Some Characterization of $L^r$-Henstock-Kurzweil Integrable Functions

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ABSTRACT: In this article, we discuss few properties of $L^r$-Henstock-Kurzweil (in short $L^r$-HK) integrable functions, introduced by Paul Musial in [8]. We re-defined $L^r$-bounded variations. We demonstrated that $L^r$-Henstock-Kurzweil integrable functions are Denjoy integrable.

Key Words: $L^r$-Henstock-Kurzweil integral, Absolute $L^r$-Henstock-Kurzweil integral, Denjoy integral.

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1. Introduction and Preliminaries

R. A. Gordon in [4] defined the Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integrals which are the extension of Dunford, Pettis, and Bochner integrals, respectively. Gordon established that a Denjoy-Dunford (Denjoy-Bochner) integrable function on $[a, b]$ is Dunford (Bochner) integrable in some interval $[a, b]$ and that for the spaces that do not contain copy $c_0$, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some sub interval of $[a, b]$. Major and minor functions were first introduced by de la Vallée Poussin in his study of the properties of the Lebesgue integral and those of functions additive of a set (see [12]). Entirely equivalent notions (of “Ober”- and “Unterfunktionen”) were introduced independently by O. Perron [11], who based on them a new definition of integral, which does not require the theory of measure. Calderón & Zygmund first gave the notion of derivation in $L^r$ and unlike the idea of the approximate derivative had proven to be quite effective in applications of Partial Differential Equation, area of surfaces, etc. (see [2]). L. Gordon defined the notion of Dini derivatives in metric $L^r$ (briefly $L^r$-derivatives) also in his work Perron integral in $L^r$ was discussed (see [6]). Gordon proved that AP-derivatives are equivalent to $L^r$ derivatives. Paul M. Musial and Yoram Sagher introduced the $L^r$- Henstock-Kurzweil integral in [8]. P. Musial and F. Tulone obtained a norm on the space of $HK_r$-integrable functions, as well as the dual and completion of this space (see [10]). Paul M. Musial defined the class of $L^r$-variational integrable functions and show that it is equivalent to the class of $L^r$-Henstock-Kurzweil integrable functions. They also define the class of functions of $L^r$-bounded variation (see [9]).

In this paper we characterize properties of $L^r$- Henstock-Kurzweil integrable functions define in $[a, b]$.

To make our presentation reasonably self-contained we recalling a few definitions and results in this section that we will use in our main section. Recalling a positive function $\delta : [a, b] \to (0, \infty)$ is a gauge (see [4]).

**Definition 1.1.** [4, Definition 9.3] A function $f : [a, b] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable on $[a, b]$ if there exists $A \in \mathbb{R}$ with the following property: given $\epsilon > 0$ there exists a gauge $\delta$ on $[a, b]$ such that

$$\left| \sum_{i=1}^{p} f(\xi_i) |J_i| - A \right| < \epsilon$$

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for each $\delta$-fine $\mathcal{P}$-partition $\{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$. We write $A$ as $\mathcal{H} \int_{[a,b]} f$

Recalling $I = [a, b]$ denote the family of all compact sub intervals $J \subset I$, a function $F : I \to X$ is additive if $F(J \cup L) = F(J) + F(L)$ for any non overlapping $J, L \in I$ such that $J \cup L \in I$. Recalling the space $L^r$, $1 \leq r < \infty$, as

$$L^r([a,b]) = \left\{ f : \left( \frac{1}{h} \int_a^b |f(x) - P(x)|^r dx \right)^{\frac{1}{r}} < \epsilon, \ 0 < h < \infty, \ for \ some \ polynomial \ P(x) \right\}.$$  

For detailed of $L^r$, $1 \leq r < \infty$ one can follow \cite{2, 8, 14}.

**Definition 1.2**. \cite{8} Let $f \in L^r(I)$ for $1 \leq r < \infty$ and $I = (a, b)$. For all $x \in I$, $r$-Dini derivative. The upper-right $L^r$- derivative:

$$D_{r+}^x f(x) = \inf \left\{ a : \left( \frac{1}{h} \int_0^h [f(x + t) - f(x) - at]_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}.$$  

The lower-right $L^r$- derivative:

$$D_{r-}^x f(x) = \sup \left\{ a : \left( \frac{1}{h} \int_0^h [f(x + t) - f(x) - at]^-_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}.$$  

The upper-left $L^r$- derivative:

$$D_{r+}^x f(x) = \inf \left\{ a : \left( \frac{1}{h} \int_0^h [-f(x - t) + f(x) - at]_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}$$  

and the lower-left $L^r$- derivative:

$$D_{r-}^x f(x) = \sup \left\{ a : \left( \frac{1}{h} \int_0^h [-f(x - t) + f(x) - at]^-_+^r dt \right)^{\frac{1}{r}} = o(h) \right\}$$

**Remark 1.3.** $D_{r+}^x f(x) = \inf \left\{ a : \int_0^h \left( \frac{f(x + t) - f(x)}{t} - a \right)_+^r dt = o(h) \right\}$, with similar results for the other $r$-Dini derivatives.

