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Preconditioned Iterative Methods for Solving a Fractional Advection-Diffusion Equation

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ABSTRACT: In this paper, we consider numerical solutions of a fractional advection-diffusion equation. We first, propose an implicit method based on Grunwald formulae and then discuss its stability and consistency. To improve the implicit method, we use a preconditioned generalized minimal residual (PGMRES) method and preconditioned conjugate gradient normal residual (PCGNR) method. Numerical experiments are given to illustrate efficiency of the method.

Key Words: Fractional diffusion equations, preconditioner, conjugate gradient normal residual of the method, generalized minimal residual method.

Contents

1. Introduction

The study on fractional calculus has a long subject, but it was not until the late 20th century that fractional differential equations (FDEs) became considerable, because of their wide applications in the fields of science and engineering such as physics [4], finance [8,13], biology [12],image processing [2]. The advection-diffusion equations comes into existence in plenty physical processes. For example, transport happens in liquids through the combination of advection and diffusion [3]. Studying the advection-diffusion equations by fractional calculus becomes favourite today, as for non-local property of fractional operators can totally explain the memory property, exponent law and anomalous behaviours in evolution of diffusion systems. There are many methods to solve fractional differential equations, for instance short memory principle [7], finite difference method [5,13], finite element method [6,10]. Individually, analytical solutions can be created in some fractional differential equations. However, due to the essential differences between classical and fractional differential equations, sometimes it is inconceivable to detect the analytical solutions of a fractional partial differential equations, and even, the numerical solutions can not be achieved easily.
In this paper, we consider the numerical solutions for solving of the following fractional advection-diffusion equation (FDE):

$$\left\{\begin{array}{l} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - b(x,t)\frac{\partial^\beta u(x,t)}{\partial_+ x^\beta} - c(x,t)\frac{\partial^\beta u(x,t)}{\partial_- x^\beta} = \nu(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ u(x,0) = \phi(x), \quad a \le x \le b, \\ u(a,t) = u(b,t) = 0, \quad 0 \le t \le T, \end{array}\right. \tag{1.1}$$

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where parameters $\alpha \in (0,1]$,$\beta \in (1,2]$, and diffusion coefficient functions $b(x,t)$ and $c(x,t)$ are non-negative. $f(x,t)$ is the source term, and $\nu(x,t)$ is the drift of the process. The time fractional derivative in (1.1) is the Caputo fractional derivative of order α [15], which is defined as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha}, \tag{1.2}$$

and the left-sided and the right-sided space fractional derivatives in (1.1) are the Riemann-Liouville fractional derivatives of order β [9,15], which is defined as follows:

$$\frac{\partial^\beta u(x,t)}{\partial_+ x^\beta} = \frac{1}{\Gamma(m-\beta)} \frac{\partial^m}{\partial x^m} \int_a^x \frac{u(s,t)}{(x-s)^{\beta-m+1}} ds, \tag{1.3}$$

$$\frac{\partial^\beta u(x,t)}{\partial_- x^\beta} = \frac{1}{\Gamma(m-\beta)} \frac{\partial^m}{\partial x^m} \int_x^b \frac{u(s,t)}{(s-x)^{\beta-m+1}} ds, \tag{1.4}$$

respectively.
This paper is organized as follows. In section 2, we discretize the Eq. (1.1) by the finite difference method and present the corresponding coefficient matrix. In section 3, we propose a stable and convergent implicit difference method. In section 4, we propose the preconditioned GMRES method and preconditioned CGNR method for solving of the implicit difference scheme. In section 5, we employ the proposed numerical schemes on some test examples.

2. Descretization of FDE

Let m and n be two positive integers, and $h = (b-a)/m$ and $\tau = T/n$ be the sizes of the spatial grid and time step, respectively. Then we define a spatial and temporal partition as follows:

$$\begin{aligned} x_i &= ih \qquad , i = 0, 1, \cdots, m; \\ t_j &= j\Delta t \qquad , j = 0, 1, \cdots, n. \end{aligned}$$

For the sake of simplicity, let:

$$\begin{aligned} &b_i^{(j)} = b(x_i, t_j), \qquad c_i^{(j)} = c(x_i, t_j), \qquad f_i^{(j)} = f(x_i, t_j) \\ &u_i^{(j)} = u(x_i, t_j) \qquad \nu_i^{(j)} = \nu(x_i, t_j) \qquad \triangle_t u(x_i, t_j) = u(x_i, t_{j+1}) - u(x_i, t_j). \end{aligned}$$

Using the forward difference formula, the time fractional derivative for $0 < \alpha < 1$ can be approximated by the following formula [1]:

$$\begin{aligned} \frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{\partial u(x_i, s)}{\partial s} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \left(\left(\frac{1}{\tau} \triangle_t u(x_i, t_j) + o(\tau) \right) \int_{tj}^{t_{j+1}} (t_{k+1}-s)^{-\alpha} \right) + O(\tau^{2-\alpha}) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} a_j \triangle_t u(x_i, t_{k-j}) + o(\tau^{2-\alpha}). \end{aligned} \tag{2.1}$$

where $a_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, ..., n$. Also, we use the shifted Grunwald-Letnikov [14] approximated the left-sided and the right-sided fractional derivatives for parameter $1 < \beta < 2$ as follows:

$$\frac{\partial^\beta u(x_i, t_{k+1})}{\partial_+ x^\beta} = \frac{1}{h^\beta} \sum_{j=0}^{i+1} g_j^{(\beta)} u_{i-j+1}^{(k+1)} + o(h), \tag{2.2}$$

$$\frac{\partial^\beta u(x_i, t_{k+1})}{\partial_- x^\beta} = \frac{1}{h^\beta} \sum_{j=0}^{m-i+1} g_j^{(\beta)} u_{i+j-1}^{(k+1)} + o(h), \tag{2.3}$$

in which $g_j^{(\beta)}$ the Grunwald coefficients and defined by:

$$g_0^{(\beta)}=1 \quad , \quad g_j^{(\beta)}=\frac{(-1)^j}{j!}\beta(\beta-1)\dots(\beta-j+1) \quad , \quad j=1,2,\dots \tag{2.4}$$

