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$\mathscr{T}$-Commuting Generalized Derivations on Ideals and Semi-Prime Ideal

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ABSTRACT: This paper's primary goal is to look at a quotient ring $\mathscr{A}/\mathscr{T}$ structure, where $\mathscr{A}$ is an arbitrary ring and $\mathscr{T}$ is a semi-prime ideal of $\mathscr{A}$. More precisely, we examine the differential identities in a semi-prime ideal of an arbitrary ring involving $\mathscr{T}$-commuting generalized derivation. Furthermore, examples are given to prove that the restrictions imposed on the hypothesis of the various theorems were not superfluous.

Key Words: Semi-prime ideal, generalized derivations, commutativity.

Contents

1. Introduction

Throughout this paper, the symbols $a \circ b$ and $[a, b]$, where $a, b \in \mathscr{A}$, stands for the anti-commutator $ab + ba$ and commutator $ab - ba$ respectively. An ideal $\mathscr{T}$ is said to be a prime ideal of $\mathscr{A}$ if $\mathscr{T} \neq \mathscr{A}$ and for any $a, b \in \mathscr{A}$, whenever $a\mathscr{A}b \subseteq \mathscr{T}$ implies $a \in \mathscr{T}$ or $b \in \mathscr{T}$ and $\mathscr{A}$ is a prime ring if $\mathscr{T} = 0$. Also, $\mathscr{T}$ is a semi-prime ideal if $\mathscr{T} \neq \mathscr{A}$ and for any $a \in \mathscr{A}$, $a\mathscr{A}a \subseteq \mathscr{T}$ implies $a \in \mathscr{T}$ and $\mathscr{A}$ is a semi-prime ring if $\mathscr{T} = 0$. A ring $\mathscr{A}$ is said to be 2-torsion free if $2a = 0$, $a \in \mathscr{A}$ implies $a = 0$. A map $f : \mathscr{D} \to \mathscr{A}$ is called a $\mathscr{S}$-commuting map on $\mathscr{D}$ if $[f(a), a] \in \mathscr{S}$ $\forall$ $a \in \mathscr{D}$ and some $\mathscr{S} \subseteq \mathscr{A}$. In particular, if $\mathscr{S} = \{0\}$, then f is called a commuting map on $\mathscr{D}$ if $[f(a), a] = 0$. If can be note that every commuting map is a $\mathscr{S}$-commuting map (put $\{0\} = \mathscr{S}$), but converse is not true in general (take $\mathscr{S}$ some a set of $\mathscr{A}$ has no zero such that $[f(a), a] \in \mathscr{S}$, then f is a $\mathscr{S}$-commuting map but it is not a commuting map. An additive map $\psi : \mathscr{A} \to \mathscr{A}$ is called a derivation of $\mathscr{A}$ if $\psi(ab) = \psi(a)b + a\psi(b)$ holds $\forall$ $a, b \in \mathscr{A}$. An additive map $\mathscr{H} : \mathscr{A} \to \mathscr{A}$ associated with a derivation $\psi : \mathscr{A} \to \mathscr{A}$ is called a generalized derivation of $\mathscr{A}$ if $\mathscr{H}(ab) = \mathscr{H}(a)b + a\psi(b)$ holds $\forall$ $a, b \in \mathscr{A}$. Because generalized derivations have generally been investigated using operator algebras and any investigation from an algebraic approach could be interesting.

Many researchers have examined the commutativity of prime and semi-prime rings admitting sufficiently limited additive mappings acting on appropriate subsets of the rings over the last two decades. Also many of the obtained results go beyond those already established for the action of the considered mapping on the entire ring. This technique has led to the publication of numerous new results on commutativity in prime and semi-prime rings admitting constrained additive mappings, such as automorphisms, derivations, skew derivations, and generalized derivations acting on appropriate subsets of the rings. Posner [6] was the first to investigate centralizing derivation, proving that a prime ring R admitting a non-zero centralizing derivation is commutative. Moreover, Bell and Martindale [2] found that if $\mathscr{A}$ is a semi-prime ring and $\mathscr{I}$ is a non-zero left ideal of $\mathscr{A}$, then $\mathscr{A}$ has a non-zero central ideal if it admits a non-zero derivation ψ such that $\psi(\mathscr{I}) \neq \{0\}$ and centralizing on $\mathscr{I}$. Mayne [5] showed that centralizing automorphisms have an analogous result. Posner and Mayne's theorems have been extended in a variety of ways by a number of authors and can be seen in [4,7,13,10,11,12,8,9,3] etc.

