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# Appell-Type Extension of the ${}_pR_q(\nu,\tau;z)$ Function

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ABSTRACT: In this paper, we define two variables Appell-type extension of the ${}_pR_q(\nu,\tau;z)$ function, also study confluence formulae, double integral representations and differentiation formulae for the Appell-type extension of the ${}_pR_q(\nu,\tau;z)$ function.

Key Words: Gamma function, Beta function, Mittag-Leffler function, Appell hypergeometric function.

## Contents



## 1. Introduction and preliminaries

In 1903, Gosta Mittag-Leffler [5] defined $E_\nu(z)$ function as

$$E_\nu(z)=\sum_{s=0}^{\infty}\frac{z^s}{\Gamma(\nu s+1)}, \tag{1.1}$$

where $\nu,z\in\mathbf{C};\Re(\nu)>0$.

Wiman [12] generalized $E_\nu(z)$ and defined as

$$E_{\nu,\tau}(z)=\sum_{s=0}^{\infty}\frac{z^s}{\Gamma(\nu s+\tau)}, \tag{1.2}$$

where $\nu,\tau,z\in\mathbf{C};\Re(\nu)>0,\Re(\tau)>0$.

Further, E. M. Wright [13] gave the following function, which can be represented in series form and valid in the whole complex plane

$$W_{\nu,\tau}(z)=\sum_{s=0}^{\infty}\frac{1}{\Gamma(\nu s+\tau)}\frac{z^s}{s!},\ \ \nu>-1,\tau\in\mathbf{C}. \tag{1.3}$$

In 1971, Prabhakar [6] generalized (1.3) as,

$$E^{\delta}_{\nu,\tau}(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\nu s+\tau)}\frac{z^s}{s!}, \tag{1.4}$$

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where $\nu, \tau, \delta, z \in \mathbf{C}; \Re(\nu) > 0, \Re(\tau) > 0, \Re(\delta) > 0$.

Sharma [9] studied the following function

$$\begin{aligned} {}_pK_q^{\nu,\tau,\gamma}(x) &= {}_pK_q^{\nu,\tau,\gamma}(\mu_1, ...., \mu_p, \delta_1, ...., \delta_q; x) \\ &= \sum_{s=0}^{\infty} \frac{(\mu_1)_s......(\mu_p)_s}{(\delta_1)_s......(\delta_q)_s} \frac{(\gamma)_s x^s}{s!\Gamma(\nu s+\tau)}, \end{aligned} \tag{1.5}$$

where $\nu, \tau, \gamma \in \mathbf{C}; \Re(\nu) > 0$.

Afterwards, Sharma [10] studied the generalized M-series as defined below

$$\begin{aligned} {}_p\overset{\nu,\tau}{M}_q(z) &= {}_p\overset{\nu,\tau}{M}_q(\mu_1, ...., \mu_p, \delta_1, ...., \delta_q; z) \\ &= \sum_{s=0}^{\infty} \frac{(\mu_1)_s......(\mu_p)_s}{(\delta_1)_s......(\delta_q)_s} \frac{z^s}{\Gamma(\nu s+\tau)}, \end{aligned} \tag{1.6}$$

where $\nu, \tau, z \in \mathbf{C}; \Re(\nu) > 0$.

In a remarkable paper, Saxena et al. [8] discussed multivariate Mittag-Leffler-type function,

$$E_{(\rho_i),\lambda}^{(\delta_i)}(z_1, z_2, ..., z_n) = \sum_{s_1,s_2,...,s_n=0}^{\infty} \frac{(\delta_1)_{s_1}(\delta_2)_{s_2}...(\delta_n)_{s_n}}{\Gamma\left(\lambda + \sum\limits_{i=1}^{n}(\rho_i s_i)\right)} \frac{{z_1}^{s_1}{z_2}^{s_2}...{z_n}^{s_n}}{(s_1)!\,(s_2)!...\,(s_n)!}, \tag{1.7}$$

where $\lambda, \delta_i, \rho_i, z_i \in \mathbf{C}; \Re(\rho_i) > 0, (i = 1, 2, ..., n)$.

Horn [1,4] investigated thirty four distinct convergent hypergeometric series of order two of which four Appell's hypergeometric series $F_1, F_2, F_3$ and $F_4$-series and seven confluent series $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$ and $\Xi_2$, are limiting cases of Appell's hypergeometric series, discussed below as,

$$F_1[\mu, \delta, \delta'; \gamma; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} x^p y^q, \qquad |x|, |y| < 1 \tag{1.8}$$

$$F_2[\mu, \delta, \delta'; \gamma, \gamma'; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}(\delta)_p(\delta')_q}{p!q!(\gamma)_p(\gamma')_q} x^p y^q, \qquad |x| + |y| < 1 \tag{1.9}$$