**Definition 1.4.** \cite{8} For $1 \leq \infty$, a real valued function $f$ is $L^r$-Henstock-Kurzweil integrable (in short $HK_r$, integrable) if there exists a function $F \in L^r([a, b])$ so that for any $\epsilon > 0$ there exists a gauge function $\delta$ so that for all finite collections $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non overlapping tagged intervals in $[a, b]$ with $P \leq \delta$, we have:

$$\sum_{i=1}^n \frac{1}{d_i - c_i} \int_{x_i}^{d_i} \left| F(y) - F(x_i) - f(x_i)(y - x_i) \right|^r dy < \epsilon. \quad (1.1)$$

The function $f$ is said to be $L^r$-Henstock-Kurzweil integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is $L^r$-Henstock-Kurzweil integrable on $[a, b]$. We write

$$(L^r - H) \int_I f\chi_E = (L^r - H) \int_E f.$$  

Recalling that a gauge $\delta$ is $HK_r$-appropriate for $\epsilon$ and for $f$ if (1.1) holds for any $\delta$-fine tagged partition $\mathcal{P}$. If $f$ is $HK_r$-integrable on $[a,b]$, the following function is well defined for all $x \in [a, b]$

$$F(x) = (HK_r) \int_a^x f(t)dt.$$  

(1.2)
Let $f \in HK_r[a,b]$. The $HK_r$ norm of $f$ as follows:

$$||f||_{HK_r} = ||F||_r,$$

where $F$ is the indefinite $HK_r$ integral of $f$ as defined in (1.2). The concept of absolute continuity which characterizes indefinite $HK_r$-integrals as follows:

**Definition 1.5.** [8, Definition 11] Let $1 \leq r < \infty$. We say that $F \in AC_r(E)$ if for all $\epsilon > 0$ there exists $\nu > 0$ and a gauge function $\delta(x)$ defined on $E$ so that for all $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ such that $\sum_{i=1}^{n}(d_i - c_i) < \nu$ we have

$$\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < \epsilon$$

2. Bounded variation of $L^r$-Henstock-Kurzweil integral

Paul Musial in [9] gave the definition of $L^r$- bounded variation. They missed the coherent concept of $L^r[a,b]$.

**Definition 2.1.** [9] Let $1 \leq r \leq \infty$, let $f : [a, b] \to \mathbb{R}$ and let $E$ be a measurable subset of $[a, b]$. We say that $f$ is $L^r$- bounded variation on $E(f \in BV_r(E))$ if there exists $M > 0$ and a gauge $\delta > 0$ defined on $E$ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$ fine tagged sub-intervals of $[a, b]$ having tags in $E$, then

$$\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M.$$

We re-write the definition of $L^r$-bounded variational as follows:

**Definition 2.2.** Let $1 \leq r \leq \infty$, let $f : [a, b] \to \mathbb{R}$ and let $E$ be a measurable subset of $[a, b]$. We say that $f$ is $L^r$- bounded variation on $E(f \in BV_r(E))$ if there exists a function $F \in L^r([a, b])$ so that for any $M > 0$ and a gauge $\delta > 0$ defined on $E$ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$ fine tagged sub-intervals of $[a, b]$ having tags in $E$, then

$$\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M.$$

Paul Musial in [9] mentioned the sketch of proof of the following Theorem. We have given the full proof here so that we can use this Theorem in our results.

**Theorem 2.3.** [9, Theorem 2] If $f \in BV_r(E)$, then we can find $\{E_i\}_{i \geq 1}$ so that $E = \bigcup_{i=1}^{\infty} E_i$ and $f \in BV(E_i)$ for all $i$.

**Proof.** Let $f \in BV_r(E)$ then for a function $F \in L^r([a, b])$ there exists $M > 0$ and a gauge $\delta > 0$ defined on $E$ so that $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$fine tagged sub intervals of $[a, b]$ having tags in $E$ then

$$\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i)|^r dy \right)^{\frac{1}{r}} < M. \quad (2.1)$$

Assume $F \in BV_r[a,b]$ and let $\epsilon > 0$, then for a gauge function $\delta$ defined on $[a,b]$ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\} < \delta$ such that the equation (2.1) holds.