Further, utilizing the forward difference formula, the space partial derivative can be approximated:

$$\frac{\partial^2 u(x_i,t_{k+1})}{\partial x^2}=\frac{u_{i+1}^{(k+1)}-2u_i^{(k+1)}+u_{i-1}^{(k+1)}}{h^2}+O(h^2). \tag{2.5}$$

Let

$$\omega_1=\frac{\Gamma(2-\alpha)\tau^\alpha}{h^\beta}, \qquad \omega_2=\frac{\Gamma(2-\alpha)\tau^\alpha}{h^2}, \qquad \omega_3=\Gamma(2-\alpha)\tau^\alpha.$$

Using (2.1)-(2.5) and to overlook truncation error, the implicit difference method is given by:

$$u_i^{(k+1)}-\omega_1\left(b_i^{(k+1)}\sum_{j=0}^{i+1}g_j^{(\beta)}u_{i-j+1}^{(k+1)}+c_i^{(k+1)}\sum_{j=0}^{m-i+1}g_j^{(\beta)}u_{i+j-1}^{(k+1)}\right)-2\omega_2\nu_i^{(k+1)}$$
$$\left(\frac{u_{i+1}^{(k+1)}}{2}-u_i^{(k+1)}+\frac{u_{i-1}^{(k+1)}}{2}\right)=u_i^{(k)}-\sum_{j=1}^{k}a_j(u_i^{(k-j+1)}-u_i^{(k-j)})+\omega_3 f_i^{(k+1)}, \tag{2.6}$$

where $i=1,...,m-1$; $k=0,...,n-1$, and the boundary and initial conditions are discretized as follows:

$$u_i^{(0)}=\phi(x_i), \qquad i=0,\dots,m; \qquad u_0^{(k)}=u_m^{(k)}=0, \qquad k=1,\dots,n.$$

Lemma 2.1 *The coefficients $a_j, g_j^{(\beta)}$, for $j=1,2,...,$ satisfy:*

(a) $1=a_0>a_1>...>a_j\to 0$ *as* $j\to\infty$*;*

(b) $g_0^{(\beta)}=1, g_1^{(\beta)}=-\beta<0,\ g_j^{(\beta)}>0$ *for* $j\neq 1$*, and* $\sum_{j=0}^{\infty}g_j^{(\beta)}=0$*;*

(c) $g_j^{(\beta)}=O(j^{-(\beta+1)})$

Proof:

(a) By using the use of $a_j=(j+1)^{1-\alpha}-j^{1-\alpha}$, it is clear that $a_0=1$. The following statement hold,

$$\lim_{j\to\infty}(j+1)=\lim_{j\to\infty}j.$$

Also,

$$\lim_{j\to\infty}a_j=\lim_{j\to\infty}\frac{(j+1)}{(j+1)^\alpha}-\lim_{j\to\infty}\frac{j}{j^\alpha}=\lim_{j\to\infty}\frac{j}{j^\alpha}-\lim_{j\to\infty}\frac{j}{j^\alpha}=0.$$

(b) It is well known that

$$(1+z)^\beta=\sum_{j=0}^{\infty}\binom{\beta}{j}z^j,$$

for any complex $|z|\le 1$ and $\beta>0$. If $z=-1$, therefore

$$\sum_{j=0}^{\infty}g_j^\beta=0,$$

Note that $g_0^{(\beta)}=1$, $g_1^{(\beta)}=-\beta<0$. Also, using the recurrence relationship (2.4), one gets:

$$g_2^{(\beta)}>g_3^{(\beta)}>...>0.$$

(c) By the Stirlings formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\; x^x \; e^{-x} \qquad as \qquad x \longrightarrow \infty$$

one can get

$$\begin{aligned}\frac{\Gamma(j-\beta)}{\Gamma(j+1)} &\sim \frac{\sqrt{2\pi(j-1-\beta)}(j-1-\beta)^{(j-1-\beta)}e^{-(j-1-\beta)}}{\sqrt{2\pi\, j}j^j e^{-j}} \\ &= e^{\beta+1}\sqrt{\frac{j-1-\beta}{j}}\frac{(j-1-\beta)^{(j-1-\beta)}}{j^j}\end{aligned}$$

where

$$\sqrt{\frac{j-1-\beta}{j}} \longrightarrow 1,$$

and

$$j^{\beta+1}\frac{(j-1-\beta)^{(j-1-\beta)}}{j^j} = \big(1-\frac{\beta+1}{j}\big)^j\big(\frac{j}{j-1-\beta}\big)^{\beta+1} \longrightarrow e^{-\beta-1},$$

as $j \to \infty$. Therefore

$$\frac{\Gamma(j-\beta)}{\Gamma(j+1)} \sim j^{-\beta-1}$$

as $j \to \infty$. So the proof is complete.