$$F_3[\mu, \mu', \delta, \delta'; \gamma; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_p(\mu')_q(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} x^p y^q, \qquad |x|, |y| < 1 \tag{1.10}$$

$$F_4[\mu, \delta; \gamma, \gamma'; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}(\delta)_{p+q}}{p!q!(\gamma)_p(\gamma')_q} x^p y^q, \qquad |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1 \tag{1.11}$$

$$\Phi_1[\mu, \delta; \gamma; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}(\delta)_p}{p!q!(\gamma)_{p+q}} x^p y^q, \quad |x| < 1 \tag{1.12}$$

$$\Phi_2[\delta, \delta'; \gamma; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} x^p y^q \tag{1.13}$$

$$\Phi_3[\delta; \gamma; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\delta)_p}{p!q!(\gamma)_{p+q}} x^p y^q \tag{1.14}$$

$$\Psi_1[\mu, \delta; \gamma, \gamma'; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}(\delta)_p}{p!q!(\gamma)_p(\gamma')_q} x^p y^q, \quad |x| < 1 \tag{1.15}$$

$$\Psi_2[\mu; \gamma, \gamma'; x, y] = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}}{p!q!(\gamma)_p(\gamma')_q} x^p y^q \tag{1.16}$$

$$\Xi_1\left[\mu,\mu',\delta;\gamma;x,y\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_p(\mu')_q(\delta)_p}{p!q!(\gamma)_{p+q}}x^py^q,\quad |x|<1 \tag{1.17}$$

$$\Xi_2\left[\mu,\delta;\gamma;x,y\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_p(\delta)_q}{p!q!(\gamma)_{p+q}}x^py^q,\quad |x|<1 \tag{1.18}$$

Desai and Shukla [2,3] modified the conditions of $K$ function [9] and renamed as the ${}_pR_q(\nu,\tau;z)$ function, which is defined by

$$\begin{aligned}{}_pR_q\left(\nu,\tau;z\right)&={}_pR_q\left(\begin{array}{c}\mu_1,\mu_2,...,\mu_p\\ \omega_1,\omega_2,...,\omega_q\end{array}\middle|\nu,\tau;z\right)\\&=\sum_{s=0}^{\infty}\frac{1}{\Gamma\left(\nu s+\tau\right)}\frac{(\mu_1)_s(\mu_2)_s...(\mu_p)_s}{(\omega_1)_s(\omega_2)_s...(\omega_q)_s}\frac{z^s}{s!},\end{aligned} \tag{1.19}$$

where $\nu,\tau,\mu_i,\omega_j\in\mathbf{C},\omega_j\notin\mathbf{Z}^-\cup\{0\},p,q\in\mathbf{Z}^+\cup\{0\},\Re(\nu)>0,\Re(\tau)>0,\Re(\mu_i)>0,\Re(\omega_j)>0$ for $1\leq i\leq p,1\leq j\leq q$.
The series (1.19) terminates to polynomial in $z$ when any $\mu_i\in\mathbf{Z}^-\cup\{0\}$.
The convergence criteria for ${}_pR_q(\nu,\tau;z)$ function is as follows

(i). The series converges for all $|z|<\infty$ when $\Re(\nu)\geq p-q$.

(ii). The series converges for all $|z|<1$ and diverges for $|z|>1$ when $\Re(\nu)=p-q-1$.

(iii). The series converges absolutely for all $|z|=1$ when $\Re(\nu)=p-q-1$ and $\Re\left(\nu+\sum\limits_{j=1}^{q}\omega_j-\sum\limits_{i=1}^{p}\mu_i\right)>0$.

For $\nu=0$ and $\tau=1$, (1.19) reduces to generalized hypergeometric function [4,7].

$${}_pR_q\left(\begin{array}{c}\mu_1,\mu_2,...,\mu_p\\ \omega_1,\omega_2,...,\omega_q\end{array}\middle|0,1;z\right)={}_pF_q\left(\begin{array}{c}\mu_1,\mu_2,...,\mu_p\\ \omega_1,\omega_2,...,\omega_q\end{array}\middle|z\right). \tag{1.20}$$

## 2. Appell-type extension of ${}_pR_q(\nu,\tau;z)$ function

We are motivated to define eleven functions of two variables, out of which four functions are Appell-type extension of the ${}_pR_q(\nu,\tau;z)$ function and remaining seven functions are confluent functions of Appell-type extension of the ${}_pR_q(\nu,\tau;z)$ function.