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In this research, we will conduct a novel study that is both an extension and a generalization of recent findings in the literature. We will examine differential identities in a semi-prime ideal of an arbitrary ring involving generalized derivation.

2. Preliminaries

Lemma 2.1. *[2] Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $\mathscr{A}$ admits a non-zero derivation ψ such that $[a, \psi(a)] = 0\ \forall\ a \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.*

Lemma 2.2. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{T}$ 2-torsion free. If $a, b \in \mathscr{A}$ and $axb + bxa \in \mathscr{T}\ \forall\ x \in \mathscr{I}$, then axb and $bxa \in \mathscr{T}\ \forall\ x \in \mathscr{I}$.*

Proof. Assume that

$$axb + bxa \in \mathscr{T} \tag{2.1}$$

$\forall\ x \in \mathscr{I}$. Replacing x by xby in (2.1), we have

$$axbyb + bxbya \in \mathscr{T} \tag{2.2}$$

$\forall\ x, y \in \mathscr{I}$. Right multiplying (2.1) by yb and then comparing it with (2.2), we get

$$bx(ayb - bya) \in \mathscr{T} \tag{2.3}$$

$\forall\ x, y \in \mathscr{I}$. Putting $x = y$ in (2.1) and then left multiplying it by bx , this gives

$$bx(ayb + bya) \in \mathscr{T} \tag{2.4}$$

$\forall\ x, y \in \mathscr{I}$. Comparing (2.3) and (2.4), we see that $2bxayb \in \mathscr{T}$ and so $bxayb \in \mathscr{T}$. Right multiplying the last relation by xa, we conclude $bxaybxa \in \mathscr{T}$ and hence $bxa\mathscr{I}bxa \subseteq \mathscr{T}$. Thus, $bxa \in \mathscr{T}$ and by using the last relation in (2.1), we have $axb \in \mathscr{T}$. □

Lemma 2.3. *[1, Lemma 2.1] Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If $\mathscr{A}$ admits a derivation ψ such that $[a, \psi(a)] \in \mathscr{T}\ \forall\ a \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Lemma 2.4. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal, $\mathscr{A}/\mathscr{T}$ 2-torsion free and $\psi : \mathscr{A} \to \mathscr{A}$ derivation. Suppose that the mapping $[\psi^2(a), a] \in \mathscr{T}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{A}$.*

Proof. We have $[\psi^2(a), a] \in \mathscr{T}$. Putting $\mathscr{H}(a) = \psi^2(a)$, we have

$$[\mathscr{H}(a), a] \in \mathscr{T} \tag{2.5}$$

$\forall\ a \in \mathscr{A}$. By linearizing (2.5), we get

$$[\mathscr{H}(a), b] + [\mathscr{H}(b), a] \in \mathscr{T} \tag{2.6}$$

$\forall\ a, b \in \mathscr{A}$. Replacing b by ba in (2.6) and noting that $\mathscr{H}(ba) = \mathscr{H}(b)a + b\mathscr{H}(a) + 2\psi(b)\psi(a)$, to get

$$[\mathscr{H}(a), b]a + b[\mathscr{H}(a), a] + [\mathscr{H}(b), a]a + [b, a]\mathscr{H}(a)$$
$$+ b[\mathscr{H}(a), a] + 2[\psi(b), a]\psi(a) + 2\psi(b)[\psi(a), a] \in \mathscr{T}.$$

Which on using (2.5) and (2.6) in the last relation, one can see that

$$[b, a]\mathscr{H}(a) + 2[\psi(b), a]\psi(a) + 2\psi(b)[\psi(a), a] \in \mathscr{T} \tag{2.7}$$

$\forall\ a, b \in \mathscr{A}$. Replacing b by ab in (2.7), we conclude

$$a[b, a]\mathscr{H}(a) + 2[\psi(a), a]b\psi(a) + 2\psi(a)[b, a]\psi(a) + 2a[\psi(b), a]\psi(a) + 2\psi(a)b[\psi(a), a]$$
$$+ 2a\psi(b)[\psi(a), a] \in \mathscr{T}.$$

By using (2.7) in the last relation, we get

$$[\psi(a), a]b\psi(a) + \psi(a)[b, a]\psi(a) + \psi(a)b[\psi(a), a] \in \mathscr{T} \tag{2.8}$$

