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New analogous of Ramanujan's remarkable product of theta-function and their explicit evaluations

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ABSTRACT: In this article, we define $\mu_{m,n}$ for any positive real numbers m and n involving Ramanujan's product of theta-functions $f(q)$ and $f(-q^2)$, which is analogous to Ramanujan's remarkable product of theta-functions and establish its several properties. We establish properties of $\mu_{m,n}$, general theorems for the explicit evaluations of $\mu_{m,n}$ and its explicit values.

Key Words: Class invariant, modular equation, theta-function, cubic continued fraction.

Contents

1. Introduction

Ramanujan's general theta-function [15] $f(a,b)$ is defined by

$$\begin{aligned} f(a,b) :&= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \\ &= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \end{aligned} \tag{1.1}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

On page 338 in his first notebook [15], [3] Ramanujan defines

$$a_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(e^{-\pi\sqrt{mn}}) \varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}}) \varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.5}$$

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He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [4]. M. S. Mahadeva Naika and B. N. Dharmendra [8], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values from it. Further results on $a_{m,n}$ can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [9], and Mahadeva Naika and M. C. Mahesh Kumar [10]. Recently Nipen Saikia [13] established new properties of $a_{m,n}$.

In [12], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [11] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1, \tag{1.7}$$

and found some particular values of $V(q)$.

Recently in [5] Dharmendra, has defined the product of theta-fuctions $d_{m,n}$ as

$$d_{m,n} = \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{m}}}\right)\varphi\left(e^{-\pi\sqrt{mn}}\right)}{e^{-\frac{(m-1)\pi}{12}\sqrt{\frac{n}{m}}} f\left(-e^{-2\pi\sqrt{mn}}\right)\varphi\left(e^{-\pi\sqrt{\frac{n}{m}}}\right)}, \tag{1.8}$$

where m and n are positive real numbers. They established several properties of the product $d_{m,n}$ and proved general formulas for explicit evaluations of $d_{m,n}$ and find its explicit values.

In [6] Dharmendra and S. Vasanth Kumar, have defined the product of theta-fuctions $E_{m,n}$ as

$$E_{m,n} = \frac{f(e^{-\pi\sqrt{\frac{n}{m}}})\psi(-e^{-\pi\sqrt{mn}})}{e^{\frac{-\pi(1-m)}{12}\sqrt{\frac{n}{m}}} f(e^{-\pi\sqrt{mn}})\psi(-e^{-\pi\sqrt{\frac{n}{m}}})}, \tag{1.9}$$

where m and n are positive real numbers. They established several properties of the product of $E_{m,n}$. They proved general formulas for explicit evaluations of $E_{m,n}$ and find its explicit values.

Motivated by above, we define

$$\mu_{m,n} = \frac{f\left(e^{-\pi\sqrt{\frac{n}{m}}}\right) f\left(-e^{-2\pi\sqrt{mn}}\right)}{e^{-\frac{(1-m)\pi}{24}\sqrt{\frac{n}{m}}} f\left(e^{-\pi\sqrt{mn}}\right) f\left(-e^{-2\pi\sqrt{\frac{n}{m}}}\right)}, \tag{1.10}$$

where m and n are positive real numbers. We establish several properties of the product $\mu_{m,n}$ and prove general formulas for explicit evaluations of $\mu_{m,n}$ and find its explicit values.

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.11}$$

Then a modular equation of degree n is a relation between k and l induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.
Define

$$\chi(q) := (-q;q^2)_\infty,$$

and

$$G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q),$$

where

$$q = e^{-\pi\sqrt{r}}.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{n}{m}}}$ and β has degree n over α, then

$$G_{\frac{n}{m}} = (4\alpha(1-\alpha))^{\frac{-1}{24}} \tag{1.12}$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{-1}{24}}. \tag{1.13}$$

