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Subspaces of the Space of Real-Valued Continuous Functions Endowed with Regular Topology

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ABSTRACT: This paper studies the regular topology and its relation with the point-open topology and the graph topology on the function space $C(X)$, where X is a Tychonoff space. Furthermore, it characterizes different subspaces of the function space $C(\mathbb{R})$ endowed with regular topology and few more results corresponding to the regular topology on $C(X)$. Moreover, it is proved that the graph topology is stronger than the regular topology and they coincide when X is a weak *cb*-space, for the function space $C(X)$.

Key Words: Function space, regular topology, point-open topology, weak *cb*-space and Tychonoff space.

Contents

1. Introduction

The set $C(X, Y)$ of all continuous functions from a topological space X to a topological space Y has been topologized in a number of ways where the most primitive one we know is the point-open topology. A lot of study has already been done on this topology [1], [3], [5], [7] (being the classical one). Apart from the point-open topology, there are other numerous function space topologies which have already been studied such as; compact-open topology, uniform topology, fine topology, graph topology etc. Out of all these topologies, point-open topology is the coarsest one and graph topology, the finest one. Since the topological significance of function spaces comes from the study of convergence of sequences of functions, and so under point-open topology, pointwise convergence is seen and under the uniform topology, uniform convergence of sequence of functions is seen. Though sometimes none of these topologies are strong enough to apply to a given situation, where a stronger topology is needed to get the strong convergence. In this way one more stronger function space topology was introduced in 2011 by Iberklied et. al. [8] called as the regular topology on $C(X)$. They introduced this topology, proved that it is stronger than the fine topology and then studied various cardinal invariants of the space $C(X)$ under regular topology. Afterwards in 2015, Azarpannah at. al [9] studied connectedness, compactness and various countability properties for the space $C(X)$ endowed with regular topology.

Furthermore, in 2019, Varun and Anubha [10] explored this regular topology for a more general function space $C(X, Y)$, for a Tychonoff space X and a metric space (Y, d). They also investigated some metrizability, countability and certain completeness properties for the space $C(X, Y)$ with regular topology.

In 2023, Aaliya and Mishra [2] studied the submetrizability, some separation axioms corresponding to the function space $C(X, Y)$ endowed with regular topology, taken Y as a metric space. Alongside proving the space $C_r(X, Y)$ to be a normed linear space, topological group and a topological vector space. Maps such as composition function, bijection, almost onto and embeddings are also studied for the space $C_r(X, Y)$.

In 2024, Aaliya and Mishra [6] elaborated the study of regular topology to another function space $H(X)$, which is the space of self-homeomorphisms on X, a metric space. Metrizability, countability and

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compactness are also discussed for this space, along with proving that the homeomorphism spaces of increasing and decreasing functions on $\mathbb{R}$ under regular topology are open subspaces of $H(\mathbb{R})$ and are homeomorphic.

In 2024, Aaliya and Mishra [4] investigated some forms of compactness such as sequential compactness, countable compactness and pseudocompactness. Moreover, some cardinal invariants such as density, character and pseudocharacter are also discussed in this paper, along with defining a type of equivalence between X and Y with respect to $C(X)$ and $C(Y)$.

As a consequence, we find that point-open topology is the coarsest, then moving to the stronger ones comes the compact-open topology, then comes the uniform topology, then fits the fine topology and then the regular topology. Out of this agreement, we reach to the point that the point-open topology is weaker than the regular topology on $C(X)$.

In this paper, we investigate the direct relation between the point-open topology and the regular topology on $C(X)$. Moreover, we then consider a special case as $C(\mathbb{R})$ and study it under the regular topology, specifically we investigate its different subspaces and some of which prove to be dense in it. The subspaces such as the set of uniformly continuous functions, set of bounded continuous functions and the set of homeomorphisms are considered.

Moreover, no work has been done regarding the relation of the regular topology with the graph topology so far. Here we also derive the relation and the condition of coincidence of the regular topology with the graph topology. Afterwards, a few more general results for the space $C_r(X)$ are derived.