$\forall\ a, b \in \mathscr{A}$. Replacing b by $b\psi(a)z$ in (2.8) and using it, we obtain

$$\psi(a)b\psi(a)[z, a]\psi(a) + \psi(a)b\psi(a)z[\psi(a), a] \in \mathscr{T}.$$

Again, by (2.8), we see that $\psi(a)b[\psi(a), a]z\psi(a) \in \mathscr{T}$. Right multiplying the last relation by $b[\psi(a), a]$, this gives $\psi(a)b[\psi(a), a]z\psi(a)b[\psi(a), a] \in \mathscr{T}$ and so $\psi(a)b[\psi(a), a] \in \mathscr{T}$. Putting in the relation above first ab for b and then multiplying the relation above from the left side by a and then subtracting the relations so obtained one from another we obtain $[\psi(a), a]b[\psi(a), a] \in \mathscr{T}$ and so $[\psi(a), a] \in \mathscr{T}$. □

Lemma 2.5. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal, $\mathscr{A}/\mathscr{T}$ 2-torsion free and let $\psi : \mathscr{A} \to \mathscr{A}$ be a derivation satisfying $[[\psi(a), a], \psi(a)] \in \mathscr{T}\ \forall\ a \in \mathscr{A}$, then $[\psi(a), a] \in \mathscr{T}$.*

Proof. Consider a mapping $\mathscr{L}(.,.) : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ where

$$\mathscr{L}(a, b) = [\psi(a), b] + [\psi(b), a],$$

$\forall\ a \in \mathscr{A}$. Clearly, the mapping $\mathscr{L}(.,.)$ is symmetric (i.e., $\mathscr{L}(a, b) = \mathscr{L}(b, a)\ \forall\ a, b \in \mathscr{A}$ and additive. Moreover, clearly to shows that $\forall\ a, b, z \in \mathscr{A}$ the following relation holds

$$\mathscr{L}(ab, z) = \mathscr{L}(a, z)b + a\mathscr{L}(b, z) + \psi(a)[b, z] + [a, z]\psi(b). \tag{2.9}$$

$\forall\ a, b, z \in \mathscr{A}$. We will write $\lambda(a)$ for $\mathscr{L}(a, a)$. Then

$$\lambda(a) = 2[\psi(a), a] \tag{2.10}$$

$\forall\ a \in \mathscr{A}$. Clearly we see that

$$\lambda(a + b) = \lambda(a) + \lambda(b) + 2\mathscr{L}(a, b) \tag{2.11}$$

$\forall\ a, b \in \mathscr{A}$. Throughout the proof we will use the relations (2.9), (2.10) and (2.11) without specific reference. The assumption of the Lemma can now be written as follows

$$[\lambda(a), \psi(a)] \in \mathscr{T} \tag{2.12}$$

$\forall\ a \in \mathscr{A}$. We have

$$\begin{aligned}
&2[[\lambda(a), a], \psi(a)]\\
=&2[\lambda(a), a]\psi(a) + \lambda(a)^2 - \lambda(a)^2 - 2\psi(a)[\lambda(a), a]\\
=&2[\lambda(a), a]\psi(a) + 2\lambda(a)[\psi(a), a] - 2[\psi(a), a]\lambda(a) - 2\psi(a)[\lambda(a), a]\\
=&2[[\lambda(a), \psi(a)], a] \in \mathscr{T}.
\end{aligned}$$

That is,

$$[[\lambda(a), a], \psi(a)] \in \mathscr{T} \tag{2.13}$$

$\forall\ a \in \mathscr{A}$. By linearizing (2.12), this gives

$$[\lambda(b), \psi(a)] + 2[\mathscr{L}(a, b), \psi(a)] + [\lambda(a), \psi(b)] + 2[\mathscr{L}(a, b), \psi(b)]$$

$\forall\ a, b \in \mathscr{A}$. Replacing $-a$ for a and comparing the so obtained relation with the last relation we obtain

$$[\lambda(a), \psi(b)] + 2[\mathscr{L}(a, b), \psi(a)] \in \mathscr{T} \tag{2.14}$$

