endowed with the norm $\|\mathfrak{X}\|_{PC} = \sup_{\varsigma\in\mathfrak{J}} \|\mathfrak{X}(\varsigma)\|$.

Definition 4.1. *A function $\mathfrak{X} \in PC([0,\mathfrak{T}],\mathcal{W})$ is called a mild solution of the impulsive system (4.1) if $\mathfrak{X}(\varsigma) = \varnothing(\varsigma)$, $\mathfrak{X}'(0) = \xi_0$, the non-instantaneous impulse conditions $\mathfrak{X}(\varsigma) = \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-))$, $\varsigma \in (\varsigma_i, \varpi_i]$, $i = 1,2,\cdots,m$, $\mathfrak{X}'(\varsigma) = \psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-))$, $\varsigma \in (\varsigma_i, \varpi_i]$, $i = 1,2,\cdots,m$, and $\mathfrak{X}(\varsigma)$ is the solution of the following integral equations*

$$\mathfrak{X}(\varsigma) = \begin{cases} \Phi(\varsigma,0)\varnothing(0) + \Psi(\varsigma,0)\xi_0 \\ \quad + \int_0^\varsigma \Psi(\varsigma,\varpi)[\mathfrak{C}u(\varpi) + \mathscr{F}(\varpi, \mathfrak{X}_{\rho(\varpi,\mathfrak{X}_\varpi)}) + \int_0^\varpi \mathscr{K}(\varsigma-\zeta)\mathscr{G}(\zeta,\mathfrak{X}(\zeta))d\zeta]d\varpi, \; \varsigma \in [0,\varsigma_1] \\ \Phi(\varsigma,\varpi_i) \; \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-)) + \Psi(\varsigma,\varpi_i)\psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-)) \\ \quad + \int_0^\varsigma \Psi(\varsigma,\varpi)[\mathfrak{C}u(\varpi) + \mathscr{F}(\varpi, \mathfrak{X}_{\rho(\varpi,\mathfrak{X}_\varpi)}) + \int_0^\varpi \mathscr{K}(\varsigma-\zeta)\mathscr{G}(\zeta,\mathfrak{X}(\zeta))d\zeta]d\varpi. \; \varsigma \in [\varpi_i, \varsigma_{i+1}]. \end{cases}$$

In order to prove the approximate controllability of integro differential system (4.1) with non-instantaneous impulsive we require the following assumptions.

($\mathbf{P_6}$) (i) There exist positive constants $c_{\psi_i^1}$ and $c_{\psi_i^2}$, $i = 1, 2, \cdots, m$ such that $\max_{\varsigma \in \mathfrak{J}_i} \|\psi_i^1(\varsigma, \cdot)\| = c_{\psi_i^1}$ and $\max_{\varsigma \in \mathfrak{J}_i} \|\psi_i^2(t, \cdot)\| = c_{\psi_i^2}$, where $\mathfrak{J}_i = [\varsigma_i, \varpi_i]$.

(ii) $\psi_i^d \in C(I_i \times \mathcal{W}, \mathcal{W})$ and there are positive constants $L_{\psi_i^d}$, $i = 1, 2, \cdots, m$, $d = 1, 2$, such that $\|\psi_i^d(\varsigma, \mathfrak{X}_1) - \psi_i^d(\varsigma, \mathfrak{X}_2)\| \leq L_{\psi_i^d} \|\mathfrak{X}_1 - \mathfrak{X}_2\|$, $\forall \varsigma \in \mathfrak{J}_i$ and $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{W}$.

($\mathbf{P_7}$)(i) $K_{\mathfrak{T}} = \int_0^\varsigma \mathscr{K}(\varpi) d\varpi$

(ii) $\|\mathscr{G}(\varsigma, .)\| \leq M_{\mathscr{G}}$

(iii) $\|\mathscr{G}(\varsigma, \mathfrak{X}_1(\varsigma)) - \mathscr{G}(\varsigma, \mathfrak{X}_2(\varsigma))\| \leq L_{\mathscr{G}} \|(\mathfrak{X}_1(\varsigma)) - (\mathfrak{X}_2(\varsigma))\|$ where $M_{\mathscr{G}}$ and $L_{\mathscr{G}}$ is a positive number.