□

We note that Lemma 2.1 implies that

$$\sum_{j=0}^{k+1} g_j^{(\beta)} < 0, \qquad for \;\; k = 0, 1, ... \tag{2.7}$$

To get the matrix form, for $1 \leq j \leq n-1$, let

$$u^{(j)} = \big[u_1^{(j)}, u_2^{(j)}, \dots, u_{m-1}^{(j)}\big]^T, \qquad f^{(j)} = \big[f_1^{(j)}, f_2^{(j)}, \dots, f_{m-1}^{(j)}\big]^T,$$

$$B^{(j)} = diag\big(b_1^{(j)}, \dots, b_{m-1}^{(j)}\big), \qquad C^{(j)} = diag\big(c_1^{(j)}, \dots, c_{m-1}^{(j)}\big),$$

and $u^{(0)} = (\phi_1^{(0)}, \dots, \phi_{m-1}^{(0)})$. Let G_β and ν^j be Toplitz and three-diagonal matrices defined by

$$G_\beta = \begin{bmatrix} g_1^{(\beta)} & g_0^{(\beta)} & 0 & 0 & \cdots & 0 \\ g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & 0 & \cdots & 0 \\ \vdots & g_2^{(\beta)} & g_1^{(\beta)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ g_{m-2}^{(\beta)} & \ddots & \ddots & \ddots & g_1^{(\beta)} & g_0^{(\beta)} \\ g_{m-1}^{(\beta)} & g_{m-2}^{(\beta)} & \cdots & \cdots & g_2^{(\beta)} & g_1^{(\beta)} \end{bmatrix} \nu^{(j)} = \begin{bmatrix} 2\nu_1^{(j)} & -\nu_1^{(j)} & 0 & 0 & \cdots \\ -\nu_2^{(j)} & 2\nu_2^{(j)} & -\nu_2^{(j)} & 0 & \cdots \\ 0 & -\nu_3^{(j)} & 2\nu_3^{(j)} & -\nu_3^{(j)} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & -\nu_{m-1}^{(j)} & 2\nu_{m-1}^{(j)} \end{bmatrix}$$

If I_{m-1} denote the identity matrix of order $m-1$, then one can see that the schemes (2.6) is equivalent to a matrix form:

$$(I_{m-1} + A^{(k+1)})u^{(k+1)} = b^{(k+1)} \tag{2.8}$$

where

$$b^{(k+1)} = \sum_{j=1}^{k}\big(a_{k-j} - a_{k-j+1}\big)u^{(j)} + a_k u^{(0)} + \omega_3 f^{(k+1)},$$

and

$$A^{(k+1)} = -\omega_1(B^{(k+1)}G_\beta + C^{(k+1)}G_\beta^T) + \omega_2\nu^{(k+1)}. \tag{2.9}$$

This coefficients, for $i = 1, ..., m-1$ and $j = 1, ..., m-1$ are defined as follows:

$$a_{i,j}^{(k+1)} = \begin{cases} -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + 2\delta_i^{(k+1)} & j = i \\ -\varepsilon_i^{(k+1)} g_2^{(\beta)} - \eta_i^{(k+1)} g_0^{(\beta)} - \delta_i^{(k+1)} & j = i-1 \\ -\varepsilon_i^{(k+1)} g_0^{(\beta)} - \eta_i^{(k+1)} g_2^{(\beta)} - \delta_i^{(k+1)} & j = i+1 \\ -\varepsilon_i^{(k+1)} g_{i-j+1}^{(\beta)} & j < i-1 \\ -\eta_i^{(k+1)} g_{j-i+1}^{\prime(\beta)} & j > i+1 \end{cases} \tag{2.10}$$

where

$$\varepsilon_i^{(k+1)} = w_1 b_i^{(k+1)}, \qquad \eta_i^{(k+1)} = w_1 c_i^{(k+1)}, \qquad \delta_i^{(k+1)} = w_2 \nu_i^{(k+1)}.$$

Obviously, for $i \neq j$, $a_{ij} \leq 0$.

Theorem 2.1 *The matrix $I_{m-1} + A^{(k+1)}$ in (2.8) is nonsingular and strictly diagonally dominated.*

Proof: Let $a_{ij}^{(k+1)}$ be the (i, j) entry of the matrix $A^{(k+1)}$ in (2.8). Hence

$$\begin{aligned} & a_{ii}^{(k+1)} - \sum_{j=0, j\neq i}^{m-1} |a_{ij}^{(k+1)}| \\ &= -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + 2\delta_i^{(k+1)} - \left(\varepsilon_i^{(k+1)} \sum_{j=0, j\neq 1}^{i} g_j^{(\beta)} + \eta_i^{(k+1)} \sum_{j=0, j\neq 1}^{m-i} g_j^{(\beta)} + 2\delta_i^{(k+1)}\right) \\ &\geq -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + 2\delta_i^{(k+1)} - \left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) \sum_{j=0, j\neq 1}^{\infty} g_j^{(\beta)} - 2\delta_i^{(k+1)} \\ &= -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + \left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} = 0. \end{aligned}$$

This observation shows that the coefficient matrix $I_{m-1} + A^{(k+1)}$ is strictly diagonally dominated and invertible. This completes the proof. □

Corollary 2.1 *The difference method given by (2.6) is uniquely solvable.*

3. Stability analysis of the implicit difference method

In this section, we derive a theorem for the stability of the method given by (2.6).