2. Preliminaries

Definition 2.1 *For a topological space X and a real line $\mathbb{R}$ with standard topology, the point-open topology is generated by the basis elements of the form :*

$$[x_1, x_2, ..., x_n; O_1, O_2, ..., O_n] = \{f \in C(X) \colon f(x_i) \in O_i, \forall i = 1, 2, ..., n\}, \tag{2.1}$$

where $x_i \in X$ and O_i are open intervals in $\mathbb{R}$ [1]. The space $C(X)$ endowed with the point-open topology is denoted by $C_p(X)$.

Definition 2.2 *For the regular topology on $C(X)$, we take X as a Tychonoff space and a real line $\mathbb{R}$ with standard topology. Then the basis elements used to generate the regular topology on $C(X)$ are of the form :*

$$B(f, r) = \{g \in C(X) \colon |f(x) - g(x)| < r(x), \forall x \in \operatorname{coz}(r)\}, \tag{2.2}$$

where r is the positive regular element of the ring $C(X)$ and $\operatorname{coz}(r)$ is the cozero set of r [8]. We denote the set of positive regular elements of the ring $C(X)$ by $r^+(X)$. An element $r \in C(X)$ is said to be the regular element if

$$Int(Z(r)) = \phi, \text{ where } Z(r) = \{x \in X \colon r(x) = 0\}. \tag{2.3}$$

Then $C(X)$ endowed with the regular topology is denoted by $C_r(X)$.

Definition 2.3 *The graph topology for a function space $C(X, Y)$ is studied for a Tychonoff space X and any topological space Y, and the basis elements that generate this topology are given by*

$$G_u = \{f \in C(X, Y) \colon f \subseteq U\}, \tag{2.4}$$

where U is an open subset of $X \times Y$. Moreover, when (Y, d) is particularly taken as a metric space, then the basis elements to generate the graph topology are of the form :

$$B(f, l) = \{g \in C(X, Y) \colon d(f(x) - g(x)) < l(x), \forall x \in X, f \in C(X, Y)\}, \tag{2.5}$$

and l is the lower semi-continuous function from X to $\mathbb{R}$ [11]. The function space $C(X, Y)$ endowed with the graph topology is denoted by $C_g(X, Y)$.

3. Results

At the first place we prove manually the interrelation of the regular topology with that of the point-open topology and the graph topology.

Theorem 3.1 *Let*

$$PC(X) = \{U \subset C(X) \colon f \in U \Rightarrow f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \subset U\},$$

for some $n \in \mathbb{N}$, where $x_1, x_2, \ldots, x_n \in X$ and each O_i is an open interval in $\mathbb{R}$. Then $PC(X)$ forms a topology on $C(X)$.

Proof: To prove that the given collection is a topology on $C(X)$, we must verify that it satisfies the three axioms of a topology. First, note that for any $f \in C(X)$, we can always find some $[x : \mathbb{R}]$ such that $f \in [x : \mathbb{R}] \subset C(X)$ for all $x \in X$. Hence, $C(X) \in PC(X)$.

Next, consider the union of any collection of sets in $PC(X)$. Let $f \in \bigcup_{i=1}^{\infty} U_i$, where each $U_i \in PC(X)$. Then f belongs to at least one of the U_i. Suppose $f \in U_1$; then there exists a set $[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n]$ such that $f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \subset U_1$. Similarly, if $f \in U_2$, then there exists a set $[x_1, x_2, \ldots, x_n; V_1, V_2, \ldots, V_n]$ with $f \in [x_1, x_2, \ldots, x_n; V_1, V_2, \ldots, V_n] \subset U_2$, and so on. Thus, there exists at least one such neighborhood $[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n]$ contained in some U_i such that $f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \subset \bigcup_{i=1}^{\infty} U_i$. Therefore, the union $\bigcup_{i=1}^{\infty} U_i$ belongs to $PC(X)$.

Finally, consider a finite intersection of members of $PC(X)$. Let $f \in \bigcap_{i=1}^{n} U_i$, where each $U_i \in PC(X)$. Then f belongs to each U_i, and for each U_i there exists a neighborhood of the form $[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n]$ such that $f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \subset U_i$. Hence, there exist open intervals $O_i, V_i, W_i, \ldots$ in $\mathbb{R}$ for each coordinate such that the intersection of these intervals, $\cap Z_i = O_i \cap V_i \cap \cdots \cap W_i$, defines a new neighborhood $[x_1, x_2, \ldots, x_n; \cap Z_1, \cap Z_2, \ldots, \cap Z_n]$ satisfying $f \in [x_1, x_2, \ldots, x_n; \cap Z_1, \cap Z_2, \ldots, \cap Z_n] \subset \bigcap_{i=1}^{n} U_i$. Thus, $\bigcap_{i=1}^{n} U_i \in PC(X)$.