The following functions are defined for $\mu,\mu',\delta,\delta',\gamma,\gamma',\nu,\tau,z_1,z_2\in\mathbf{C},\gamma,\gamma'\notin\mathbf{Z}^-\cup\{0\},\Re(\nu)\geq 0$, $\Re(\tau)>0$ and $|z_1|,|z_2|<\infty$.

$$R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p(\delta')_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{2.1}$$

$$R_2\left[\mu,\delta,\delta';\gamma,\gamma';\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p(\delta')_q}{(\gamma)_p(\gamma')_q}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{2.2}$$

$$R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_p(\mu')_q(\delta)_p(\delta')_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{2.3}$$

$$R_4\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_{p+q}}{(\gamma)_p(\gamma')_q}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{2.4}$$

$$R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.5}$$

$$R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\delta)_p(\delta')_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.6}$$

$$R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.7}$$

$$R\Psi_1\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p}{(\gamma)_p(\gamma')_q}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.8}$$

$$R\Psi_2\left[\mu;\gamma,\gamma';\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}}{(\gamma)_p(\gamma')_q}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.9}$$

$$R\Xi_1\left[\mu,\mu',\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_p(\mu')_q(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.10}$$

$$R\Xi_2\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_p(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\tag{2.11}$$

For $\nu=0$ and $\tau=1$, these functions reduce to the Appell functions $F_1,F_2,F_3$ and $F_4$ and seven confluent functions $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Xi_1$ and $\Xi_2$.

## 3. Some Confluence formulae

In this section, we give the functions (2.5)-(2.11) as confluent function of (2.1)-(2.4).

The following confluence formulae for (2.5)-(2.11) hold true.

$$\lim_{\lambda\to 0}R_1\left[\mu,\delta,\frac{1}{\lambda};\gamma;\nu,\tau,z_1,\lambda z_2\right]=R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\tag{3.1}$$

$$\lim_{\lambda\to 0}R_1\left[\frac{1}{\lambda},\delta,\delta';\gamma;\nu,\tau,\lambda z_1,\lambda z_2\right]=R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\tag{3.2}$$

$$\lim_{\lambda\to 0}R_1\left[\frac{1}{\lambda},\delta,\frac{1}{\lambda};\gamma;\nu,\tau,\lambda z_1,\lambda^2 z_2\right]=R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]\tag{3.3}$$

$$\lim_{\lambda\to 0}R_2\left[\mu,\delta,\frac{\delta'}{\lambda};\gamma,\gamma';\nu,\tau,z_1,\lambda z_2\right]=R\Psi_1\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,\delta' z_2\right]\tag{3.4}$$

$$\lim_{\lambda\to 0}R_2\left[\mu,\frac{\delta}{\lambda},\frac{\delta'}{\lambda};\gamma,\gamma';\nu,\tau,\lambda z_1,\lambda z_2\right]=R\Psi_2\left[\mu;\gamma,\gamma';\nu,\tau,\delta z_1,\delta' z_2\right]\tag{3.5}$$

$$\lim_{\lambda\to 0}R_3\left[\mu,\mu',\delta,\frac{\delta'}{\lambda};\gamma;\nu,\tau,z_1,\lambda z_2\right]=R\Xi_1\left[\mu,\mu',\delta;\gamma;\nu,\tau,z_1,\delta' z_2\right]\tag{3.6}$$

$$\lim_{\lambda\to 0}R_3\left[\mu,\frac{\mu'}{\lambda},\delta,\frac{\delta'}{\lambda};\gamma;\nu,\tau,z_1,\lambda^2 z_2\right]=R\Xi_2\left[\mu,\delta;\gamma;\nu,\tau,z_1,\mu'\delta' z_2\right]\tag{3.7}$$

On using the relation $\lim\limits_{\lambda\to 0}\lambda^m\left(\frac{1}{\lambda}\right)_m=1$, the proof of identity (3.1) follows as

$$\begin{aligned}\lim_{\lambda\to 0}R_1\left[\mu,\delta,\frac{1}{\lambda};\gamma;\nu,\tau,z_1,\lambda z_2\right]&=\lim_{\lambda\to 0}\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p\left(\frac{1}{\lambda}\right)_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{(\lambda z_2)^q}{q!}\\&=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!}\\&=R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\end{aligned}$$

Similarly, we can prove other confluence formulae.

## 4. Double Integral Representations

In this section, we obtain double integral representations of the functions (2.1)-(2.11). The following well-known integral results [7] are used to obtain various double integral representations.

(i). $\int\limits_0^\infty x^{z-1}e^{-x}dx=\Gamma\left(z\right),\quad \Re\left(z\right)>0.$

(ii). $\int\limits_0^a x^{\mu-1}(a-x)^{\delta-1}dx=a^{\mu+\delta-1}B\left(\mu,\delta\right)=a^{\mu+\delta-1}\Gamma\left(\begin{array}{c}\mu,\delta\\ \mu+\delta\end{array}\right).$

**Theorem 4.1** *The following double integral representation for (2.1) holds true for* $\Re\left(\delta\right)>0,\Re\left(\delta'\right)>0,\Re\left(\gamma\right)>0,\Re\left(\gamma-\delta-\delta'\right)>0$

$$\begin{aligned}&\Gamma\left(\begin{array}{c}\delta,\delta',\gamma-\delta-\delta'\\ \gamma\end{array}\right)R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\\&=\int\int x^{\delta-1}y^{\delta'-1}(1-x-y)^{\gamma-\delta-\delta'-1}E^{\mu}_{\nu,\tau}\left(xz_1+yz_2\right)dxdy,\end{aligned}\tag{4.1}$$

*taken over the triangle* $x\geq 0,y\geq 0,x+y\leq 1$.