The main purpose of this paper is to obtain several properties of the product $\mu_{m,n}$ and several general theorems for the explicit evaluations of analogous of Ramanujan's product of theta-function of $\mu_{m,n}$ and also some new explicit evaluations from it.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1 *[1, Ch. 17, Entry 10(i) and Entry 11(iii), pp. 122 and 124] We have,*

$$2^{1/6}\, e^{-\alpha/24}\, f(e^{-\alpha}) = \sqrt{z_1}\, \{\alpha(1-\alpha)\}^{1/24}, \tag{2.1}$$

$$2^{1/6}\, e^{-m\alpha/24}\, f(e^{-m\alpha}) = \sqrt{z_m}\, \{\beta(1-\beta)\}^{1/24}, \tag{2.2}$$

$$2^{1/3}\, e^{-\alpha/12}\, f(-e^{-2\alpha}) = \sqrt{z_1}\, \{\alpha(1-\alpha)\}^{1/12}, \tag{2.3}$$

$$2^{1/3}\, e^{-m\alpha/12}\, f(-e^{-2m\alpha}) = \sqrt{z_m}\, \{\beta(1-\beta)\}^{1/12}. \tag{2.4}$$

Lemma 2.2 *[1, Ch. 16, Entry 27 (i) and (iii), pp. 43] We have,*

$$e^{-\alpha/24}\, \sqrt[4]{\alpha}\, f(e^{-\alpha}) = e^{-\beta/24}\, \sqrt[4]{\beta}\, f(e^{-\beta}), \quad if\ \alpha\beta = \pi^2, \tag{2.5}$$

$$e^{-\alpha/12}\, \sqrt[4]{\alpha}\, f(-e^{-2\alpha}) = e^{-\beta/12}\, \sqrt[4]{\beta}\, f(-e^{-2\beta}), \quad if\ \alpha\beta = \pi^2. \tag{2.6}$$

Lemma 2.3 *[1, Ch. 19, Entry 5(xii), pp. 231] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$, *then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left(\frac{1}{P} - P\right). \tag{2.7}$$

Lemma 2.4 *[1, Ch. 19, Entry 13(xiv), pp. 282] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ *and* $Q := \left\{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$, *then*

$$Q + \frac{1}{Q} = 2\left(\frac{1}{P} - P\right). \tag{2.8}$$

Lemma 2.5 *[1, Ch. 19, Entry 19(ix), pp. 315] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/6}$, *then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right). \tag{2.9}$$

3. Some Properties of $\mu_{m,n}$

In this section, we establish some properties of the product $\mu_{m,n}$.

Theorem 3.1

$$\mu_{m,n} = \mu_{n,m}. \tag{3.1}$$

Proof: Substituting the equations (2.5) and (2.6) in (1.10), we obtain (3.1). □

Theorem 3.2

$$\mu_{m,n}\mu_{m,1/n} = 1. \tag{3.2}$$

Proof: Using the equations (2.5) and (2.6) in (1.8), we obtain (3.2). □

Corollary 3.1

$$\mu_{m,1} = 1. \tag{3.3}$$

Proof: Substituting $n = 1$ in the equation (3.2), we get (3.3). □

Remark 3.1 By using the definition of $f(q)$, $f(-q^2)$ and $\mu_{m,n}$, it can be seen that $\mu_{m,n}$ has positive real value and that the values of $\mu_{m,n}$ increases as n increase when $m > 1$. Thus by the above corollary, $\mu_{m,n} > 1$ for all $n > 1$ if $m > 1$.

Theorem 3.3

$$\frac{\mu_{km,n}}{\mu_{nm,k}} = \mu_{m,\frac{n}{k}}. \tag{3.4}$$

Proof: Employing the definition of $\mu_{m,n}$, we obtain

$$\frac{\mu_{km,n}}{\mu_{nm,k}} = e^{\frac{\pi\left(\sqrt{\frac{k}{mn}}-\sqrt{\frac{n}{mk}}\right)}{12}} \frac{f\left(e^{-\pi\sqrt{\frac{n}{mk}}}\right) f\left(-e^{-2\pi\sqrt{\frac{k}{mn}}}\right)}{f\left(e^{-\pi\sqrt{\frac{k}{mn}}}\right) f\left(-e^{-2\pi\sqrt{\frac{n}{mk}}}\right)}. \tag{3.5}$$

Using the Lemma **2.2** in the above equation (3.5) and simplifying using the Theorems **3.1** and **3.2**, we obtain (3.4). □