Theorem 4.1. *The system (4.1) has solution on J if the assumptions* ($\mathbf{P_1}$) – ($\mathbf{P_4}$) *and* ($\mathbf{P_6}$), ($\mathbf{P_7}$) *are satisfied and for all $\alpha > 0$*

$$(1 + \frac{1}{\alpha} N^2 M_{\mathfrak{C}}^2 \mathfrak{T})(\tilde{\aleph} \mathscr{H}_2 \delta + \mathfrak{T} M_{\mathscr{G}} K \mathfrak{T}) < 1.$$

Proof: Consider $\hat{\mathscr{Z}} = \{\mathfrak{X} \in PC : \mathfrak{X}(0) = \varnothing(0)\}$ be the space endowed with uniform norm convergence. In space $\hat{\mathscr{Z}}$, we take a subset $\hat{\mathscr{Q}} = \{\mathfrak{X} \in \hat{\mathscr{Z}} : \|\mathfrak{X}\| \leq \hat{\mathfrak{r}}\}$ of $\hat{\mathscr{Z}}$, where $\hat{\mathfrak{r}} > 0$.
Define the operator $\hat{\mathbb{F}}_\alpha : \hat{\mathscr{Z}} \to \hat{\mathscr{Z}}$ given by

$$\begin{aligned} \hat{\mathbb{F}}_\alpha \mathfrak{X}(\varsigma) &= \Phi(\varsigma, \varpi_i) \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-)) + \Psi(\varsigma, \varpi_i) \psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-)) + \int_{\varpi_i}^{\varsigma_{i+1}} \Psi(\varsigma, \varpi_i)[\mathfrak{C} u(\varpi) + \mathscr{F}(\varsigma, \bar{\mathfrak{X}}_{\rho(\varsigma, \bar{\mathfrak{X}}_\varsigma)}) \\ &+ \int_0^\varpi \mathscr{K}(\varsigma - \zeta) \mathscr{G}(\zeta, \mathfrak{X}(\zeta)) d\zeta] d\varpi \end{aligned}$$

where

$$v(\varsigma) = \mathfrak{C}^* \Psi^*(T, \varsigma) \mathscr{R}(\alpha, \Gamma_{\varpi_i}^{\varsigma_{i+1}}) \mathfrak{q}(\mathfrak{X}(\cdot))$$

and

$$\begin{aligned} \mathfrak{q}(\mathfrak{X}(\cdot)) &= \mathfrak{X}_{\mathfrak{T}} - \Phi(\mathfrak{T}, 0) \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-)) - \Psi(\mathfrak{T}, 0) \psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-)) \\ &- \int_{\varpi_i}^{\varsigma_{i+1}} \Psi(\mathfrak{T}, \varpi)[\mathscr{F}(\varsigma, \bar{\mathfrak{X}}_{\rho(\varpi, \bar{\mathfrak{X}}_\varpi)}) + \int_0^\varpi \mathscr{K}(\varsigma - \zeta) \mathscr{G}(\zeta, \mathfrak{X}(\zeta)) d\zeta] d\varpi. \end{aligned}$$