Since the collection $PC(X)$ contains $C(X)$, is closed under arbitrary unions, and under finite intersections, it satisfies all the axioms of a topology. Therefore, $PC(X)$ defines a topology on $C(X)$. □

Theorem 3.2 *The collection $\mathbb{B} = \{[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \colon x_i \in X, \forall i = 1, 2, \ldots, n;$ and $O_i, i = 1, 2, \ldots, n$ are open intervals in $\mathbb{R}\}$ is base for $C(X)$, that generates the point-open topology on it.*

Proof: To prove that the given collection acts as basis, we are required to prove the conditions of basis to be satisfied by this collection as : Suppose $f \in C(X)$, then we can always find at least one element such as $B = [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n]$ such that $f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \subset \mathbb{B}$.

Now, let $f \in [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \cap [y_1, y_2, \ldots, y_m; U_1, U_2, \ldots, U_m]$, where $[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \in \mathbb{B}$ and $[y_1, y_2, \ldots, y_m; U_1, U_2, \ldots, U_m] \in \mathbb{B}$. Then $f(x_i) \in O_i, i = 1, 2, \ldots, n$ and $f(y_i) \in U_i, i = 1, 2, \ldots, m$. Since O_i and U_i are open intervals in $\mathbb{R}$, so we can define $V_i = O_i \cap U_i$ such that $V_i \subset O_i, V_i \subset U_i, i = 1, 2, \ldots, m$. Therefore, for some $x_i (i < n)$ and some $y_j (j < m)$, we have, $f \in [x_1, x_2, v, x_i, y_1, y_2, \ldots, y_j; V_1, V_2, \ldots, V_i, \ldots, V_j, \ldots, V_k]$ so that $[x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j; V_1, V_2, \ldots, V_i, \ldots, V_j, \ldots, V_k]$ is contained by $[x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \cap [y_1, y_2, \ldots, y_m; U_1, U_2, \ldots, U_m]$. Thus, we have, $[x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j; V_1, V_2, \ldots, V_i, \ldots, V_j, \ldots, V_k] \subseteq [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n] \cap [y_1, y_2, \ldots, y_m; U_1, U_2, \ldots, U_m]$. Since f is arbitrary and therefore the given collection is a base for $C(X)$. □

Theorem 3.3 *The collection $\mathbb{B} = \{B(f, r) \colon f \in C(X); r \in r^+(X)\}$ forms a base for $C(X)$ that generates regular topology on it.*

Proof: To prove that the given collection forms a basis for $C(X)$, suppose $f \in C(X)$. Then we can write $B(f, r) = \{g \in C(X) : |f(x) - g(x)| < r(x), \ \forall x \in \operatorname{coz}(r)\}$, where $r \in r^+(X)$. Consider two basis elements $B(f_1, r_1)$ and $B(f_2, r_2)$ and let $h \in B(f_1, r_1) \cap B(f_2, r_2)$. Then $|f_1(x) - h(x)| < r_1(x)$ for all $x \in \operatorname{coz}(r_1)$ and $|f_2(x) - h(x)| < r_2(x)$ for all $x \in \operatorname{coz}(r_2)$. Define $s_i(x) = r_i(x) - |f_i(x) - h(x)|$ for $i = 1, 2$, so that $\operatorname{coz}(s_i) = \operatorname{coz}(r_i)$. Then $B(h, s_1) \subseteq B(f_1, r_1)$ and $B(h, s_2) \subseteq B(f_2, r_2)$, and letting $s = \min\{s_1, s_2\}$, we have $B(h,s) \subseteq B(f_1,r_1) \cap B(f_2,r_2)$. This shows that the collection satisfies the basis condition and hence forms a basis for $C(X)$.

□

Although the literature establishes that the regular topology is stronger than the point-open topology on a continuous function space, we provide a direct proof of this relation for the space $C(X)$ in the following result.