Corollary 3.2

$$\mu_{m^2,n} = \mu_{nm,n}\mu_{m,\frac{n}{m}}. \tag{3.6}$$

Proof: Substituting $n = m$ in the above Theorem **3.3** and simplifying using the equation (3.2), we get

$$\mu_{m^2,k} = \mu_{mk,m}\mu_{m,\frac{k}{m}}. \tag{3.7}$$

Replace k by n, we obtain (3.6). □

Theorem 3.4 *If* $mn = rs$,

$$\mu_{m,n}\mu_{kr,ks} = \mu_{r,s}\mu_{km,kn}. \tag{3.8}$$

Proof: Using the definition of $\mu_{m,n}$ and letting $mn = rs$ for positive real numbers m, n, r, s and k, we find that

$$\frac{\mu_{km,kn}}{\mu_{m,n}} = \frac{\mu_{kr,ks}}{\mu_{r,s}}. \tag{3.9}$$

On rearranging the above equation (3.9), we obtain the required result. □

Corollary 3.3 *If* $mn = rs$,

$$\mu_{np,np} = \mu_{np^2,n}\mu_{p,p}. \tag{3.10}$$

Proof: Letting $m = p^2$, $n = 1$, $r = s = p$ and $k = n$ in above Theorem **3.4**, we deduced the equation (3.10). □

Theorem 3.5 *For all positive real numbers* m, n, r *and* s, *then*

$$\mu_{m/n,r/s} = \frac{\mu_{ms,nr}}{\mu_{mr,ns}}. \tag{3.11}$$

Proof: Employing the equation (3.2) in equation (3.4), we find that, for all positive real numbers m, n and k

$$\mu_{m/n,k} = \mu_{m,nk}\mu_{n,mk}^{-1}. \tag{3.12}$$

Letting $k = r/s$ and again using the equation (3.4) and (3.1) in (3.12), we get (3.11). □

Theorem 3.6

$$\mu_{m/n,m/n} = \mu_{n,n}\mu_{m,m/n^2}. \tag{3.13}$$

Proof: Using the Theorems **3.2** and **3.5**, we get (3.13). □

Theorem 3.7

$$\mu_{m,m}\mu_{m,n^2/m} = \mu_{n,n}\mu_{m,n^2/m}. \tag{3.14}$$

Proof: Substituting $k = m/n$ in the equation (3.12) and employing Theorems **3.2** and **3.6**, we obtain (3.14). □

Theorem 3.8

$$\mu_{m,m}\mu_{n,m^2n} = \mu_{n,n}\mu_{m,mn^2}. \tag{3.15}$$

Proof: Employing the Theorems **3.1, 3.2, 3.6** and **3.7**, we obtain (3.15). □

4. Some General Theorems on $\mu_{m,n}$ and their explicit evaluations

In this section, we established some general theorems on $\mu_{m,n}$ and their explicit evaluations.

Theorem 4.1 *If* $P := \{G_{n/3}G_{3n}\}^{-3}$ *and* $Q := \mu_{3,n}^6$, *then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left\{\frac{1}{P} - P\right\}. \tag{4.1}$$

Proof: Using the Lemma **2.1** with the definition of $\mu_{m,n}$, we obtain

$$\mu_{m,n} = \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}. \tag{4.2}$$

Employing the above equation (4.2) and definition of class invariant (1.12), (1.13) in the Lemma (2.3) with $m = 3$, we obtain (4.1). □

Corollary 4.1

$$\mu_{3,3} = \left\{2 - \sqrt{3}\right\}^{1/6} \tag{4.3}$$

Proof: Substituting $n = 3$ in the above Theorem **4.1**, we obtain

$$\mu_{3,3}^{6} + \mu_{3,3}^{-6} = 2\sqrt{2}\left\{G_1^3 G_9^3 - G_1^{-3} G_9^{-3}\right\}. \tag{4.4}$$

Solving the above equation (4.4) with from the table of Chapter 34 of Ramanujan notebooks [3, p.189] $G_1 = 1$ and $G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}$, we obtain (4.3). □