Theorem 3.1 *The implicit difference method given by (2.6), based on the Grunwald approximation to the time-space fractional diffusion equation is unconditionally stable.*

Proof: According to the Greshgorin theorem, the eigenvalues of the matrix A are all inside the disks centered at

$$a_{ii}^{(k+1)} = -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + 2\delta_i^{(k+1)}$$

with radius

$$r_i^{(k+1)} = \sum_{j=0,j\neq i}^{m-1} |a_{ij}| = \varepsilon_i^{(k+1)} \sum_{j=0,j\neq 1}^{i} g_j^{(\beta)} + \eta_i^{(k+1)} \sum_{j=0,j\neq 1}^{m-i} g_j^{(\beta)} + 2\delta_i^{(k+1)}$$
$$\geq -\left(\varepsilon_i^{(k+1)} + \eta_i^{(k+1)}\right) g_1^{(\beta)} + 2\delta_i^{(k+1)}.$$

therefore

$$0 \leq \lambda_i \leq 2\left(2\delta_i^{(k+1)} - (\varepsilon_i^{(k+1)} + \eta_i^{(k+1)})g_1^{(\beta)}\right).$$

λ_i is an eigenvalue, if and only if $\frac{1-\lambda_i/2}{1+\lambda_i/2}$ is an eigenvalue of $(I+A/2)^{-1}(I-A/2)$. Since $\lambda_i \geq 0$, then

$$-1 < -1 + \frac{2}{1+\lambda_i/2} = \frac{1-\lambda_i/2}{1+\lambda_i/2} = 1 - \frac{1}{1+\lambda_i/2} < 1$$

Therefore $|\frac{1-\lambda_i/2}{1+\lambda_i/2}| < 1$, and the implicit method defined is unconditionally stable. □

Remark 3.1 The implicit method defined (2.6) is consistent with local truncation error of the form $O(\tau^{2-\alpha}+h^2+h)$. On the other hand, according to the above theroem, the implicit method is stable. Thus, according to the Lax Equivalence Theorem [11], the method is convergent.

4. Preconditioned iterative methods

Preconditioners are useful in iterative methods to solve a linear system $Ax = b$ since the rate of convergence for most iterative linear solvers increases because the condition number of a matrix decreases as a result of preconditioning.

Lemma 4.1 *The Toeplitz matrice $-G_\beta$ is an M-matrice.*

Proof: Let $F = -G_\beta$, therefore

$$F = D - (D - F) = D\left(I - (I - D^{-1}F)\right),$$
$$B = I - D^{-1}F,$$

where D is a diametrical section of matrice $-G_\beta$, as regards:

$$(I - D^{-1}F)x = \lambda x \quad \Longrightarrow \quad (D-F)x = \lambda D x,$$

So,

$$\lambda g_1^{(\beta)} x_i = g_1^{(\beta)} x_i - \sum_{j=0}^{m-1} g_j^{(\beta)} x_j; \quad \forall i = 1, 2, ..., m-1.$$

Therefore

$$|-\sum_{j=0,j\neq 1}^{m-1} g_j^{(\beta)} x_j| = |\lambda g_1^{(\beta)} x_i|,$$

$$|\lambda||g_1^{(\beta)}||x_j| \leq \sum_{j=0,j\neq 1}^{m-1} |g_j^{(\beta)}||x_j|,$$

$$|\lambda| \leq \frac{\sum_{j=0,j\neq 1}^{m-1} |g_j^{(\beta)}|}{|g_1^{(\beta)}|}.$$

According to (2.7), we have

$$|g_1^{(\beta)}| \geq \sum_{j=0,j\neq 1}^{m-1} g_j^{(\beta)}.$$

Thus the matrix F is strictly diagonally dominanted, so $\rho(B) < 1$. Hence, we have

$$I - D^{-1}F = B \quad \Longrightarrow \quad I - B = D^{-1}F, \tag{4.1}$$

$$(D^{-1}F)^{-1} = (I-B)^{-1} = \sum_{i=0}^{\infty} B^i \geq 0, \tag{4.2}$$

$$(I-B)^{-1} = F^{-1}D \geq 0. \tag{4.3}$$

The (4.1) shows that:

$$F^{-1} = -G_\beta \geq 0.$$

in which $[-G_\beta]$ is a L-matrice and $[-G_\beta]^{-1} \geq 0$, which is an M-matrice. □

To derive a preconditioner, we now split matrices G_β as

$$G_{\beta,l} = \begin{bmatrix} g_1^{(\beta)} & g_0^{(\beta)} & & & & \\ \vdots & g_1^{(\beta)} & g_0^{(\beta)} & & & \\ g_l^{(\beta)} & & \ddots & & \ddots & \\ & \ddots & & & \ddots & g_0^{(\beta)} \\ & & & g_l^{(\beta)} & \cdots & g_1^{(\beta)} \end{bmatrix} + \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & g_{l+1}^{(\beta)} & & \\ & & & & \ddots & \\ & & & & & \sum_{j=l+1}^{m-1} g_j^{(\beta)} \end{bmatrix}$$

Similarly, this implies that the matrice $G_{\beta,l}$ is strictly diagonally dominanted M-matrices. Therefore, the matrix $A^{(k+1)}$ can be decomposed as:

$$A^{(k+1)} = A_l^{(k+1)} + B_l^{(k+1)},$$

where

$$A_l^{(k+1)} = -\omega_1(B^{(k+1)}G_{\beta,l} + C^{(k+1)}G_{\beta,l}^T) + \omega_2\nu^{(k+1)},$$

$$B_l^{(k+1)} = A^{(k+1)} - A_l^{(k+1)}.$$

If the matrix $I + A_l^{(k+1)}$ is an approximation of the matrix $I + A^{(k+1)}$, the relative error is equal to:

$$\frac{\|(I_{m-1} + A^{(k+1)}) - (I_{m-1} + A_l^{(k+1)})\|_\infty}{\|I_{m-1} - A^{(k+1)}\|_\infty} \leq \frac{\|E^{(k+1)}(G_\beta - G_{\beta,l}) + C^{(k+1)}(G_\beta - G_{\beta,l})^T\|_\infty}{\|B^{(k+1)}G_\beta + C^{(k+1)}G_\beta^T\|_\infty}$$
$$= o(k^{-\beta}).$$

The relative difference between $I_{m-1} + A^{(k+1)}$ and $I_{m-1} + A_l^{(k+1)}$ may become very small as k becomes large enough.