Theorem 3.4 *The point-open topology on $C(X)$ is weaker than the regular topology on it; that is, $C_p(X) < C_r(X)$.*

Proof: To prove the result, we must show that every open set in $C_p(X)$ is open in $C_r(X)$, but the converse is not true. Consider an open set $S(x,U)$ in $C_p(X)$ and let $f \in S(x,U)$, where U is an open interval in $\mathbb{R}$ and $f(x) \in U$. Then we can find some $\epsilon > 0$ such that $B(f(x),\epsilon) \subset U$, where $B(f(x),\epsilon) = \{y \in \mathbb{R} : |f(x) - y| < \epsilon\}$. Define $r(x) = \epsilon/2$ for all $x \in X$, where $r \in r^+(X)$ and $x \in \mathrm{coz}(r)$. Then every element of $B_r(f,r)$ must also lie in $S(x,U)$, where $B_r(f,r)$ is the basic open set in $C_r(X)$. Thus, it follows that $B_r(f,r) \subset S(x,U)$. Therefore, every open set in $C_p(X)$ is open in $C_r(X)$, i.e., $C_p(X) < C_r(X)$.

Now, we show that the converse does not hold, i.e., $C_r(X) \not< C_p(X)$ (equivalently, $B_p \subsetneq B_r$). To prove this, we need to show that not every open set in $C_r(X)$ is open in $C_p(X)$. Consider an open set $S = [x_1, x_2, \ldots, x_n; O_1, O_2, \ldots, O_n]$ in $C_p(X)$ and let $f \in S$. Then, for each x_i $(i = 1, 2, \ldots, n)$, we have $f(x_i) \in O_i$. In particular, $f(x_1) \in O_1$, which implies that there exists some $\epsilon > 0$ (chosen as the least such value) such that $B_p(f(x_i),\epsilon) \subset O_i$, where $B_p(f(x_i),\epsilon) = (f(x_i) - \epsilon, f(x_i) + \epsilon) = \{y \in \mathbb{R} : |f(x_i) - y| < \epsilon\}$.

Next, we show that $f \notin B_r(g,r)$ for any $g \in C(X)$, where $r \in r^+(X)$ is chosen such that $r(x) = \epsilon/2$ for all $x \in \mathrm{coz}(r)$. We proceed by contradiction. Suppose that $f \in B_r(g,r)$. Then, by definition, $|g(x) - f(x)| < r(x)$ for all $x \in \mathrm{coz}(r)$. Since $x_i \in \mathrm{coz}(r)$, it follows that $|g(x_i) - f(x_i)| < \epsilon/2$. This contradicts the choice of ϵ as the least positive number such that $B_p(f(x_i),\epsilon) \subset O_i$. Hence, our assumption is false, and we conclude that $f \notin B_r(g,r)$. Consequently, $S \subsetneq B_r(f,r)$. Therefore, not every open set in $C_r(X)$ is open in $C_p(X)$, establishing that $C_p(X) < C_r(X)$ and $C_r(X) \not< C_p(X)$. □

Theorem 3.5 *For a Tychonoff space X, we have $C_r(X) \leq C_g(X)$.*

Proof: The basis elements for the regular topology on $C(X)$ are of the form $R(f,r) = \{g \in C(X) : |f(x) - g(x)| < r(x),\ x \in \mathrm{coz}(r)\}$, where r is a regular element from the ring $C(X)$. On the other hand, the basis elements for the graph topology are given by $B(f,l) = \{g \in C(X) : |f(x) - g(x)| < l(x),\ x \in X\}$, where l is a lower semi-continuous function on X. Since every continuous function is lower semi-continuous, and every regular element is continuous (hence also lower semi-continuous), it follows that the regular topology on $C(X)$ is weaker than the graph topology.