For $\varsigma \in \bigcup_{i=0}^m [\varpi_i, \varsigma_{i+1}]$, we have

$$\begin{aligned} \hat{\mathbb{F}}_\alpha \mathfrak{X}(\varsigma) &= \Phi(\varsigma, \varpi_i) \psi_i^1(\varsigma, \mathfrak{X}(\varsigma_i^-)) + \Psi(\varsigma, 0) \psi_i^2(\varsigma, \mathfrak{X}(\varsigma_i^-)) \\ &+ \int_{\varpi_i}^{\varsigma_{i+1}} \Psi(\varsigma, \varpi_i)[\mathfrak{C} u(\varpi) + \mathscr{F}(\varsigma, \bar{\mathfrak{X}}_{\rho(\varsigma, \bar{\mathfrak{X}}_\varsigma)}) + \int_0^\varpi \mathscr{K}(\varsigma - \zeta) \mathscr{G}(\zeta, \mathfrak{X}(\zeta)) d\zeta] d\varpi. \end{aligned}$$

The function $\bar{\mathfrak{X}} : (-\infty, \mathfrak{T}] \to \mathcal{W}$ is the extension of $\mathfrak{X}$ to $(-\infty, \mathfrak{T}]$ such that $\bar{\mathfrak{X}}_0 = \phi$ We assume that there exist $\hat{\mathfrak{r}} > 0$ such that $\hat{\mathbb{F}}_\alpha$ is self map in $\hat{\mathscr{Q}}$. Suppose that our assumption is false, then there exist $\alpha > 0$ such that for all $\hat{\mathfrak{r}} > 0$, there exist $\mathfrak{X}' \in \hat{\mathscr{Q}}$ and $\varsigma' \in J$ such that $\hat{\mathfrak{r}} < \|\hat{\mathbb{F}}_\alpha \mathfrak{X}'(\varsigma')\|$. we get

$$\begin{aligned} \hat{\mathfrak{r}} &< \|\mathbb{F}_\alpha \mathfrak{X}'(\varsigma')\| \\ &\leq \aleph \tilde{\mathscr{H}} c_{\psi_i^1} + \tilde{\aleph} c_{\psi_i^2} + \tilde{\aleph} M_{\mathfrak{C}} \int_{\varpi_i}^{\varsigma_{i+1}} \|v(\varpi)\| d\varpi \\ &+ \tilde{\aleph} \int_{\varpi_i}^{\varsigma_{i+1}} \|[\mathscr{F}(\varpi, \bar{\mathfrak{X}}'_{\rho(\varsigma, \bar{\mathfrak{X}}'_\varpi)}) + \int_0^\varpi K(\varsigma - \zeta) \mathscr{G}(\zeta, \mathfrak{X}(\zeta)) d\zeta]\| d\varpi, \end{aligned}$$

for any $\mathfrak{X} \in \hat{Q}$, it follows from above Lemma (2.5).

$$\|\bar{\mathfrak{X}}'_{\rho(\varsigma,\bar{\mathfrak{X}}'_{\varpi})}\| \leq (\mathscr{H}_3 + \mathscr{H})\|\phi\|_{\mathscr{P}} + \mathscr{H}_2\hat{r} = \hat{\mathfrak{r}}^*,$$