$\forall\ a, b \in \mathscr{A}$. Substituting bz for b in the last relation, gives

$$\begin{aligned}
&[\lambda(a), \psi(bz)] + 2[\mathscr{L}(bz, a), \psi(a)] \in \mathscr{T} \\
&= [\lambda(a), \psi(b)z + b\psi(z)] + 2[\mathscr{L}(b, a)z \\
&+ b\mathscr{L}(z, a) + \psi(b)[z, a] + [b, a]\psi(z), \psi(a)] \\
&= [\lambda(a), \psi(b)]z + \psi(b)[\lambda(a), z] + [\lambda(a), b]\psi(z) + b[\lambda(a), \psi(z)] \\
&+ 2[\mathscr{L}(b, a), \psi(a)]z + 2\mathscr{L}(b, a)[z, \psi(a)] \\
&+ 2[b, \psi(a)]\mathscr{L}(z, a) + 2b[\mathscr{L}(z, a), \psi(a)] \\
&+ 2[\psi(b), \psi(a)][z, a] + 2\psi(b)[[z, a], \psi(a)] \\
&+ 2[[b, a], \psi(a)]\psi(z) + 2[b, a][\psi(z), \psi(a)].
\end{aligned}$$

By using (2.14) in the last relation, we have

$$\begin{aligned}
&\psi(b)[\lambda(a), z] + [\lambda(a), b]\psi(z) + 2\mathscr{L}(b, a)[z, \psi(a)] \\
&+ 2[b, \psi(a)]\mathscr{L}(z, a) + 2[\psi(b), \psi(a)][z, a] + 2\psi(b)[[z, a], \psi(a)] \\
&+ 2[[b, a], \psi(a)]\psi(z) + 2[b, a][\psi(z), \psi(a)] \in \mathscr{T},
\end{aligned} \tag{2.15}$$

$\forall\ a, b \in \mathscr{A}$. Taking $b = a$ and $z = b$

$$\begin{aligned}
&\psi(a)[\lambda(a), b] + [\lambda(a), a]\psi(b) + 2\lambda(a)[b, \psi(a)] - \lambda(a)\mathscr{L}(b, a) \\
&+ 2\psi(a)[[b, a], \psi(a)] \in \mathscr{T},
\end{aligned} \tag{2.16}$$

$\forall\ a, b \in \mathscr{A}$. Substituting ba for b in (2.16), we get

$$\begin{aligned}
&\psi(a)[\lambda(a), ba] + [\lambda(a), a]\psi(ba) + 2\lambda(a)[ba, \psi(a)] \\
&- \lambda(a)\mathscr{L}(ba, a) + 2\psi(a)[[b, a]a, \psi(a)] \in \mathscr{T} \\
&= \psi(a)[\lambda(a), b]a + \psi(a)b[\lambda(a), a] + [\lambda(a), a]\psi(b)a \\
&+ [\lambda(a), a]b\psi(a) + 2\lambda(a)[b, \psi(a)]a + 2\lambda(a)b[a, \psi(a)] \\
&- \lambda(a)\mathscr{L}(b, a)a - \lambda(a)b\lambda(a) - \lambda(a)[b, a]\psi(a) \\
&+ 2\psi(a)[[b, a], \psi(a)]a + 2\psi(a)[b, a][a, \psi(a)],
\end{aligned}$$

$\forall\ a, b \in \mathscr{A}$. By using (2.16) in the last relation, we obtain

$$\begin{aligned}
&\psi(a)b[\lambda(a), a] + [\lambda(a), a]b\psi(a) - 2\lambda(a)b\lambda(a) \\
&- \lambda(a)[b, a]\psi(a) - \psi(a)[b, a]\lambda(a) \in \mathscr{T},
\end{aligned} \tag{2.17}$$

$\forall\ a, b \in \mathscr{A}$. Writing $\psi(a)b$ instead of b, this gives

$$\begin{aligned}
&\psi(a)^2b[\lambda(a), a] + [\lambda(a), a]\psi(a)b\psi(a) - 2\lambda(a)\psi(a)b\lambda(a) \\
&- \lambda(a)[\psi(a)b, a]\psi(a) - \psi(a)[\psi(a)b, a]\lambda(a) \in \mathscr{T} \\
&= \psi(a)^2b[\lambda(a), a] + [\lambda(a), a]\psi(a)b\psi(a) - 2\lambda(a)\psi(a)b\lambda(a) \\
&- \lambda(a)[\psi(a), a]b\psi(a) - \lambda(a)\psi(a)[b, a]\psi(a) \\
&- \psi(a)[\psi(a), a]b\lambda(a) - \psi(a)^2[b, a]\lambda(a).
\end{aligned}$$

$\forall\ a, b \in \mathscr{A}$. That is

$$\begin{aligned}
&\psi(a)^2b[\lambda(a), a] + [\lambda(a), a]\psi(a)b\psi(a) \\
&- 2\lambda(a)\psi(a)b\lambda(a) - \lambda(a)[\psi(a), a]b\psi(a) \\
&- \lambda(a)\psi(a)[b, a]\psi(a) - \psi(a)[\psi(a), a]b\lambda(a) \\
&- \psi(a)^2[b, a]\lambda(a) \in \mathscr{T},
\end{aligned} \tag{2.18}$$