**Proof:** In the light of (1.4), we have

$$E^{\delta}_{\nu,\tau}\left(z_1+z_2\right)=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\delta)_{p+q}}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}.\tag{4.2}$$

From the right hand side of the equation (4.1), we get

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1-x}x^{\delta+p-1}y^{\delta'+q-1}(1-x-y)^{\gamma-\delta-\delta'-1}dxdy$$

$$\begin{aligned}&=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}B\left(\delta+p,\gamma-\delta+q\right)B\left(\delta'+q,\gamma-\delta-\delta'\right)\\&=\Gamma\left(\begin{array}{c}\delta,\delta',\gamma-\delta-\delta'\\ \gamma\end{array}\right)R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right].\end{aligned}$$

This leads to the proof of theorem 4.1. □

Similarly, the following double integral representation for (2.6) holds true for $\Re(\delta) > 0, \Re(\delta') > 0$, $\Re(\gamma) > 0, \Re(\gamma - \delta - \delta') > 0$

$$\Gamma\left(\begin{array}{c}\delta, \delta', \gamma - \delta - \delta' \\ \gamma\end{array}\right) R\Phi_2\left[\delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right]$$

$$= \int\int x^{\delta-1} y^{\delta'-1} (1-x-y)^{\gamma-\delta-\delta'-1} {}_0R_0\left[-; -; \nu, \tau, xz_1 + yz_2\right] dxdy, \tag{4.3}$$

taken over the triangle $x \geq 0, y \geq 0, x + y \leq 1$

**Theorem 4.2** *For $\Re(\delta) > 0, \Re(\delta') > 0$, the following double integral representation for (2.1) holds true*

$$\Gamma(\delta, \delta') R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right] = \int\int x^{\delta-1} y^{\delta'-1} e^{-(x+y)} {}_1R_1\left(\mu; \gamma; \nu, \tau, xz_1 + yz_2\right) dxdy, \tag{4.4}$$

*taken over first quadrant.*

**Proof:** From the result due to Thakkar and Shukla [?] and (1.19), we have

$${}_1R_1\left(\mu; \gamma; \nu, \tau, z_1 + z_2\right) = \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}}{(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{p!q!\Gamma(\nu(p+q)+\tau)}.$$

From the righthand side of (4.4), we get

$$\sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}}{(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{p!q!\Gamma(\nu(p+q)+\tau)} \int\limits_{x=0}^{x=\infty}\int\limits_{y=0}^{y=\infty} x^{\delta+p-1} y^{\delta'+q-1} e^{-(x+y)} dxdy$$

$$\begin{aligned}&= \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\mu)_{p+q}}{(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{p!q!\Gamma(\nu(p+q)+\tau)} \Gamma(\delta+p)\Gamma(\delta'+q) \\ &= \Gamma(\delta, \delta') R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right].\end{aligned}$$

□

Similarly, the following double integral representation for (2.6) holds true for $\Re(\delta) > 0, \Re(\delta') > 0$

$$\Gamma(\delta, \delta') R\Phi_2(\delta, \delta'; \gamma; \nu, \tau, z_1, z_2) = \int\int x^{\delta-1} y^{\delta'-1} e^{-(x+y)} {}_0R_1\left(-; \gamma; \nu, \tau, xz_1 + yz_2\right) dxdy, \tag{4.5}$$

taken over first quadrant.

**Theorem 4.3** *The following double integral representation for (2.2) holds true for $\Re(\gamma) > \Re(\delta) > 0$ and $\Re(\gamma') > \Re(\delta') > 0$*

$$\Gamma\left(\begin{array}{c}\delta, \delta', \gamma - \delta, \gamma' - \delta' \\ \gamma, \gamma'\end{array}\right) R_2\left[\mu, \delta, \delta'; \gamma, \gamma'; \nu, \tau, z_1, z_2\right]$$

$$= \int\limits_0^1\int\limits_0^1 x^{\delta-1} y^{\delta'-1} (1-x)^{\gamma-\delta-1} (1-y)^{\gamma'-\delta'-1} E^{\mu}_{\nu,\tau}(xz_1 + yz_2) dxdy \tag{4.6}$$