where $\hat{\mathfrak{r}}^* > 0$ is a constant. We obtain

$$\begin{aligned}
\hat{\mathfrak{r}} &\leq \aleph\tilde{\mathscr{H}} c_{\psi_i^1} + \tilde{\aleph} c_{\psi_i^2} + \tilde{\aleph} M_{\mathfrak{C}}\mathfrak{T}\|v(s)\| + \tilde{\aleph}\int_0^{\mathfrak{T}} \lambda_{\hat{\mathfrak{r}}^*}(\varpi)d\varpi + \tilde{\aleph}\mathfrak{T}K_{\mathfrak{T}}M_{\mathscr{G}} \\
\hat{\mathfrak{r}} &\leq \aleph\tilde{\mathscr{H}} c_{\psi_i^1} + \tilde{\aleph} c_{\psi_i^2} + \tilde{\aleph} M_{\mathfrak{C}}\mathfrak{T}\Big\{\mathfrak{C}^*\Psi^*(\mathfrak{T}-\varpi_i,\varsigma)\mathscr{R}(\alpha,\Gamma_{\varpi_i}^{\varsigma_{i+1}})\Big[\mathfrak{X}_{i+1} - \Phi(\varsigma,\varpi_i)\psi_i^1(\varsigma,\mathfrak{X}(\varsigma_i^-)) \\
&\quad - \Psi(\varsigma,\varpi_i)\psi_i^2(\varsigma,\mathfrak{X}(\varsigma_i^-)) - \int_{\varpi_i}^{\varsigma_{i+1}} \Psi(\varsigma,\varpi_i)[\mathscr{F}(\varpi,\bar{\mathfrak{X}}'_{\rho(\varpi,\bar{\mathfrak{X}}'_{\varpi})}) + \int_0^{\varpi} \mathscr{K}(\varsigma-\zeta)\mathscr{G}(\zeta,\mathfrak{X}(\zeta))d\zeta]d\varpi\Big]\Big\} \\
&\quad + \tilde{N}\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}(\varpi)d\varpi + \tilde{N}\mathfrak{T}K_{\mathfrak{T}}M_g \\
\hat{\mathfrak{r}} &\leq \aleph\tilde{\mathscr{H}} c_{\psi_i^1} + \tilde{\aleph} c_{\psi_i^2} + \tilde{\aleph} M_{\mathfrak{C}}\mathfrak{T}\frac{1}{\alpha}\tilde{\aleph}M_c\mathfrak{T}\Big[\hat{K} + \tilde{\aleph}\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}d\varpi + \tilde{\aleph}K_{\mathfrak{T}}M_{\mathscr{G}}\mathfrak{T}\Big] \\
&\quad + \tilde{\aleph}\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}d\varpi + \tilde{\aleph}K_{\mathfrak{T}}M_{\mathscr{G}}\mathfrak{T} \\
\hat{r} &\leq \aleph\tilde{\mathscr{H}} c_{\psi_i^1} + \tilde{\aleph} c_{\psi_i^2} + \tilde{\aleph}^2 M_{\mathfrak{C}}^2\mathfrak{T}\hat{K}\frac{1}{\alpha} + \tilde{\aleph}^2 M_c^2 T\frac{1}{\alpha}\tilde{\aleph}\Big\{\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}d\varpi + \tilde{\aleph}K_{\mathfrak{T}}M_{\mathscr{G}}\mathfrak{T}\Big\} \\
&\quad + \tilde{\aleph}\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}d\varpi + \tilde{\aleph}K_{\mathfrak{T}}M_{\mathscr{G}}\mathfrak{T} \\
\hat{\mathfrak{r}} &\leq \bar{K} + (1 + \frac{1}{\alpha}\tilde{\aleph}^2\tilde{M_c}^2\mathfrak{T})\tilde{\aleph}\Big(\int_{\varpi_i}^{\varsigma_{i+1}} \lambda_{\hat{\mathfrak{r}}^*}d\varpi + K_{\mathfrak{T}}M_{\mathscr{G}}\mathfrak{T}\Big) \\
\hat{\mathfrak{r}} &\leq \bar{K} + (1 + \frac{1}{\alpha}\tilde{\aleph}^2\tilde{M_c}^2\mathfrak{T})\tilde{\aleph}(\delta\mathscr{H}_2 + \mathfrak{T}K_{\mathfrak{T}}M_{\mathscr{G}}).
\end{aligned}$$

We note that $\bar{K}$ is independent of $\hat{r}$ and $\bar{K} \to \infty$ as $\hat{\mathfrak{r}} \to \infty$. Now

$$\lim_{\hat{\mathfrak{r}}\to\infty} \inf \int_0^{\mathfrak{T}} \frac{\lambda_{\hat{\mathfrak{r}}^*}(\varpi)}{\hat{\mathfrak{r}}}d\varpi = \lim_{\hat{\mathfrak{r}}\to\infty} \inf \int_0^{\mathfrak{T}} \frac{\lambda_{\hat{\mathfrak{r}}^*}(\varpi)\hat{\mathfrak{r}}^*}{\hat{\mathfrak{r}}^*\hat{\mathfrak{r}}}d\varpi = \delta\mathscr{H}_2.$$