□

Definition 3.1 *[12] A space X is said to be a weak cb-space if every locally bounded, lower semi-continuous function on X is bounded above by a continuous function. Moreover, it is equivalent to say that X is a weak cb-space if and only if, for every positive normal lower semi-continuous function g on X, there exists a function $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$.*

Definition 3.2 *[13] For a function $f \in C^*(X)$, let $I(f)$ denote the lower limit function of f, and let $S(f)$ denote the upper limit function of f, defined respectively by*

$$I(f)\colon X \to \mathbb{R}, \quad I(f)(x) = \sup_{V \in N_x} \inf_{y \in V} f(y),$$

$$S(f)\colon X \to \mathbb{R}, \quad S(f)(x) = \inf_{V \in N_x} \sup_{y \in V} f(y),$$

where N_x denotes the family of all neighborhoods of the point $x \in X$. Then we have

$$I(f)(x) \leq f(x) \leq S(f)(x), \quad \text{for all } x \in X.$$

The class of normal lower semi-continuous functions on X is defined as

$$NLSC(X) = \{f \in LSC(X) : I(S(f)) = f\}.$$

Theorem 3.6 *For a Tychonoff space X, then $C_r(X) = C_g(X)$ if and only if X is a weak cb-space.*

Proof: First, suppose that $C_r(X) = C_g(X)$. Let $\eta \in NLSC^+(X)$ and let $h\colon [0,1] \to \mathbb{R}$ be a continuous function such that $h(z) \neq h(0)$ for all $z \neq 0$. Define $f(x) = h(0)$ for all $x \in X$, so that $f \in C(X)$. Set $\lambda = \min\{\eta, |h(0) - h(1)|/2\}$, which implies $\lambda \in NLSC^+(X)$. Since $C_r(X) = C_g(X)$, there exists $r \in r^+(X)$ such that $B_r(f,r) \subseteq B_g(f,\lambda)$. We claim that $r(x) \leq \lambda(x)$ for all $x \in X$. Suppose, on the contrary, that $\lambda(x_0) < r(x_0)$ for some $x_0 \in X$. Let $O(x_0)$ be an open neighborhood of x_0 such that $\lambda(x_0) < r(x)$ for every $x \in O(x_0)$. The set $\{z \in [0,1] : |h(0) - h(z)| \geq \lambda(x_0)\}$ is a non-empty compact subset of $[0,1]$ and hence has a minimum point $b > 0$ with $|h(0) - h(b)| = \lambda(x_0)$. Since X is Tychonoff, there exists a continuous function $H\colon X \to [0,b]$ such that $H(x_0) = b$ and $H(x) = 0$ for all $x \notin O(x_0)$. Define $G\colon X \to \mathbb{R}$ by $G(z) = h(H(z))$. Then G is continuous and distinct from f. For $x \in O(x_0)$, we have $|f(x) - G(x)| = |h(0) - h(H(x))| \leq \lambda(x_0) < r(x)$, and for $x \notin O(x_0)$, $|f(x) - G(x)| = 0 < r(x)$. Hence $G \in B_r(f,r)$, implying $G \in B_g(f,\lambda)$, a contradiction since $|f(x_0) - G(x_0)| = \lambda(x_0)$. Thus, $r(x) \leq \lambda(x)$ for all $x \in X$, and consequently $r \leq \lambda \leq \eta$. Therefore, X is a weak *cb*-space.

Conversely, suppose that X is a weak *cb*-space. To show that $C_r(X) = C_g(X)$, it suffices to prove that $C_g(X) \leq C_r(X)$. Let $B_r(f,r)$ be an open set in $C_r(X)$ and let $g \in B_r(f,r)$. Then $|f(x) - g(x)| < r(x)$ for all $x \in \operatorname{coz}(r)$, where $r \in r^+(X)$. Since X is a weak *cb*-space, for every $l \in NLSC^+(X)$ there exists $\varphi \in U^+(X)$ such that $\varphi(x) \leq l(x)$ for all $x \in X$. Because $U^+(X) \subseteq r^+(X)$, it follows that for some $l \in LSC^+(X)$, we have $r(x) \leq l(x)$ for all $x \in X$. Hence $|f(x) - g(x)| < l(x)$ for all $x \in X$, which means $g \in B_g(f,l)$. Therefore, $C_r(X) = C_g(X)$. □

Theorem 3.7 *There exists a subspace of $C_r(\mathbb{R})$ that is homeomorphic to $C_r(\mathbb{I})$, where $\mathbb{I} = [-1,1]$.*