The function $F$ is $L^r-$continuous and so clearly approximately continuous, using the [8, Theorem 5] there exists $\mathcal{P}_i = \{(x_{i,j}, [c_{i,j}, d_{i,j}])\} < \delta$, where $[c_{i,j}, d_{i,j}] \subseteq [c_i, d_i]$ for all $i$ and $j$, so that
\[
\sum_{i=1}^{n} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})|dy \geq \frac{1}{2} |F(d_i) - F(c_i)|.
\]

Since \( P = \bigcup_{i=1}^{n} P_i \) is sub-ordinates to \( \delta \), we have
\[
\sum_{i=1}^{n} |F(d_i) - F(c_i)| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})|dy < \frac{1}{2} \epsilon.
\]

So, \( F \in BV(E_i) \). Hence we can find \( f \in BV(E_i) \). \( \square \)

3. \( L^r \)-Henstock-Kurzweil integral and properties

In this section we discuss few properties of \( L^r \)-Henstock-Kurzweil integrals in real space \( \mathbb{R} \). The collection of all function that are \( L^r \)-Henstock integrable on \( I = [a, b] \), will be denoted by \( HK_r(I) \). In the begining of the section, we discuss few properties of \( BV_r[a, b] \).

**Proposition 3.1.** 1. Let \( F \in BV_r[a, b] \) then \( F \) is bounded variation on every sub interval of \([a, b]\) and
\[
BV_r(F, [a, b]) = BV_r(F, [a, c]) + BV_r(F, [c, b])
\]
for each \( c \in (a, b) \).

2. If \( F \) is in \( BV_r[a, c] \) and \( F \) is in \( BV_r[c, b] \) then \( F \) is in \( BV_r[a, b] \).

**Theorem 3.2.** The function \( F \in AC_r[a, b] \) is in \( BV_r[a, b] \).

**Proof.** Let \( F \in AC_r[a, b] \) and let \( \epsilon > 0 \). There exists \( \nu > 0 \) and a gauge function \( \delta \) defined on \([a, b]\) so that if \( \mathcal{P} = \{(x_n, [c_n, d_n])\} \) \( < \delta \) and
\[
\sum_{n=1}^{q} (d_n - c_n) < \nu
\]
then
\[
\sum_{n=1}^{q} \frac{1}{d_n - c_n} \int_{c_n}^{d_n} |F(y) - F(x_n)|dy < \epsilon. \]

\( \square \)

**Theorem 3.3.** For \( 1 \leq r < \infty \), \( BV_r[a, b] = BV[a, b] \).

**Proof.** Let us assume \( F \in BV[a, b] \). If \( \{[c_i, d_i]\} \) is a finite collection of non overlapping intervals that have end points in \( E \), there exists \( M > 0 \) such that
\[
\sup_{j=1}^{q} \sum_{j=1}^{q} |F(d_j) - F(c_j)| < M.
\]

This implies that for any \( \nu > 0 \) if \( \sum_{j=1}^{q} (d_j - c_j) < \nu \) then
\[
\sum_{j=1}^{q} \left( \max_{x \in [c_j, d_j]} F(x) - \min_{x \in [c_j, d_j]} F(x) \right) < M.
\]

For any choice of \( x_j \in [c_j, d_j] \),
\[
\sum_{j=1}^{q} \frac{1}{d_j - c_j} \int_{c_j}^{d_j} |F(y) - F(x_j)|^r dy \leq \sum_{j=1}^{q} \left( \max_{x \in [c_j, d_j]} F(x) - \min_{x \in [c_j, d_j]} F(x) \right) < M \text{ for any gauge function } \delta.
\]

So, \( BV[a, b] \subseteq BV_r[a, b] \). also from the Theorem(2.3) \( BV_r[a, b] \subseteq BV[a, b] \). Hence \( BV_r[a, b] = BV[a, b] \). \( \square \)
Theorem 3.10. If $f : I = [a, b] \to \mathbb{R}$ are $L^r$-Henstock-Kurzweil integrable on $I$. If $f \geq 0$ a.e. on $I$ then $(L^r - H) \int_I f \geq 0$.

Proof. Let $f$ be $L^r$-Henstock-Kurzweil integrable on $I = [a, b]$ then there exists a function $F \in L^r[I]$ such that for any $\epsilon > 0$ there exists a gauge function $\delta$ such that for all finite collection $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non-overlapping tagged intervals in $I$ with $\mathcal{P} < \delta$ implies

$$\sum_{i=1}^{n} \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy < \epsilon.$$ 

That is,

$$\sum_{i=1}^{n} \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - S(f, \mathcal{P})|^r \, dy < \epsilon.$$ 

Now by the [8, Theorem 21], $f \in L^1[a, b]$. From the fact of Lebesgue integral we get the proof. \qed

Remark 3.5. The linearity of $L^r$-Henstock-Kurzweil integral (see [9]) and the Theorem 3.4, gives if $f \geq g$ a.e. on $I$ then $(L^r - H) \int_I f \geq (L^r - H) \int_I g$.