Hence we have for $\alpha > 0$.

$$(1 + \frac{1}{\alpha}\tilde{\aleph^2} M_{\mathfrak{C}}^2\mathfrak{T})\tilde{\aleph}(\mathscr{H}_2\delta + \mathfrak{T}K_{\mathfrak{T}}M_{\mathscr{G}}) > 1,$$

which is contradiction to our assumption. Thus $\alpha > 0$, there exist $\hat{\mathfrak{r}} > 0$ such that $\hat{\mathbb{F}}_\alpha$ map $\hat{\mathscr{Q}}$ into itself. Further, one can easily prove that for all $\alpha > 0$, $\hat{\mathbb{F}}_\alpha$ has a fixed point by applying Schauder's fixed point theorem.

5. Examples

Example 5.1. *In this section, we illustrate an example to show our abstract results. We need to introduce some technical terms in order to apply the abstract results to a partial differential equation. In view of Eq. (2.5), we take $\mathscr{A}(\varsigma) = \mathscr{A} + \tilde{\mathscr{A}}(\varsigma)$, where the operator $\mathscr{A}$ is the infinitesimal generator of $\Phi(\varsigma)$ associated with $\Psi(\varsigma)$, $\tilde{\mathscr{A}}(\varsigma) : \mathcal{D}(\tilde{\mathscr{A}}(\varsigma)) \to \mathcal{W}$ is closed linear operator.*

Let the space $\mathcal{W} = L^2(\mathbb{P},\mathbb{C})$, where $\mathbb{P}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{P}$ and 2π periodic functions on $\mathbb{R}$. The space of $2\pi-$periodic integrable functions from $\mathbb{R}$ into $\mathbb{C}$ is denoted by $\mathcal{W} = L^2(\mathbb{P},\mathbb{C})$. Also, we take $\mathbb{H}^2(\mathbb{P},\mathbb{C})$, the Sobolev space of $2\pi-$periodic functions, $\mathfrak{X} : \mathbb{R} \to \mathbb{C}$ such that $\mathfrak{X}'' \in L^2(\mathbb{P},\mathbb{C})$. Define the operator $\mathscr{A}\mathfrak{X}(\varsigma) = \mathfrak{X}''(\varsigma)$ with domain $\mathcal{D}(\mathscr{A}) = \mathbb{H}^2(\mathbb{P},\mathbb{C})$ and $\mathscr{A}$ is infinitesimal generator of strongly continuous cosine family $\Phi(\varsigma)$ on $\mathcal{W}$ (see [4]). Also $\mathscr{A}$ has discrete spectrum which consists eigenvalues $-\lambda^2$ for $\lambda \in \mathbb{Z}$, with associated eigenvectors $\xi_\lambda(\varsigma) = \frac{1}{\sqrt{2\pi}}e^{i\lambda\varsigma}$ and the set $\{\xi_\lambda,\ \lambda \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{W}$.

In particular $\mathscr{A}\mathfrak{X} = -\sum_{\lambda=1}^{\infty}\lambda^2\langle\mathfrak{X},\xi_\lambda\rangle\xi_\lambda$ *for* $\mathfrak{X}\in\mathcal{D}(\mathscr{A})$. *The cosine function* $\Phi(\varsigma)$ *is given by* $\Phi(\varsigma)\mathfrak{X} = \sum_{\lambda=1}^{\infty}\cos(\lambda\varsigma)\langle\mathfrak{X},\xi_\lambda\rangle\xi_\lambda, \quad \varsigma\in\mathbb{R}$ *with associated sine function* $\Psi(\varsigma)\mathfrak{X} = \sum_{\lambda=1}^{\infty}\frac{\sin(\lambda\varsigma)}{\lambda}\langle\mathfrak{X},\xi_\lambda\rangle\xi_\lambda, \quad \varsigma\in\mathbb{R}$. *It is clear that for each* $\varsigma\in\mathbb{R}$, $\|\Phi(\varsigma)\|\le 1$. *Hence, it is uniformly bounded on* $\mathbb{R}$.