**Proof:** From right hand side of (4.6) and using (4.2), we get

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1}x^{\delta+p-1}y^{\delta'+q-1}(1-x)^{\gamma-\delta-1}(1-y)^{\gamma'-\delta'-1}dxdy$$

$$=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}B\left(\delta+p,\gamma-\delta\right)B\left(\delta'+q,\gamma'-\delta'\right)$$
$$=\Gamma\left(\begin{array}{c}\delta,\delta',\gamma-\delta,\gamma'-\delta'\\ \gamma,\gamma'\end{array}\right)R_2\left[\mu,\delta,\delta';\gamma,\gamma';\nu,\tau,z_1,z_2\right].$$

□

**Theorem 4.4** *For $\Re\left(\delta\right)>0,\Re\left(\delta'\right)>0,\Re\left(\gamma\right)>0,\Re\left(\gamma-\delta-\delta'\right)>0$, the following double integral representation for (2.3) holds true*

$$\Gamma\left(\begin{array}{c}\delta,\delta',\gamma-\delta-\delta'\\ \gamma\end{array}\right)R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]$$
$$=\int\int x^{\delta-1}y^{\delta'-1}(1-x-y)^{\gamma-\delta-\delta'-1}E^{(\mu,\mu')}_{(\nu,\nu),\tau}\left(xz_1,yz_2\right)dxdy, \tag{4.7}$$

*taken over the triangle $x\geq 0,y\geq 0,x+y\leq 1$.*

**Proof:** From right hand side of (4.7) and using (1.7), we get

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_p(\mu')_q}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1-x}x^{\delta+p-1}y^{\delta'+q-1}(1-x-y)^{\gamma-\delta-\delta'-1}dxdy$$

$$=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_p(\mu')_q}{p!q!}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}B\left(\delta+p,\gamma-\delta+q\right)B\left(\delta'+q,\gamma-\delta-\delta'\right)$$
$$=\Gamma\left(\begin{array}{c}\delta,\delta',\gamma-\delta-\delta'\\ \gamma\end{array}\right)R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right].$$

□

**Theorem 4.5** *The following double integral representation for (2.3) holds true for $\Re\left(\mu\right)>0,\Re\left(\mu'\right)>0$, $\Re\left(r\right)>0,\Re\left(r-\mu-\mu'\right)>0$*

$$\Gamma\left(\begin{array}{c}\mu,\mu',r-\mu-\mu'\\ r\end{array}\right)R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]$$
$$=\int\int x^{\mu-1}y^{\mu'-1}(1-x-y)^{r-\mu-\mu'-1}R_1\left[r,\delta,\delta';\gamma;\nu,\tau,xz_1,yz_2\right]dxdy, \tag{4.8}$$

*taken over the triangle $x\geq 0,y\geq 0,x+y\leq 1$.*

**Proof:** From right hand side of (4.8) and using (2.1), we get

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{(r)_{p+q}(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1-x}x^{\mu+p-1}y^{\mu'+q-1}(1-x-y)^{r-\mu-\mu'-1}dxdy$$

$$= \sum_{p\geq 0}\sum_{q\geq 0} \frac{(r)_{p+q}(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)} B\left(\mu+p, r-\mu+q\right) B\left(\mu'+q, r-\mu-\mu'\right)$$
$$= \Gamma\left(\begin{array}{c} \mu, \mu', r-\mu-\mu' \\ r \end{array}\right) R_3\left[\mu, \mu', \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right].$$

□

Similarly, the following double integral representation for (2.10) holds true for $\Re(\mu) > 0, \Re(\mu') > 0$, $\Re(r) > 0, \Re(r-\mu-\mu') > 0$

$$\Gamma\left(\begin{array}{c} \mu, \mu', r-\mu-\mu' \\ r \end{array}\right) R\Xi_1\left[\mu, \mu', \delta; \gamma; \nu, \tau, z_1, z_2\right]$$
$$= \int\int x^{\mu-1}y^{\mu'-1}(1-x-y)^{r-\mu-\mu'-1} R\Phi_1\left[r, \delta; \gamma; \nu, \tau, xz_1, yz_2\right] dxdy, \tag{4.9}$$

taken over the triangle $x \geq 0, y \geq 0, x+y \leq 1$.

**Theorem 4.6** *For $\Re(\mu) > 0, \Re(\mu') > 0$, the following double integral representation for (2.3) holds true*

$$\Gamma\left(\mu, \mu'\right) R_3\left(\mu, \mu', \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right) = \int\int x^{\mu-1}y^{\mu'-1}e^{-(x+y)} R\Phi_2\left(\delta, \delta'; \gamma; \nu, \tau, xz_1, yz_2\right) dxdy, \tag{4.10}$$