$\forall\ a, b \in \mathscr{A}$. Left multiplying (2.17) by $\psi(a)$, we see that

$$\begin{aligned} &\psi(a)^2 b[\lambda(a), a] + \psi(a)[\lambda(a), a]b\psi(a) \\ &- 2\psi(a)\lambda(a)b\lambda(a) - \psi(a)\lambda(a)[b, a]\psi(a) \\ &- \psi(a)^2[b, a]\lambda(a) \in \mathscr{T}, \end{aligned} \tag{2.19}$$

$\forall\ a, b \in \mathscr{A}$. Subtracting (2.19) from (2.18) we obtain

$$\begin{aligned} &[[\lambda(a), a], \psi(a)]b\psi(a) - 2[\lambda(a), \psi(a)]b\lambda(a) \\ &- \lambda(a)[\psi(a), a]b\psi(a) - [\lambda(a), \psi(a)][b, a]\psi(a) \\ &- \psi(a)[\psi(a), a]b\lambda(a) \in \mathscr{T}, \end{aligned}$$

$\forall\ a, b \in \mathscr{A}$, and by using (2.12) and (2.13) in the last relation, we conclude

$$\lambda(a)^2 b\psi(a) + \psi(a)\lambda(a)b\lambda(a) \in \mathscr{T} \tag{2.20}$$

$\forall\ a, b \in \mathscr{A}$. Right multiplying (2.20) by $\lambda(a)$, this gives

$$\lambda(a)^2 b\psi(a)\lambda(a) + \psi(a)\lambda(a)b\lambda(a)^2 \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$, by using Lemma 2.2, we get

$$\lambda(a)^2 b\psi(a)\lambda(a) \in \mathscr{T} \tag{2.21}$$

$\forall\ a, b \in \mathscr{A}$. Substituting $b\psi(a)$ for b in (2.20) we get

$$\lambda(a)^2 b\psi(a)^2 + \psi(a)\lambda(a)b\psi(a)\lambda(a) \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$. Replacing b by $b\psi(a)\lambda(a)z\psi(a)\lambda(a)b$ in the last relation, we have

$$\begin{aligned} &\lambda(a)^2 b\psi(a)\lambda(a)z\psi(a)\lambda(a)b\psi(a)^2 \\ &+ \psi(a)\lambda(a)b\psi(a)\lambda(a)z\psi(a)\lambda(a)b\psi(a)\lambda(a) \in \mathscr{T} \end{aligned}$$

$\forall\ a, b, z \in \mathscr{A}$, by using (2.21) in the last relation, we obtain

$$\psi(a)\lambda(a)b\psi(a)\lambda(a)z\psi(a)\lambda(a)b\psi(a)\lambda(a) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$, and by semi-primeness of $\mathscr{T}$, we have

$$\psi(a)\lambda(a) \in \mathscr{T} \tag{2.22}$$

$\forall\ a \in \mathscr{A}$. Now, we have from (2.12)

$$\lambda(a)\psi(a) \in \mathscr{T} \tag{2.23}$$

$\forall\ a \in \mathscr{A}$. By linearizing the last relation and then replacing a by $-a$, we see that

$$\lambda(a)\psi(b) + 2\mathscr{L}(a, b)\psi(a) \in \mathscr{T} \tag{2.24}$$

$\forall\ a, b \in \mathscr{A}$. Writing bz instead of b in the last relation, we get

$$\begin{aligned} &\lambda(a)\psi(bz) + 2\mathscr{L}(a, bz)\psi(a) \in \mathscr{T} \\ &= \lambda(a)\psi(b)z + \lambda(a)b\psi(z) + 2\mathscr{L}(a, b)z\psi(a) \\ &+ 2b\mathscr{L}(a, z)\psi(a) + 2[b, a]\psi(z)\psi(a) + 2\psi(b)[z, a]\psi(a) \end{aligned}$$

$\forall\ a, b, z \in \mathscr{A}$. By (2.24) we can replace in the last relation $2b\mathscr{L}(a,z)\psi(a)$ by $-b\lambda(a)\psi(z)$ and $\lambda(a)\psi(b)z$ by $-2\mathscr{L}(a,b)\psi(a)z$ this gives

$$[\lambda(a), b]\psi(z) + 2\mathscr{L}(a,b)[z,\psi(a)] + 2[b,a]\psi(z)\psi(a) + 2\psi(b)[z,a]\psi(a) \in \mathscr{T} \tag{2.25}$$