Corollary 4.2

$$\mu_{3,9} = \left\{1 + 2^{2/3} - 2^{4/3}\right\}^{1/6} \tag{4.5}$$

Proof: Substituting $n = 9$ in the above Theorem **4.1**, we obtain

$$\mu_{3,9}^{6} + \mu_{3,9}^{-6} = 2\sqrt{2}\left\{G_3^3 G_{27}^3 - G_3^{-3} G_{27}^{-3}\right\}. \tag{4.6}$$

Solving the above equation (4.6) with from the table of Chapter 34 of Ramanujan notebooks [3, p.189,190] $G_3 = 2^{1/12}$ and $G_{27} = 2^{1/12}\left(\sqrt[3]{2} - 1\right)^{-1/3}$, we obtain (4.5). □

Theorem 4.2 *If* $P := \{G_{n/5} G_{5n}\}^{-2}$ *and* $Q := \mu_{5,n}^3$, *then*

$$Q + \frac{1}{Q} = 2\left\{\frac{1}{P} - P\right\}. \tag{4.7}$$

Proof: Using the equation (4.2) and definition of class invariant (1.12), (1.13) in the Lemma **2.7** with $m = 5$, we obtain (4.7). □

Corollary 4.3

$$\mu_{5,5} = \left\{\sqrt{5} - 2\right\}^{1/3}. \tag{4.8}$$

Proof: Substituting $n = 5$ in the above Theorem **4.2**, we obtain

$$\mu_{5,5}^{3} + \mu_{5,5}^{-3} = 2\left\{G_1^2 G_{25}^2 - G_1^{-2} G_{25}^{-2}\right\}. \tag{4.9}$$

Solving the above equation (4.9) with from the table of Chapter 34 of Ramanujan notebooks [3, p.189] $G_1 = 1$ and $G_{25} = \frac{1+\sqrt{5}}{2}$, we obtain (4.8). □

Theorem 4.3 *If* $P := \{G_{n/7} G_{7n}\}^{-3}$ *and* $Q := \mu_{7,n}^4$, *then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left\{P + \frac{1}{P}\right\}. \tag{4.10}$$

Proof: Using the equation (4.2) and definition of class invariant (1.12), (1.13) in the Lemma **2.8** with $m = 7$, we obtain (4.10). □

Corollary 4.4

$$\mu_{7,7} = \left\{\frac{\sqrt{2}\left(4\sqrt{7} + 9\right) - 3\left(7^{3/4} + 3\,7^{1/4}\right)}{2\sqrt{2}}\right\}^{1/4}. \tag{4.11}$$

Proof: Substituting $n = 7$ in the above Theorem **4.3**, we obtain

$$\mu_{7,7}^4 + \mu_{7,7}^{-4} + 7 = 2\sqrt{2}\left\{G_1^3 G_{49}^3 + G_1^{-3} G_{49}^{-3}\right\}. \tag{4.12}$$

Solving the above equation (4.12) with from the table of Chapter 34 of Ramanujan notebooks [3, p.189,191] $G_1 = 1$ and $G_{49} = \dfrac{7^{1/4} + \sqrt{4+\sqrt{7}}}{2}$, we obtain (4.11). □

Theorem 4.4

$$\mu_{m,n} = \left\{\frac{G_{n/m}}{G_{mn}}\right\}. \tag{4.13}$$

Proof: Employing the Lemma **2.1** in the definition of $\mu_{m,n}$, we obtain

$$\mu_{m,n} = \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}. \tag{4.14}$$

Using the equation (1.12) and (1.13), we get

$$\frac{G_{nm}}{G_{n/m}} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{4.15}$$

By observing the equations (4.14) and (4.15), we obtain (4.13). □

Corollary 4.5

$$\mu_{n,n} = \frac{1}{G_{n^2}}. \tag{4.16}$$

Proof: Setting $m = n$ in the above Theorem **4.4** with the value $G_1 = 1$, we obtain required result. □

Corollary 4.6

$$(i)\ \mu_{3,3} = \left\{2-\sqrt{3}\right\}^{1/6}, \tag{4.17}$$

$$(ii)\ \mu_{5,5} = \left\{\sqrt{5}-2\right\}^{1/3}. \tag{4.18}$$

Proof: For $(i) - (ii)$, we use corresponding values of G_n from [1, p.189-193]. □

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