Now, we consider the following delay system:

$$\begin{aligned}
\mathfrak{X}_{\varsigma\varsigma}(\varsigma,\varpi) &= \mathfrak{X}_{\varpi\varpi}(\varsigma,\varpi)+b(\varsigma)\mathfrak{X}_{\varpi}(\varsigma,\varpi)+\int_{-\infty}^{\varsigma}\zeta_1(r-\varsigma)\mathfrak{X}\left(\mathfrak{r}-\rho_1(\varsigma)\rho_2\left(\|\mathfrak{X}\|\right),\varpi\right)dr, \qquad (5.1)\\
\mathfrak{X}(\varsigma,0) &= \mathfrak{X}(\varsigma,\pi)=0, \quad \varsigma\in[0,\mathfrak{T}],\ 0<\mathfrak{T}<\infty,\\
\mathfrak{X}(\varsigma,\varpi) &= \varnothing(\varsigma,\varpi), \quad \mathfrak{X}_{\varsigma}(0,\varpi)=\mathfrak{X}_1(\varpi), \quad \varpi\in[0,\pi],
\end{aligned} \qquad (5.2)$$

where, $\rho_1,\rho_2\colon[0,\infty]\to[0,\infty]$ *are continuous functions,* $b:\mathbb{R}\to\mathbb{R}$ *is continuous function with* $\beta = \sup_{0\le\varsigma\le a}|b(\varsigma)|$ *for fix number* $a>0$. *Now, we define the phase space* $\mathscr{P}=C((-\infty,\mathfrak{T}],\mathcal{W})$, *which is space of all functions* $\varnothing:(-\infty,0]\to\mathcal{W}$, *such that* ϕ *is left continuous,* $\varnothing|_{[-\mathfrak{r},0]}\in C([-\mathfrak{r},0];\mathcal{W})$, *for* $\mathfrak{r}>0$, *and*

$$\int_{-\infty}^{0}\frac{\|\phi(\theta)\|_{\mathcal{W}}}{\zeta_2(\theta)}d(\theta)>\infty.$$

The norm $\|\cdot\|_{\mathscr{P}}$ *is defined as*

$$\|\varnothing\|_{\mathscr{P}} := \int_{-\infty}^{0}\frac{\|\varnothing(\theta)\|_{\mathcal{W}}}{\zeta_2(\theta)}d(\theta),$$

where $\zeta_2:(-\infty,0]\to[1,\infty)$ *is a continuous function with* $\zeta_2(0)=1$, *and*

$$\lim_{\theta\to-\infty}\zeta_2(\theta)=\infty.$$

The function

$$G(\varsigma) := \sup_{-\infty<\theta\le-\varsigma}\frac{\zeta_2(\varsigma+\theta)}{\zeta_2(\theta)} \text{ is locally bounded for } \varsigma\ge 0.$$

Now, one can easily seen that $\mathscr{P}=C((-\infty,\mathfrak{T}],\mathcal{W})$ *forms a phase space which satisfies the axioms* $(\mathbf{R_1})$-$(\mathbf{R_3})$ *with* $\mathscr{K}(\varsigma)=\varsigma$ *and* $\tilde{\mathscr{M}}=G(\varsigma)$ *and function* $\varnothing:(-\infty,0]\to\mathcal{W}$, *is defined as*

$$\varnothing(\varsigma)(\varpi)=\varphi(\varsigma,\varpi),\ \varpi\in[0,\pi].$$

The functions $f,\rho:J\times\mathscr{P}\to\mathcal{W}$ *are defined as*

$$f(\varsigma,\varphi)\varpi = \int_{-\infty}^{0}\zeta_1(-\theta)\varphi(\theta,\varpi)d\theta,$$

$$\rho(\varsigma,\varphi)=\varsigma-\rho_1(s)\rho_2(\|\varphi(0)\|_{\mathcal{W}}).$$