$\forall\ a, b, z \in \mathscr{A}$. Putting $z = \psi(a)$ in (2.25) and by using (2.23) in (2.25), we obtain

$$[\lambda(a), b]\psi^2(a) + 2[b,a]\psi^2(a)\psi(a) \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$. Substituting ab for b, this gives

$$[\lambda(a), a]b\psi^2(a) \in \mathscr{T} \tag{2.26}$$

$\forall\ a, b \in \mathscr{A}$. Taking $z = a$ in (2.25) and using (2.23), we have

$$\lambda(a)b\psi(a) - \mathscr{L}(a,b)\lambda(a) + 2[b,a]\psi(a)^2 \in \mathscr{T} \tag{2.27}$$

$\forall\ a, b \in \mathscr{A}$. Putting ba for b in the last relation, we obtain

$$\begin{aligned}&\lambda(a)ba\psi(a) - \mathscr{L}(a,ba)\lambda(a) + 2[ba,a]\psi(a)^2 \in \mathscr{T}\\ &= \lambda(a)ba\psi(a) - \mathscr{L}(a,b)a\lambda(a) - b\lambda(a)^2 - [b,a]\psi(a)\lambda(a)\\ &+ 2[b,a]a\psi(a)^2\end{aligned}$$

$\forall\ a, b \in \mathscr{A}$, by using (2.23) in the last relation, we get

$$\lambda(a)ba\psi(a) - \mathscr{L}(a,b)a\lambda(a) - b\lambda(a)^2 + 2[b,a]a\psi(a)^2 \in \mathscr{T} \tag{2.28}$$

$\forall\ a, b \in \mathscr{A}$. Right multiplying (2.27) by a, this gives

$$\lambda(a)b\psi(a)a - \mathscr{L}(a,b)\lambda(a)a + 2[b,a]\psi(a)^2 a \in \mathscr{T} \tag{2.29}$$

$\forall\ a, b \in \mathscr{A}$. Subtracting (2.28) from (2.29) we obtain

$$\begin{aligned}&\lambda(a)b[\psi(a), a] - \mathscr{L}(a,b)[\lambda(a), a] + b\lambda(a)^2\\ &+ 2[b,a]([\psi(a), a]\psi(a) + \psi(a)[\psi(a), a]) \in \mathscr{T}\end{aligned}$$

$\forall\ a, b \in \mathscr{A}$, by using (2.22) and (2.23) in the last relation, we conclude

$$\lambda(a)b[\psi(a), a] - \mathscr{L}(a,b)[\lambda(a), a] + b\lambda(a)^2 \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$. Right multiplying the above relation by $z\psi^2(a)$ and according to (2.26), gives

$$\lambda(a)b\lambda(a)z\psi^2(a) + 2b\lambda(a)^2 z\psi^2(a) \in \mathscr{T} \tag{2.30}$$

$\forall\ a, b, z \in \mathscr{A}$. Left multiplying (2.30) by $\psi(a)$ and using (2.22), we see that

$$2\psi(a)b\lambda(a)^2 z\psi^2(a) \in \mathscr{T} \tag{2.31}$$

$\forall\ a, b, z \in \mathscr{A}$. Putting ab for b in (2.31) then left multiplying (2.31) by a and subtracting the relations so obtained one from another we obtain

$$\lambda(a)b\lambda(a)^2 z\psi^2(a) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$. Replacing b by $\lambda(a)z\psi^2(a)b$ in the last relation, this gives

$$(\lambda(a)^2 z\psi^2(a))b(\lambda(a)^2 z\psi^2(a)) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$, and hence

$$\lambda(a)^2 z\psi^2(a) \in \mathscr{T}$$

$\forall\ a, z \in \mathscr{A}$, by using the last relation in (2.30), we have

$$\lambda(a)b\lambda(a)z\psi^2(a) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$. Substituting $z\psi^2(a)b$ for b in the last relation, this gives

$$(\lambda(a)z\psi^2(a))b(\lambda(a)z\psi^2(a)) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$. Thus,

$$\lambda(a)b\psi^2(a) \in \mathscr{T} \tag{2.32}$$

$\forall\ a, b \in \mathscr{A}$. By linearizing the last relation and then replacing a by $-a$, we get