Proof: Define a function $\varphi\colon \mathbb{R} \to \mathbb{I}$ by $\varphi(x) = x$ if $|x| < 1$, $\varphi(x) = 1$ if $x > 1$, and $\varphi(x) = -1$ if $x < -1$. For any $f \in C_r(\mathbb{I})$, define $\varphi^*(f) = f \circ \varphi$, giving a map $\varphi^*\colon C_r(\mathbb{I}) \to C_r(\mathbb{R})$. To show that φ^* is an embedding, note that since $\varphi(x) = x$ for all $x \in \mathbb{I}$, it is injective on $\mathbb{I}$. If $f, g \in C_r(\mathbb{I})$ with $f \neq g$, then there exists $x \in \mathbb{I}$ such that $f(x) \neq g(x)$, implying $\varphi^*(f)(x) = f(\varphi(x)) = f(x) \neq g(x) = g(\varphi(x)) = \varphi^*(g)(x)$, and hence $\varphi^*(f) \neq \varphi^*(g)$. Thus, φ^* is injective. To prove continuity, let $f_0 \in C_r(\mathbb{I})$ and $U \in \tau(C_r(\mathbb{R}))$ such that $\varphi^*(f_0) \in U$. Then there exists a basic open set $B = B(\varphi^*(\varphi^{-1}), r)$ with $r \in r^+(\mathbb{R})$ and $r = \varphi^*(\acute{r})$ for some $\acute{r} \in r^+(\mathbb{I})$, such that $\varphi^*(f_0) \in B \subset U$. If we define $\acute{B} = \acute{B}(\varphi^{-1}, \acute{r})$, then $f_0 \in \acute{B}$ and $\varphi^*(\acute{B}) \subset B$, showing that φ^* is continuous at f_0. To show that $(\varphi^*)^{-1}$ is continuous, consider the map $\pi\colon C_r(\mathbb{R}) \to C_r(\mathbb{I})$ defined by $\pi(f) = f_{|\mathbb{I}}$. It is clear that π, when restricted to $\varphi^*(C_r(\mathbb{I}))$, acts as the inverse of φ^*. For $f_0 \in C_r(\mathbb{R})$ and $g_0 = \pi(f_0)$, if $g_0 \in U \in \tau(C_r(\mathbb{I}))$, then there exists a basic open set $B = B(f_{|\mathbb{I}}, r_{|\mathbb{I}})$ for some $f \in C_r(\mathbb{R})$, where $r_{|\mathbb{I}} = \pi(\acute{r})$ for some $\acute{r} \in r^+(\mathbb{R})$, such that $g_0 \in B(f_{|\mathbb{I}}, r_{|\mathbb{I}})$. Setting $\acute{B} = \acute{B}(f, \acute{r})$ gives $f_0 \in \acute{B}$ and $\pi(\acute{B}) \subset B$, establishing the continuity of π. Therefore, $(\varphi^*)^{-1}$ is continuous, and φ^* is a homeomorphism onto its image. □

The above result can be generalized to any $C(\mathbb{R})$ and $C(\mathbb{I})$, independent of the topology as:

Theorem 3.8 *There exists a subspace of $C(\mathbb{R})$ that is homeomorphic to $C(\mathbb{I})$, where $\mathbb{I} = [-1,1]$.*

Proof: Define a function $\varphi\colon \mathbb{R} \to \mathbb{I}$ by $\varphi(x) = x$ if $|x| < 1$, $\varphi(x) = 1$ if $x > 1$, and $\varphi(x) = -1$ if $x < -1$. For any $f \in C(\mathbb{I})$, define $\varphi^*(f) = f \circ \varphi$, which gives a map $\varphi^*\colon C(\mathbb{I}) \to C(\mathbb{R})$. It is sufficient to show that φ^* is an embedding. To see that φ^* is injective, let $f, g \in C(\mathbb{I})$ with $f \neq g$. Then there exists some $x \in \mathbb{I}$ such that $f(x) \neq g(x)$, since φ is injective on $\mathbb{I}$. Hence, $\varphi^*(f)(x) = f(x) \neq g(x) = \varphi^*(g)(x)$, showing that φ^* is injective. Furthermore, since φ^* is a composition of continuous functions, it is continuous. To show that $(\varphi^*)^{-1}$ is also continuous, consider the map $\pi\colon C(\mathbb{R}) \to C(\mathbb{I})$ defined by $\pi(f) = f_{|\mathbb{I}}$. It is clear that π, when restricted to $\varphi^*(C(\mathbb{I}))$, acts as the inverse of φ^*. Since π is the restriction of a continuous function, it is itself continuous. Therefore, φ^* is an embedding, and we have $C(\mathbb{I}) \cong \varphi^*(C(\mathbb{I}))$, which can be regarded as a subspace of $C(\mathbb{R})$. □