*taken over first quadrant.*

**Proof:** From right hand side of (4.10) and using (2.6), we get

$$\sum_{p\geq 0}\sum_{q\geq 0} \frac{(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)} \int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} x^{\mu+p-1}y^{\mu'+q-1}e^{-(x+y)} dxdy$$

$$= \sum_{p\geq 0}\sum_{q\geq 0} \frac{(\delta)_p(\delta')_q}{p!q!(\gamma)_{p+q}} \frac{{z_1}^p {z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)} \Gamma\left(\mu+p\right)\Gamma\left(\mu'+q\right)$$
$$= \Gamma\left(\mu, \mu'\right) R_3\left(\mu, \mu', \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right).$$

□

Similarly, the following double integral representation taken over first quadrant for (2.3), (2.10), (2.11) holds true for $\Re(\mu) > 0, \Re(\delta) > 0, \Re(\mu') > 0, \Re(\delta') > 0$

$$\Gamma\left(\mu', \delta'\right) R_3\left(\mu, \mu', \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right) = \int\int x^{\mu'-1}y^{\delta'-1}e^{-(x+y)} R\Xi_2\left(\mu, \delta; \gamma; \nu, \tau, z_1, xyz_2\right) dxdy. \tag{4.11}$$

$$\Gamma\left(\mu, \mu'\right) R\Xi_1\left(\mu, \mu', \delta; \gamma; \nu, \tau, z_1, z_2\right) = \int\int x^{\mu-1}y^{\mu'-1}e^{-(x+y)} R\Phi_3\left(\delta; \gamma; \nu, \tau, xz_1, yz_2\right) dxdy. \tag{4.12}$$

$$\Gamma\left(\mu, \delta\right) R\Xi_2\left(\mu, \delta; \gamma; \nu, \tau, z_1, z_2\right) = \int\int x^{\mu-1}y^{\delta-1}e^{-(x+y)} {}_0R_1\left(-; \gamma; \nu, \tau, xyz_1+z_2\right) dxdy. \tag{4.13}$$

**Theorem 4.7** *For $\Re(\gamma) > \Re(r) > 0$ and $\Re(\gamma') > \Re(r') > 0$, the following double integral representation for (2.4) holds true*

$$\Gamma\left(\begin{array}{c} r, r', \gamma-r, \gamma'-r' \\ \gamma, \gamma' \end{array}\right) R_4\left[\mu, \delta; \gamma, \gamma'; \nu, \tau, z_1, z_2\right]$$
$$= \int_0^1\int_0^1 x^{r-1}y^{r'-1}(1-x)^{\gamma-r-1}(1-y)^{\gamma'-r'-1} R_4\left[\mu, \delta; r, r'; \nu, \tau, xz_1, yz_2\right] dxdy. \tag{4.14}$$

**Proof:** From right hand side of (4.14) and using (2.4), we get

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{(\delta)_{p+q}}{(r)_p(r')_q}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1}x^{r+p-1}y^{r'+q-1}(1-x)^{\gamma-r-1}(1-y)^{\gamma'-r'-1}dxdy$$

$$=\sum_{p\geq 0}\sum_{q\geq 0}\frac{(\mu)_{p+q}}{p!q!}\frac{(\delta)_{p+q}}{(r)_p(r')_q}\frac{{z_1}^p{z_2}^q}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}B\left(r+p,\gamma-r\right)B\left(r'+q,\gamma'-r'\right)$$

$$=\Gamma\left(\begin{array}{c} r,r',\gamma-r,\gamma'-r' \\ \gamma,\gamma' \end{array}\right)R_4\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right].$$

□

## 5. Some Differentiation formulae

In this section, we give some differentiation formulae of functions (2.1)-(2.11).

In view of the symmetry relations

$$R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=R_1\left[\mu,\delta',\delta;\gamma;\nu,\tau,z_2,z_1\right]$$
$$R_2\left[\mu,\delta,\delta';\gamma,\gamma';\nu,\tau,z_1,z_2\right]=R_2\left[\mu,\delta',\delta;\gamma',\gamma;\nu,\tau,z_2,z_1\right]$$
$$R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=R_3\left[\mu',\mu,\delta',\delta;\gamma;\nu,\tau,z_2,z_1\right]$$
$$R_4\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]=R_4\left[\delta,\mu;\gamma',\gamma;\nu,\tau,z_2,z_1\right]$$
$$R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=R\Phi_2\left[\delta',\delta;\gamma;\nu,\tau,z_2,z_1\right]$$
$$R\Psi_2\left[\mu;\gamma,\gamma';\nu,\tau,z_1,z_2\right]=R\Psi_2\left[\mu;\gamma',\gamma;\nu,\tau,z_2,z_1\right]$$

the following differentiation formulae are discussed only for one of the function variables.