$$([\psi(a), z] + [\psi(z), a])b\psi^2(a) + [\psi(a), a]b\psi^2(z) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$. Putting $z = \psi(a)$ in the last relation, we obtain

$$[\psi^2(a), a]b\psi^2(a) + [\psi(a), a]b\psi^3(a) \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$. Replacing b by $z\psi^2(a)b$ in above expression and using (2.32), we obtain

$$[\psi^2(a), a]z\psi^2(a)b\psi^2(a) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$. Substituting $b[\psi^2(a), a]z$ for b in the last relation, this gives

$$([\psi^2(a), a]z\psi^2(a))b([\psi^2(a), a])z\psi^2(a)) \in \mathscr{T}$$

$\forall\ a, b, z \in \mathscr{A}$, and hence

$$[\psi^2(a), a]b\psi^2(a) \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$, and so

$$[\psi^2(a), a]b[\psi^2(a), a] \in \mathscr{T}$$

$\forall\ a, b \in \mathscr{A}$. Thus,

$$[\psi^2(a), a] \in \mathscr{T}$$

$\forall\ a \in \mathscr{A}$. By using Lemma 2.4 we conclude $[\psi(a), a] \in \mathscr{T}$. □

We'll use the following remark without explicitly mentioning it in our proofs.

Remark 2.6. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. Suppose that $s \in \mathscr{I}$ such that $s\mathscr{I}s \subseteq \mathscr{T}$. Then $s \in \mathscr{T}$. In particular, if $s \in \mathscr{I}$ such that $s\mathscr{I} \subseteq \mathscr{T}$ or $\mathscr{I}s \subseteq \mathscr{T}$, then $s \in \mathscr{T}$.*

Proof. Assume that $s \in \mathscr{I}$ such that $s\mathscr{I}s \subseteq \mathscr{T}$. That is, $sts \in \mathscr{T}\ \forall\ t \in \mathscr{I}$. Replacing t by asa in the last relation, where $a \in \mathscr{A}$, we have $sasas \in \mathscr{T}\ \forall\ a \in \mathscr{A}$. That is, $s\mathscr{A}s\mathscr{A}s \subseteq \mathscr{T}$. By semi-primeness of $\mathscr{T}$, we obtain $s \in \mathscr{T}$, as desired.

Now, if $s \in \mathscr{I}$ such that $s\mathscr{I} \subseteq \mathscr{T}$ or $\mathscr{I}s \subseteq \mathscr{T}$, then $s\mathscr{I}s \subseteq \mathscr{T}$, and so the same as in the above, we get $s \in \mathscr{T}$, as desired. □

3. The Main Results

Theorem 3.1. *Let $\mathscr{A}$ be a ring, $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $(\mathscr{H}, \psi)$, $(\mathscr{G}, \varphi)$ are generalized derivations of $\mathscr{A}$ satisfying one of the following conditions*

1. $\mathscr{H}(a)a \pm a\mathscr{G}(a) \in \mathscr{T}\ \forall\ a \in \mathscr{I}$
2. $[\mathscr{H}(a), b] \pm [a, \mathscr{G}(b)] \in \mathscr{T}\ \forall\ a, b \in \mathscr{I}$,

then φ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\mathscr{H}(a)a - a\mathscr{G}(a) \in \mathscr{T} \tag{3.1}$$

$\forall\ a \in \mathscr{I}$. Linearizing (3.1), we get

$$\mathscr{H}(a)b + \mathscr{H}(b)a - a\mathscr{G}(b) - b\mathscr{G}(a) \in \mathscr{T} \tag{3.2}$$

$\forall\ a, b \in \mathscr{I}$. Replacing ab for a in (3.2), we get

$$\mathscr{H}(a)b^2 + a\psi(b)b + \mathscr{H}(b)ab - ab\mathscr{G}(b) - b\mathscr{G}(a)b - ba\varphi(b) \in \mathscr{T} \tag{3.3}$$