4.1. Preconditioned GMRES method

The GMRES method is an itterative method for finding the numerical solution of a nonsymetric systeams, as was presented by Saad and Schultz [17] in 1986. The method approximates the solution by the vector in a Krylov subspace with minimal residual. Although, this method is theoretically well founded, but it is likely to suffer from slow convergence. To overcom this problem, preconditioning

techniques have been introduced to improve the convergence rate.
Let $p_l^{(k+1)} = I_{m-1} + A_l^{(k+1)}$ be a preconditioned linear systeams

$$(p_l^{(k+1)})^{-1}\left(I_{m-1} + A_l^{(k+1)}\right)u^{(k+1)} = (p_l^{(k+1)})^{-1}b^{(k+1)}. \tag{4.4}$$

By the GMRES method. The left-preconditioned GMRES algorithm described below is given in [18].

Algorithm 1 *: GMRES with Left Preconditioning*

Compute $r_0 = M^{-1}(b - Ax_0), \beta = \parallel r_0 \parallel_2$ *and* $v_1 = \frac{r_0}{\beta}$
for $j = 1, 2, \ldots, m$ ***do***
  $w = M^{-1}Av_j$
  for $i = 1 : j$ ***do***
    $h_{ij} = (w, v_i)$
    $w := w - h_{ij}v_i$
  end for
  $h_{j+1,j} = \parallel w \parallel_2$
  $v_{j+1} = \frac{w}{h_{j+1,j}}$
end for
Define $V_m := [v_1, \ldots, v_m], \tilde{H_m} = \{h_{ij}\}_{1\le i\le j+1, 1\le j\le m}$
Compute y_k *the* $\min_y \parallel \beta e_1 - \tilde{H_m}y \parallel_2$ *and* $x_m = x_0 + V_m y_m$
if *satisfied* ***then***
  stop
else
  set $x_0 := x_m$ *and go to* 1
end if

4.2. Preconditioned CGNR method

The conjugate gradient method can be applied on the normal equations. The CGNR method is a variant of this approach which is the simplest method for nonsymmetric or indefinite systeams. Let $C^{(k+1)} = I + A^{(k+1)}$ and the normal equation of (2.8) be

$$(C^{(k+1)})^T C^{(k+1)} u^{(k+1)} = (C^{(k+1)})^T b^{(k+1)}, \tag{4.5}$$

This can be solved by CGNR method efficiently if the matrix $(C^{(k+1)})^T C^{(k+1)}$ is well-conditioned and has eigenvalues clustering around 1. However, it is not the case when the diffusion coefficients $b(x,t)$ and $c(x,t)$ are not sufficiently small [16]. Therefore, it is required to introduce a preconditioner to speed up the convergence. So, we use $(p_l^{(k+1)})^T(p_l^{(k+1)})$ to precondition (4.5), where

$$p_l^{(k+1)} = I + A_l^{(k+1)}.$$

5. Numerical result

In this section, we present two example to compare the effectiveness of the iterative methods with preconditioned and without preconditioned. We choose the initial value as follows:

$$v_0 = \begin{cases} u^{(0)} := [\phi(x_1), ..., \phi(x_{m-1})]^T & k = 1, \\ 2u^{(k)} - u^{(k-1)} & k > 1, \end{cases}$$

when the stopping condition is:

$$\frac{\|r_j\|_2}{\|b^{(k+1)}\|_2} < 10^{-7},$$

in which r_j is the residual vector in $j-$th iteration.

Example 5.1 *Consider the equation (1.1) with $\alpha = 0.6$, $\beta = 1.9$ and the diffusion coefficients which are given by*

$$b(x,t) = x + 0.9t(2-x)^2, \qquad c(x,t) = x + 0.9t(2-x)^2,$$

with the spatial interval $\Omega = (0,1) \times (0,1)$ and the time interval $[0,T] = [0,1]$. The source term and the drift of the process and initial condition are given by

$$f(x,t) = \frac{2}{\Gamma(2.15)}(x^2 - x)t^{1.15} + 9\pi^2 \sin(3\pi x) - 2t^2,$$

$$\nu(x,t) = 0.6x^3,$$

$$u(x,0) = \sin(3\pi x).$$

By direct computation, we can show that the true solution to the fractional diffusion equation is

$$u(x,t) = \sin(3\pi x) + x(x-1)t^2.$$

We obtained the solution using MATLAB R2016b on a HP - Laptop with configuration Inter(R) Core (TM)i$3-5005$ U with a 2.00 GHz CPU and 4-GB RAM. We take the bandwidth l of the preconditioner $p_l^{(k+1)}$ equal to 8. The average number of iterations required by the GMRES(20), PGMRES(20), CGNR, PCGNR methods, are shown in Tables 1, and the CPU times of methods, are shown in Table 2. Also, in Table 3 the condition number of relevant matrices is shown. The condition numbers of the preconditioned matrices are very small. The distribution of the eigenvalues of matrices $\hat{A}$, $\hat{A}^T\hat{A}$, $(p_8^{(1)})^{-1}\hat{A}$ and