Theorem 3.9 *The closure of $C^*(\mathbb{R})$ in $C(\mathbb{R})$ is equal to $C_r(\mathbb{R})$, that is, $\overline{C^*(\mathbb{R})} = C_r(\mathbb{R})$.*

Proof: As we know, for a topological space X, $\overline{A} = X$ for some $A \subset X$ if and only if there exists a basis $\mathbb{B}$ of X such that $A \cap U \neq \varnothing$ for all $U \in \mathbb{B}$. Therefore, to prove that $\overline{C^*(\mathbb{R})} = C_r(\mathbb{R})$, it is sufficient to show that $C^*(\mathbb{R}) \cap B \neq \varnothing$ for every basis element B of $C_r(\mathbb{R})$. Let $B = B(f, r)$ be a basis element of $C_r(\mathbb{R})$, where $f \in C(\mathbb{R})$ and $r \in r^+(\mathbb{R})$. Since $\mathbb{R}$ is a Tychonoff space, there exists a function $g \in C(\mathbb{R})$ such that $g(x_i) = s_i < r(x_i)$, where $s_i \in [0, 1]$, $r \in r^+(\mathbb{R}, [0, 1])$, and $x_i \in \operatorname{coz}(r)$. Let $s = |x_1| + |x_2| + \cdots + |x_i| + 1$, where x_i $(i = 1, 2, \ldots, n) \in \operatorname{coz}(r)$. Define a function $h \in C(\mathbb{R})$ by setting $h(x) = g(x)$ for $|x| < s$, $h(x) = g(s)$ for $x > s$, and $h(x) = g(-s)$ for $x < -s$. This ensures that h is continuous on $\mathbb{R}$. Then $h \in C^*(\mathbb{R})$, since $h(x)$ is bounded and defined in terms of the continuous function $g(x)$. Moreover, $h \in B(f, r)$, which implies that $C^*(\mathbb{R}) \cap B \neq \varnothing$. Hence, $\overline{C^*(\mathbb{R})} = C_r(\mathbb{R})$. □

Theorem 3.10 *Let $H(\mathbb{R}) \subset C_r(\mathbb{R})$ be the set of all homeomorphisms of $\mathbb{R}$ onto itself. Then the closure of $H(\mathbb{R})$ in $C_r(\mathbb{R})$ is not equal to $C_r(\mathbb{R})$, that is, $\overline{H(\mathbb{R})} \neq C_r(\mathbb{R})$.*

Proof: It has been established for $C_p(\mathbb{R})$ that this result does not hold [S.042, [1]]. This means that for every basis element B_p of $C_p(\mathbb{R})$, we have $H(\mathbb{R}) \cap B_p = \phi$.

Since $C_p(\mathbb{R}) < C_r(\mathbb{R})$, it follows that $B_r \subset B_p$. Therefore, there exist basis elements such that $B_r = B(f, r) \subset B_p$. Consequently, we obtain $H(\mathbb{R}) \cap B_r = \phi$. Hence, we conclude that $\overline{H(\mathbb{R})} \neq C_r(\mathbb{R})$. □

The above result holds for every other topology stronger than point-open topology, because every stronger topology than point-open topology has basic open sets contained in the basis elements of point-open topology.

Theorem 3.11 *Let U be the set of all uniformly continuous functions from $C(\mathbb{R})$ (that is, $f \in U$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$, whenever $|x - y| < \delta$). Then $\bar{U} = C_r(\mathbb{R})$.*

Proof: We will prove that the statement holds. Let $B(f, r)$ be a basic open set in $C_r(\mathbb{R})$, where $f \in C(\mathbb{R})$ and r is a positive regular element of the ring $C_r(\mathbb{R})$. Choose $a, b \in \mathbb{R}$ such that $a < b$ and $\operatorname{coz}(r) \subset [a, b]$.