Lemma 3.6. For $1 \leq r < \infty$, $ACG_r[a, b] = ACG[a, b]$.

Proof. Let $E \subseteq [a, b]$. From the known fact that $ACG_r[a, b] = \bigcup AC_r[E_n]$ where $E = \bigcup_{n=1}^{\infty} E_n$. Also $AC_r[E_n] = AC[E_n]$. Therefore,

$$ACG_r[a, b] = \bigcup AC[E_n]$$

$$= ACG[E].$$

Consequently, $ACG_r[a, b] = ACG[a, b]$. \qed

We can find from the known fact that $HK_r(I)$ is contained in $L^1(I)$, then any function in $HK_r(I)$ is Denjoy integrable. That is:

Theorem 3.7. Let $f : I = [a, b] \to \mathbb{R}$. For $1 \leq r < \infty$, if $f$ is $L^r$-Henstock-Kurzweil integrable function is Denjoy integrable.

Theorem 3.8. Let $f : I \to \mathbb{R}$ be $L^r$-Henstock-Kurzweil integrable on $I$. Then $|f| \in HK_r(I)$ if and only if the indefinite integral $F(x) = \int_a^x f$ has $BV_r(I)$.

Proof. The proof is immediate. Since $f$ is in $HK_r(I)$, then $f$ is in $L^1(I)$. Therefore, $F(x)$ is of bounded variation, which tell us that $f$ is in $BV_r(I)$. See [1, Theorem 7.5]. \qed

Corollary 3.9. Let $f : [a, b] \to \mathbb{R}$ be $L^r$-Henstock-Kurzweil integrable function on $[a, b]$. $L^r$-Henstock-Kurzweil integrable function are absolutely integrable function on $[a, b]$.

Theorem 3.10. The function $f : I = [a, b] \to \mathbb{R}$.

1. If $f$ is $L^r$-Henstock-Kurzweil integrable then $f$ is measurable.

2. If $f$ is $L^r$-Henstock-Kurzweil integrable on $[a, b]$ and $f \geq 0$ a.e then $f$ is Lebesgue integrable on $[a, b]$. 
Proof. For (1) Let \( f \) be \( L^r \)-Henstock-Kurzweil integrable on \( I = [a, b] \) and \( F \) is the \( L^r \)-Henstock-Kurzweil integral of \( f \), then [8, Theorem 14] there exists \( F \in ACG_r[a, b] \) so that \( F'_r = f \) a.e. so that \( I \) is the sum of a sequence \( \{E_n\} \) of closed sets on each of which \( F \) is \( L^r \)-AC. Again [8, Theorem 15] gives \( F \) is AC. [13, Lemma 4.1 of Ch VII] there exists for each \( n \) a function \( E_n \) of bounded variation on \( I \), which coincides with \( F \) on \( E_n \). We therefore have a.e. on \( E_n \) the relation \( f(x) = F'_r(x) = F'_e(x) \) where \( F'_e(x) \) is \( L^r \) derivative of \( F \) and since the derivative of a function is bounded variation is measurable and a.e. finite, it follows that \( f \) is measurable and a.e. finite on each \( E_n \) and consequently on the whole interval \( I = [a, b] \).

For (2), follows [8, Theorem 21]. \( \square \)

Corollary 3.11. If \( f : [a, b] \to \mathbb{R} \) be \( L^r \)-Henstock-Kurzweil integrable on \( [a, b] \). The following are holds:

a) If \( f \) is bounded on \( [a, b] \) then \( f \) is clearly Lebesgue integrable on \( [a, b] \).

b) If \( f \geq 0 \) a.e. is \( L^r \)-Henstock integrable on every measurable subset of \( [a, b] \) then \( f \) is Lebesgue integrable on \( [a, b] \).

Theorem 3.12. Let \( f : [a, b] \to \mathbb{R} \). If \( f \) is \( L^r \)-Henstock-Kurzweil integrable on \( [a, b] \) then every perfect set in \( [a, b] \) contains a portion on which \( f \) is Lebesgue integrable.

Proof. Let \( f \) be \( L^r \)-Henstock-Kurzweil integrable on \( [a, b] \) then the Theorem(3.7), \( f \) is Denjoy integrable on \( [a, b] \). Using [4, Theorem 12(c)], we found every perfect set in \( [a, b] \) a portion on which \( f \) is Lebesgue integrable. \( \square \)

4. Bibliography

References


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