Table 1: The average number of iterations for Example 5.1

m=n	GMRES(20)	PGMRES(20)	CGNR	PCGNR
16	15	3.4375	17	5.1250
32	34.4375	4.1875	44.75	7
64	64.3750	5.0156	148.8125	8.9063
128	118.7578	4.9219	569.6484	10.6328
256	229.9766	4.7695	1.8855e+03	12.5625

Table 2: The required CPU times for Example 5.1

m=n	GMRES(20)	PGMRES(20)	CGNR	PCGNR
16	0.1357	0.0786	0.0732	0.0285
32	0.3696	0.1035	0.1631	0.0792
64	1.7912	0.3603	1.3662	0.2777
128	8.9245	1.5120	23.7710	1.2099
256	36.4335	8.5655	715.9216	7.1104

$((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A}$ with $m = n = 256$ are shown Figure 1. Also, Figure 2 shows the behaviour of the exact solution and the numerical solution with $m = n = 256$.

Example 5.2 *Consider the equation (1.1) with $\alpha = 0.8$, $\beta = 1.84$ and the diffusion coefficients which are given by*

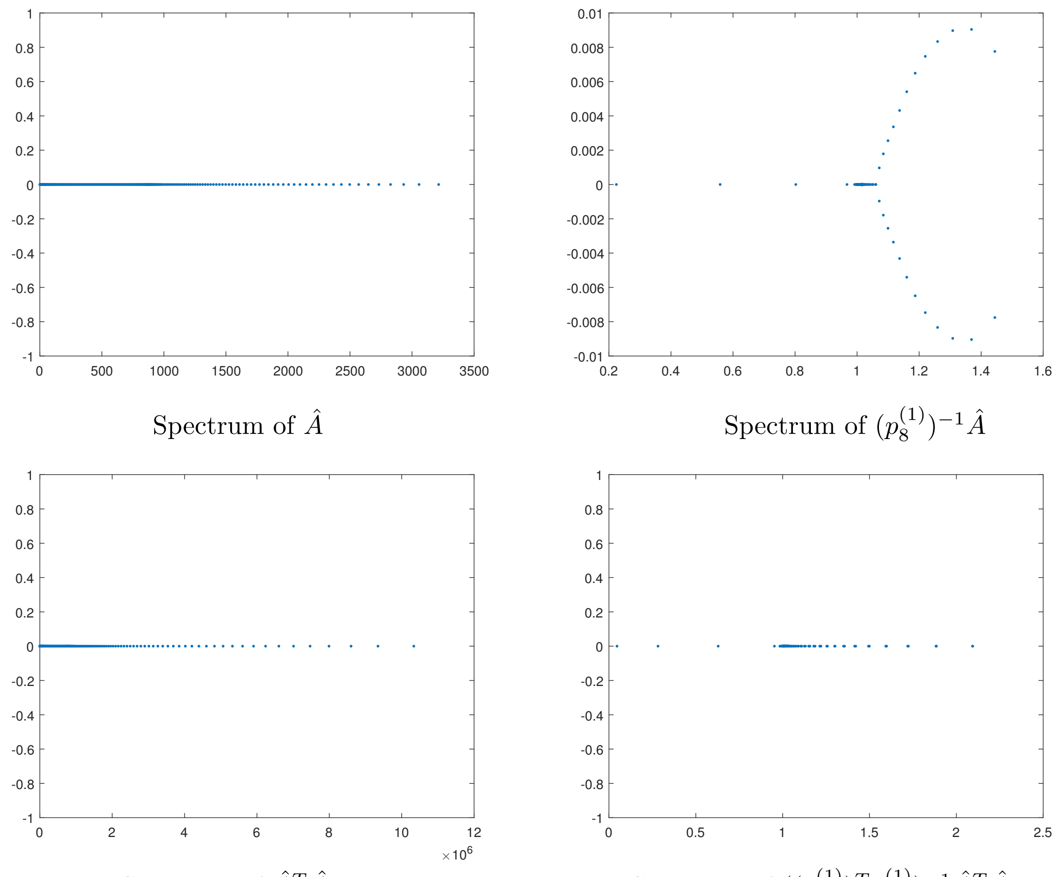
$$b(x,t) = x + 4(1+t)x^{0.8}, \qquad c(x,t) = x + 4(1+t)(1-x)^{1.8},$$

with the spatial interval $\Omega = (0,1) \times (0,1)$ and the time interval $[0,T] = [0,1]$. The source term and the drift of the process and initial condition are given by

$$f(x,t) = e^t\Big[6(1+t)\Big((\frac{\Gamma(4)}{\Gamma(3.4)} - \frac{\Gamma(4)}{\Gamma(2.2)})(x^3 + (1-x)^3) - (\frac{3\Gamma(5)}{\Gamma(4.4)} - \frac{3\Gamma(5)}{\Gamma(3.2)})$$
$$(x^4 + (1-x)^4) + (\frac{3\Gamma(6)}{\Gamma(5.4)} - \frac{3\Gamma(6)}{\Gamma(4.2)})(x^5 + (1-x)^5) - (\frac{\Gamma(7)}{\Gamma(6.4)} - \frac{\Gamma(7)}{\Gamma(5.2)})$$
$$(x^6 + (1-x)^6)\Big)\Big],$$

Table 3: Condition numbers for relevant matrices for Example 5.1

m=n	16	32	64	128	256
$\kappa(\hat{A})$	54.7910	164.0701	445.9048	1.1505e+03	2.8929e+03
$\kappa((p_8^{(1)})^{-1}\hat{A})$	1.0281	1.0979	1.1645	1.2168	1.2601
$\kappa(\hat{A}^T\hat{A})$	3.0021e+03	2.6919e+04	1.9883e+05	1.3236e+06	8.3690e+06
$\kappa(((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A})$	1.3415	2.9271	10.1159	47.5521	241.2591



Spectrum of $\hat{A}$ | Spectrum of $(p_8^{(1)})^{-1}\hat{A}$

Spectrum of $\hat{A}^T\hat{A}$ | Spectrum of $((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A}$

Figure 1: The spectra of the unpreconditioned and preconditioned coefficient matrices at time t_1 with $m = n = 256$.