**Theorem 5.1** *The following differentiation formula for (2.1) holds true*

$$D^n_{z_2}\left\{R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta')_n}{(\gamma)_n}R_1\left[\mu+n,\delta,\delta'+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.1}$$

**Proof:** We have, $D^n_{z_2}\left\{R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}$

$$=\sum_{m\geq 0}\sum_{p\geq 0}\frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^mD^n_{z_2}\left({z_2}^p\right)$$

$$=\sum_{m\geq 0}\sum_{p\geq n}\frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m(p-n+1)_n{z_2}^{p-n}$$

On setting $p=k+n$

$$=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_{m+k+n}(\delta)_m(\delta')_{k+n}}{m!\left(k+n\right)!(\gamma)_{m+k+n}\Gamma\left(\nu\left(m+k+n\right)+\tau\right)}{z_1}^m(k+1)_n{z_2}^k$$

$$=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_n(\mu+n)_{m+k}(\delta)_m(\delta')_n(\delta'+n)_k}{m!k!(\gamma)_n(\gamma+n)_{m+k}\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k$$

$$=\frac{(\mu)_n(\delta')_n}{(\gamma)_n}\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu+n)_{m+k}(\delta)_m(\delta'+n)_k}{m!k!(\gamma+n)_{m+k}\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k$$

$$=\frac{(\mu)_n(\delta')_n}{(\gamma)_n}R_1\left[\mu+n,\delta,\delta'+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right].$$

□

Similarly, the following differentiation formulae for (2.4)-(2.6) hold true

$$D_{z_1}^n\left\{R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta)_n}{(\gamma)_n}R\Phi_1\left[\mu+n,\delta+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.2}$$

$$D_{z_2}^n\left\{R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n}{(\gamma)_n}R\Phi_1\left[\mu+n,\delta;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.3}$$

$$D_{z_2}^n\left\{R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\delta')_n}{(\gamma)_n}R\Phi_2\left[\delta,\delta'+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.4}$$

$$D_{z_1}^n\left\{R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\delta)_n}{(\gamma)_n}R\Phi_3\left[\delta+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.5}$$

$$D_{z_2}^n\left\{R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{1}{(\gamma)_n}R\Phi_3\left[\delta;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.6}$$

**Theorem 5.2** *The following differentiation formula for (2.2) holds true*

$$D_{z_2}^n\left\{R_2\left[\mu,\delta,\delta';\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta')_n}{(\gamma')_n}R_2\left[\mu+n;\delta,\delta'+n;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.7}$$

**Proof:** We have, $D_{z_2}^n\left\{R_2\left[\mu,\delta,\delta';\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}$

$$=\sum_{m\geq 0}\sum_{p\geq 0}\frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_m(\gamma')_p\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m D_{z_2}^n\left({z_2}^p\right)$$

$$=\sum_{m\geq 0}\sum_{p\geq n}\frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_m(\gamma')_p\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m(p-n+1)_n{z_2}^{p-n}$$

On setting $p=k+n$

$$=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_{m+k+n}(\delta)_m(\delta')_{k+n}}{m!\,(k+n)!(\gamma)_m(\gamma')_{k+n}\Gamma\left(\nu\left(m+k+n\right)+\tau\right)}{z_1}^m(k+1)_n{z_2}^k$$

$$=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_n(\mu+n)_{m+k}(\delta)_m(\delta')_n(\delta'+n)_k}{m!k!(\gamma)_n(\gamma')_n(\gamma'+n)_k\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k$$

$$=\frac{(\mu)_n(\delta')_n}{(\gamma')_n}\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu+n)_{m+k}(\delta)_m(\delta'+n)_k}{m!k!(\gamma)_n(\gamma'+n)_k\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k$$

$$=\frac{(\mu)_n(\delta')_n}{(\gamma')_n}R_2\left[\mu+n;\delta,\delta'+n;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right].$$

□

Similarly, the following differentiation formulae for (2.8)-(2.9) hold true

$$D_{z_1}^n\left\{R\Psi_1\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta)_n}{(\gamma)_n}R\Psi_1\left[\mu+n;\delta+n;\gamma+n,\gamma';\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.8}$$

$$D_{z_2}^n\left\{R\Psi_1\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n}{(\gamma')_n}R\Psi_1\left[\mu+n;\delta;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.9}$$

$$D_{z_2}^n\left\{R\Psi_2\left[\mu;\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n}{(\gamma')_n}R\Psi_2\left[\mu+n;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.10}$$

**Theorem 5.3** *The following differentiation formula for (2.3) holds true*

$$D^n_{z_2}\left\{R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu')_n(\delta')_n}{(\gamma)_n}R_3\left[\mu,\mu'+n,\delta,\delta'+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.11}$$

**Proof:** We have, $D^n_{z_2}\left\{R_3\left[\mu,\mu',\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}$

$$\begin{aligned}
&=\sum_{m\geq 0}\sum_{p\geq 0}\frac{(\mu)_m(\mu')_p(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m D^n_{z_2}\left({z_2}^p\right)\\
&=\sum_{m\geq 0}\sum_{p\geq n}\frac{(\mu)_m(\mu')_p(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m(p-n+1)_n{z_2}^{p-n}
\end{aligned}$$