Suppose that $\tilde{\mathscr{A}}(\varsigma)\mathfrak{X}(\varpi)=b(\varsigma)\mathfrak{X}'(\varpi)$ *defined on* $H^1(\mathbb{P},\mathbb{C})$. *We can see that* $\mathscr{A}(\varsigma)=\mathscr{A}+\tilde{\mathscr{A}}(\varsigma)$ *is a closed linear operator. We will initially demonstrate that the evolution operator is generated by* $\mathscr{A}+\tilde{\mathscr{A}}(\varsigma)$. *Now, the solution of the scalar initial value problem*

$$\mathfrak{X}''(\varsigma)=-\lambda^2\mathfrak{X}(\varsigma)+z(\varsigma).$$

$$\mathfrak{X}(\mathscr{S})=0, \quad \mathfrak{X}'(\mathscr{S})=\mathfrak{X}_1,$$

is given by

$$\mathfrak{X}(\varsigma)=\frac{\mathfrak{X}_1}{\lambda}\sin\lambda(\varsigma-\mathscr{S})+\frac{1}{\lambda}\int_{\mathscr{S}}^{\varsigma}\sin\lambda(\varsigma-\varpi)z(\varpi)d\varpi.$$

Thus, the solution of the scalar initial value problem

$$\begin{aligned} \mathfrak{X}''(\varsigma) &= -\lambda^2\mathfrak{X}(\varsigma) + i\lambda b(\varsigma)\mathfrak{X}(\varsigma), \\ \mathfrak{X}(\mathscr{S}) = 0, &\quad \mathfrak{X}'(\mathscr{S}) = \mathfrak{X}_1, \end{aligned} \tag{5.3}$$

is given by

$$\mathfrak{X}(\varsigma) = \frac{\mathfrak{X}_1}{\lambda}\sin\lambda(\varsigma - \mathscr{S}) + i\int_{\mathscr{S}}^{\varsigma}\sin\lambda(\varsigma-\varpi)b(\varpi)\mathfrak{X}(\varpi)d\varpi.$$

From Gronwall-Bellman lemma, we get

$$|\mathfrak{X}(\varsigma)| \leq \frac{|\mathfrak{X}_1|}{\lambda}e^{a(\varsigma-\mathscr{S})} \tag{5.4}$$

for $\mathscr{S} \leq \varsigma$ *and* a *is a constant. Now define the evolution operator as follows*

$$\Psi(\varsigma,\mathscr{S})\mathfrak{X} = \sum_{\lambda=1}^{\infty}\mathfrak{X}_\lambda(\varsigma,\mathscr{S})\langle\mathfrak{X},\xi_\lambda\rangle\xi_\lambda.$$

From the estimate (5.4), it follows that $\mathcal{C}(\varsigma,\mathscr{S}) : \mathcal{W} \to \mathcal{W}$ *is well defined and satisfies the conditions of definition (2.1). Now the Delay non-autonomous system (1.1) can be formulated abstractly as:*

$$\begin{aligned} &\mathfrak{X}''(\varsigma) = \mathscr{A}(\varsigma)\mathfrak{X}(\varsigma) + \mathcal{C}u(\varsigma) + \mathscr{F}(\varsigma, \mathfrak{X}_{\rho(\varsigma,\mathfrak{X}_\varsigma)}), \quad \varsigma \in \mathfrak{J} = [0,\mathcal{T}], \\ &\mathfrak{X}(0) = \varnothing \in \mathscr{P}, \quad \mathfrak{X}'(0) = \xi_0, \quad \varsigma \in (-\infty, 0]. \end{aligned}$$

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