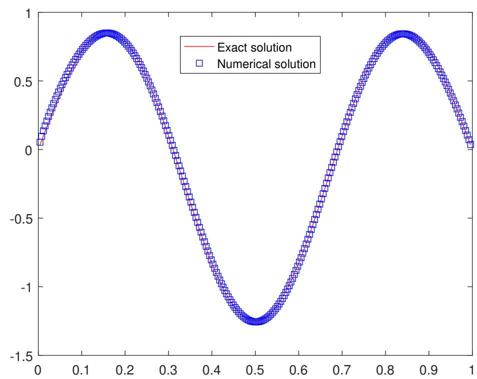


Figure 2: Numerical and exact solutions at time t_n with $m = n = 256$

$$\nu(x,t) = 0.2x^2,$$
$$u(x,0) = x^3(1-x)^3.$$

By direct computation, we can show that the true solution to the fractional diffusion equation is

$$u(x,t) = e^t + x^3(1-x)^3.$$

We take the bandwidth l of the preconditioner $p_l^{(k+1)}$ equal to 8. The average number of iterations required by the GMRES(20), PGMRES(20), CGNR, PCGNR methods are shown in Tables 4 and the CPU times of methods are shown in Table 5. Also in Table 6 the condition number of relevant matrices is shown. The condition numbers of the preconditioned matrices are very small. The distribution of the eigenvalues of matrices $\hat{A}$, $\hat{A}^T\hat{A}$, $(p_8^{(1)})^{-1}\hat{A}$ and $((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A}$ with $m = n = 256$ are shown Figure 3. Also, Figure 4 shows the behaviour of the exact solution and the numerical solution with $m = n = 256$.

Table 4: The average number of iterations for Example 5.2

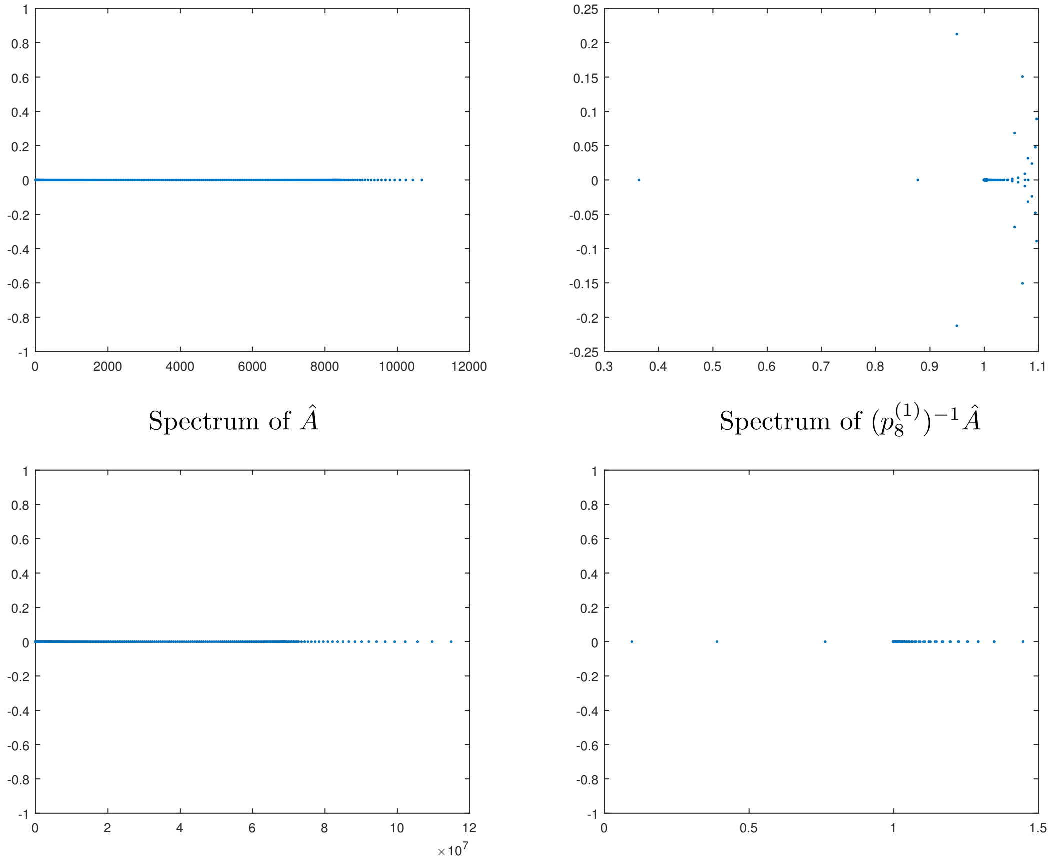
m=n	GMRES(20)	PGMRES(20)	CGNR	PCGNR
16	15	3	16.1250	5.2.5625
32	36.0625	3.1563	43.5938	3.3125
64	158.2188	4.0625	127.0625	3.2813
128	324.6094	4.4688	405.2578	4.1484
256	637.2422	5.1016	1.3681e+03	4.6641

Table 5: The required CPU times for Example 5.2

m=n	GMRES(20)	PGMRES(20)	CGNR	PCGNR
16	0.1732	0.1096	0.0448	0.0285
32	0.5172	0.1101	0.1732	0.0890
64	2.8827	0.3982	1.3315	0.3133
128	17.7450	1.6572	18.3676	1.3261
256	90.0183	9.7151	601.1096	7.8043

6. Conclusions

In this paper, we first introduce a numerical method for solving a fractional advection-diffusion equation. In other to improve the proposed numerical method, we discretize the system by the implicit



Spectrum of $\hat{A}$ — Spectrum of $(p_8^{(1)})^{-1}\hat{A}$

Spectrum of $\hat{A}^T\hat{A}$ — Spectrum of $((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A}$

Figure 3: The spectra of the unpreconditioned and preconditioned coefficient matrices at time t_1 with $m = n = 256$.