Let $p(x)$ be a polynomial satisfying $|p(x) - f(x)| < r(x)$ for all $x \in \operatorname{coz}(r)$. The polynomial p is not necessarily uniformly continuous. To remedy this, define a function g by setting $g(x) = p(x)$ for $x \in [a, b]$, $g(x) = p(b)$ for $x > b$, and $g(x) = p(a)$ for $x < a$. This construction ensures that g is continuous on $\mathbb{R}$.

It is easy to see that g is continuous. Moreover, it is a well-known result of calculus that g must be uniformly continuous on $[a, b]$; that is, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$, we have $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$.

We claim that the same δ also proves the uniform continuity of g on the whole real line $\mathbb{R}$. Indeed, if $|x - y| < \delta$ and $x, y \in [a, b]$, then $|g(x) - g(y)| < \epsilon$ because g is uniformly continuous on $[a, b]$. If $x < a$, then $|a - y| < \delta$, and therefore $|g(x) - g(y)| = |g(a) - g(y)| < \epsilon$. Analogously, if $x > b$, then $|b - y| < \delta$, and hence $|g(x) - g(y)| = |g(b) - g(y)| < \epsilon$. The cases where $y < a$ or $y > b$ are treated in the same way.

Since g is defined in terms of p, which belongs to $B(f, r)$, we have $g \in U \cap B(f, r)$, where $B(f, r)$ was arbitrary. Hence, $\overline{U} = C_r(\mathbb{R})$. □

Since it is evident to see that every basis element of the form $B(f, r)$ for $C(X)$ with regular topology acts as a local base for $f \in C(X)$.

Theorem 3.12 *Let $A, B \subset C_r(X)$, and define $A + B = \{f + g\colon f \in A,\, g \in B\}$. If A is an open subset of $C_r(X)$ and B is an arbitrary subset of $C_r(X)$, then $A + B$ is an open set in $C_r(X)$.*

Proof: We first establish that $A + b = \{a + b \colon a \in A\}$ is an open set for any $b \in B$. Consider an open set $U = B(f, r)$ for some function $f \in C_r(X)$ and for a regular element r of the ring $C(X)$. It is easy to observe that $B(f, r) + g = B(f + g, r)$ for all $f, g \in C_r(X)$. Consequently, $U + g$ is open for any open set U, since each $B(f, r)$ is a basic open set in $C_r(X)$. Now, for every $a \in A$, there exists a neighborhood U_a of the form $B(f, r)$ such that $a \in U_a \subset A$. Therefore, $A = \bigcup_{a \in A} U_a$. It follows that $A + b = \bigcup_{a \in A}(U_a + b)$ for each $b \in B$, and hence $A + b$ is open as a union of open sets. Finally, we observe that $A + B = \bigcup_{b \in B}(A + b)$, which is open, being a union of open sets. □

Theorem 3.13 *For any subsets $A, B \subset C_r(X)$, we have $\overline{A} + \overline{B} \subset \overline{A + B}$.*

Proof: Observe first that for any $f, g \in C_r(X)$, we have $B(f, r) + B(g, r) \subset B(f + g, 2r)$. Here, the sets $B(f, r) = \{h \in C(X) \colon |f(x) - h(x)| < r(x),\, x \in \mathrm{coz}(r)\}$ and $B(g, r) = \{t \in C(X) \colon |g(x) - t(x)| < r(x),\, x \in \mathrm{coz}(r)\}$ serve as local bases at f and g, respectively.

Now, suppose that $f \in \overline{A} + \overline{B}$ and that $f \in U$, where U is an open set in $\tau(C_r(X))$. Then there exists a basic neighborhood $B(f, r) \subset U$. Since $f = a + b$ for some $a \in \overline{A}$ and $b \in \overline{B}$, choose $\acute{a} \in B(a, r/2) \cap A$ and $\acute{b} \in B(b, r/2) \cap B$. It follows that $\acute{a} + \acute{b} \in (A + B) \cap B(f, r) \subset (A + B) \cap U$. Hence, for any arbitrary $f \in U$, we have $U \cap (A + B) \neq \varnothing$. Therefore, $f \in \overline{A + B}$, and thus $\overline{A} + \overline{B} \subset \overline{A + B}$.

□

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