On setting $p=k+n$

$$\begin{aligned}
&=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_m(\mu')_{k+n}(\delta)_m(\delta')_{k+n}}{m!\,(k+n)!(\gamma)_{m+k+n}\Gamma\left(\nu\left(m+k+n\right)+\tau\right)}{z_1}^m(k+1)_n{z_2}^k\\
&=\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_m(\mu')_n(\mu'+n)_k(\delta)_m(\delta')_n(\delta'+n)_k}{m!k!(\gamma)_n(\gamma+n)_{m+k}\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k\\
&=\frac{(\mu')_n(\delta')_n}{(\gamma)_n}\sum_{m\geq 0}\sum_{k\geq 0}\frac{(\mu)_m(\mu'+n)_k(\delta)_m(\delta'+n)_k}{m!k!(\gamma+n)_k\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)}{z_1}^m{z_2}^k\\
&=\frac{(\mu')_n(\delta')_n}{(\gamma)_n}R_3\left[\mu,\mu'+n,\delta,\delta'+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right].
\end{aligned}$$

□

Similarly, the following differentiation formulae for (2.10)-(2.11) hold true

$$D^n_{z_1}\left\{R\Xi_1\left[\mu,\mu',\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta)_n}{(\gamma)_n}R\Xi_1\left[\mu+n,\mu',\delta+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.12}$$

$$D^n_{z_2}\left\{R\Xi_1\left[\mu,\mu',\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu')_n}{(\gamma)_n}R\Xi_1\left[\mu,\mu'+n,\delta;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.13}$$

$$D^n_{z_1}\left\{R\Xi_2\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta)_n}{(\gamma)_n}R\Xi_2\left[\mu+n,\delta+n;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.14}$$

$$D^n_{z_2}\left\{R\Xi_2\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]\right\}=\frac{1}{(\gamma)_n}R\Xi_2\left[\mu,\delta;\gamma+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.15}$$

**Theorem 5.4** *The following differentiation formula for (2.4) holds true*

$$D^n_{z_2}\left\{R_4\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}=\frac{(\mu)_n(\delta)_n}{(\gamma')_n}R_4\left[\mu+n,\delta+n;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right]. \tag{5.16}$$

**Proof:** We have, $D^n_{z_2}\left\{R_4\left[\mu,\delta;\gamma,\gamma';\nu,\tau,z_1,z_2\right]\right\}$

$$\begin{aligned}
&=\sum_{m\geq 0}\sum_{p\geq 0}\frac{(\mu)_{m+p}(\delta)_{m+p}}{m!p!(\gamma)_m(\gamma')_p\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m D^n_{z_2}\left({z_2}^p\right)\\
&=\sum_{m\geq 0}\sum_{p\geq n}\frac{(\mu)_{m+p}(\delta)_{m+p}}{m!p!(\gamma)_m(\gamma')_p\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m(p-n+1)_n{z_2}^{p-n}
\end{aligned}$$

On setting $p = k + n$

$$= \sum_{m\geq 0}\sum_{k\geq 0} \frac{(\mu)_{m+k+n}(\delta)_{m+k+n}}{m!\,(k+n)!(\gamma)_m(\gamma')_{k+n}\Gamma\left(\nu\left(m+k+n\right)+\tau\right)} {z_1}^m (k+1)_n {z_2}^k$$

$$= \sum_{m\geq 0}\sum_{k\geq 0} \frac{(\mu)_n(\mu+n)_{m+k}(\delta)_n(\delta+n)_{m+k}}{m!k!(\gamma)_m(\gamma')_n(\gamma'+n)_k\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)} {z_1}^m {z_2}^k$$

$$= \frac{(\mu)_n(\delta)_n}{(\gamma')_n} \sum_{m\geq 0}\sum_{k\geq 0} \frac{(\mu+n)_{m+k}(\delta+n)_{m+k}}{m!k!(\gamma)_m(\gamma'+n)_k\Gamma\left(\nu\left(m+k\right)+\nu n+\tau\right)} {z_1}^m {z_2}^k$$

$$= \frac{(\mu)_n(\delta)_n}{(\gamma')_n} R_4\left[\mu+n,\delta+n;\gamma,\gamma'+n;\nu,\nu n+\tau,z_1,z_2\right].$$

□

## 6. Concluding Remarks

In present paper, we defined two variables Appell-type extension of ${}_pR_q(\nu,\tau;z)$ function as (2.1)-(2.11). We also obtained some confluence formulae, double integral representations and differentiation formulae. We deduced the results for the Appell function $F_1, F_2, F_3$ and $F_4$ and its confluent functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$ and $\Xi_2$. These formulae may be very useful in the theory of classical analysis and special functions.

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