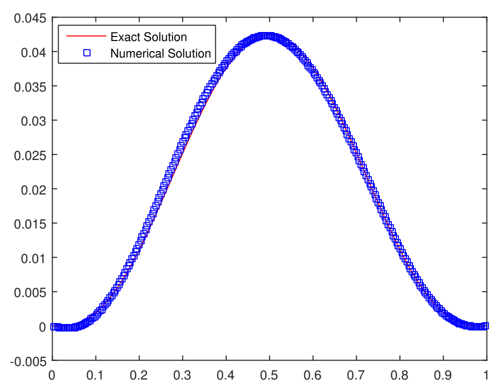


Figure 4: Numerical and exact solutions at time t_n with $m = n = 256$

Table 6: Condition numbers for relevant matrices for Example 5.2

m=n	16	32	64	128	256
$\kappa(\hat{A})$	64.7599	216.1674	660.5243	1.8430e+03	4.7295e+03
$\kappa((p_8^{(1)})^{-1}\hat{A})$	1.0247	1.1111	1.2202	1.3513	1.5370
$\kappa(\hat{A}^T\hat{A})$	4.1938e+03	4.6728e+04	4.3629e+05	3.3967e+06	2.2368e+07
$\kappa(((p_8^{(1)})^T p_8^{(1)})^{-1}\hat{A}^T\hat{A})$	1.3600	7.9993	99.9245	783.7281	4.2464e+03

difference method. Then, we introduce iterative methods of GMRES and CGNR to solve the corresponding matrix equation by introducing a preconditioner matrix. Finally, we examine the efficiency of our preconditioners by numerical results.

References

1. Liu, F., Zhuang, P., Anh, V., Turner, I., & Burrage, K., *Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation*, Applied Mathematics and Computation. 191(1), 12-20, (2007).
2. Bai, J., & Feng, X. C., *Fractional-order anisotropic diffusion for image denoising*, IEEE transactions on image processing. 16(10), 2492-2502, (2007).
3. Van Beinum, W., Meeussen, J. C., Edwards, A. C., & Van Riemsdijk, W. H., *Transport of ions in physically heterogeneous systems; convection and diffusion in a column filled with alginate gel beads, predicted by a two-region model*, Water Research. 34(7), 2043-2050, (2000).
4. Sokolov, I. M., Klafter, J., & Blumen, A., *Fractional kinetics*, Physics Today, 55(11), 48-54, (2002).
5. Savović, S., & Djordjevich, A., *Finite difference solution of the one-dimensional advection–diffusion equation with variable coefficients in semi-infinite media*, International Journal of Heat and Mass Transfer. 55(15-16), 4291-4294 (2012).
6. Dhawan, S., Kapoor, S., & Kumar, S., *Numerical method for advection diffusion equation using FEM and B-splines*, Journal of Computational Science, 3(5), 429-437, (2012).
7. Xu, Y., & He, Z., *The short memory principle for solving Abel differential equation of fractional order*, Computers & Mathematics with Applications, 62(12), 4796-4805, (2011).
8. Samko, S. G., Kilbas, A. A., Marichev, O. I., *Fractional integrals and derivatives, Gordon and Breach Science Publishers*, theory and applications. (1993).
9. Zheng, Y., Li, C., & Zhao, Z., *A note on the finite element method for the space-fractional advection diffusion equation*, Computers & Mathematics with Applications. 59(5), 1718-1726, (2010).
10. Lax, P. D., & Richtmyer, R. D., *Survey of the stability of linear finite difference equations*, In Selected Papers Volume I, Springer, New York, NY. 125-151, (2005).
11. Magin, R. L., *Fractional calculus in bioengineering begell house publishers*, Inc., Connecticut. (2006).
12. Tadjeran, C., Meerschaert, M. M., & Scheffler, H. P., *A second-order accurate numerical approximation for the fractional diffusion equation*, Journal of computational physics. 213(1), 205-213, (2006).
13. Meerschaert, M. M., & Tadjeran, C., *Finite difference approximations for two-sided space-fractional partial differential equations*, Applied numerical mathematics. 56(1), 80-90, (2006).
14. Meerschaert, M. M., & Tadjeran, C., *Finite difference approximations for fractional advection–dispersion flow equations*, Journal of computational and applied mathematics. 172(1), 65-77, (2004).
15. Podlubny, I., *Fractional differential equations*, mathematics in science and engineering, (1999).
16. Pang, H. K., & Sun, H. W., *Multigrid method for fractional diffusion equations*, Journal of Computational Physics. 231(2), 693-703, (2012).
17. Saad, Y., & Schultz, M. H., *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM Journal on scientific and statistical computing. 7(3), 856-869, (1986).
18. Saad, Y., *Iterative methods for sparse linear systems*, Society for Industrial and Applied Mathematics. (2003).

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