$\forall\ a, b \in \mathscr{I}$. Right multiplying (3.2) by b and then comparing it with (3.3), we obtain

$$a\psi(b)b - ba\varphi(b) - a[b, \mathscr{G}(b)] \in \mathscr{T} \tag{3.4}$$

$\forall\ a, b \in \mathscr{I}$. Replacing a by ra in (3.4), where $r \in \mathscr{A}$, we see that

$$ra\psi(b)b - bra\varphi(b) - ra[b, \mathscr{G}(b)] \in \mathscr{T} \tag{3.5}$$

$\forall\ a, b \in \mathscr{I}$, $r \in \mathscr{A}$. Left multiplying (3.4) by r, we conclude

$$ra\psi(b)b - rba\varphi(b) - ra[b, \mathscr{G}(b)] \in \mathscr{T} \tag{3.6}$$

$\forall\ a, b \in \mathscr{I}$, $r \in \mathscr{A}$. From (3.5) and (3.6), we have $[b, r]a\varphi(b) \in \mathscr{T}$. Putting $r = \varphi(b)$ in the last relation, we get $[b, \varphi(b)]a\varphi(b) \in \mathscr{T}$. First taking a by ab and then a by ba, we obtain $[b, \varphi(b)]a[b, \varphi(b)] \in \mathscr{T}$ and so $[b, \varphi(b)] \in \mathscr{T}$. In case $\mathscr{H}(a)a + a\mathscr{G}(a) \in \mathscr{T}$ using the same approaches as in the above proof, we get the required result.

(2) Assume that

$$[\mathscr{H}(a), b] - [a, \mathscr{G}(b)] \in \mathscr{T} \tag{3.7}$$

$\forall\ a, b \in \mathscr{I}$. Replacing b by ba in (3.7), we have

$$b[\mathscr{H}(a), a] - [a, b]\varphi(a) - b[a, \varphi(a)] \in \mathscr{T} \tag{3.8}$$

$\forall\ a, b \in \mathscr{I}$. Again, replacing b by tb in (3.8) and using it, we get $[a, t]b\varphi(a) \in \mathscr{T}\ \forall\ a, b, t \in \mathscr{I}$. Replacing t by $\varphi(a)t$, we obtain $[a, \varphi(a)]tb\varphi(a) \in \mathscr{T}$ and so $[a, \varphi(a)]tb[a, \varphi(a)]t \in \mathscr{T}$ and hence $[a, \varphi(a)]t \in \mathscr{T}$. It follows that $[a, \varphi(a)]t[a, \varphi(a)] \in \mathscr{T}$. That is, $[a, \varphi(a)] \in \mathscr{T}$. In case $[\mathscr{H}(a), b] + [a, \mathscr{G}(b)] \in \mathscr{T}$ using the same approaches as in the above proof, we get the required result. □

By using Lemma 2.3 and Theorem 3.1, we easily get the following corollary:

Corollary 3.2. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If $(\mathscr{H}, \psi)$, $(\mathscr{G}, \varphi)$ are generalized derivations of $\mathscr{A}$ satisfying any one of the conditions*

1. $\mathscr{H}(a)a \pm a\mathscr{G}(a) \in \mathscr{T}\ \forall\ a \in \mathscr{A}$
2. $[\mathscr{H}(a), b] \pm [a, \mathscr{G}(b)] \in \mathscr{T}\ \forall\ a, b \in \mathscr{A}$,

then $\varphi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

According to Lemma 2.1 and Theorem 3.1, $\mathscr{A}$ has a non-zero central ideal. Therefore, we get the following result.

Corollary 3.3. *Let* $\mathscr{A}$ *be a semi-prime ring and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$*. If* $(\mathscr{H}, \psi)$*,* $(\mathscr{G}, \varphi)$ *are generalized derivations of* $\mathscr{A}$ *satisfying any one of the conditions*

1. $\mathscr{H}(a)a \pm a\mathscr{G}(a) = 0 \ \forall \ a \in \mathscr{I}$,

2. $[\mathscr{H}(a), b] = \pm[a, \mathscr{G}(b)] \ \forall \ a, b \in \mathscr{I}$,

then $\mathscr{A}$ *has a non-zero central ideal.*

Note that if we take $\mathscr{G} = \mathscr{H}$ or $\mathscr{G} = -\mathscr{H}$ in Theorem 3.1, with necessary variations, we can prove the following:

Theorem 3.4. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$*. If* $(\mathscr{H}, \psi)$ *is a generalized derivation of* $\mathscr{A}$ *satisfying any one of the conditions*

1. $[\mathscr{H}(a), a] \in \mathscr{T} \ \forall \ a \in \mathscr{I}$ *or* $\mathscr{H}(a) \circ a \in \mathscr{T} \ \forall \ a \in \mathscr{I}$,

2. $[\mathscr{H}(a), b] \pm [a, \mathscr{H}(b)] \in \mathscr{T} \ \forall \ a, b \in \mathscr{I}$,

then ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$*.*

Corollary 3.5. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a prime ideal of* $\mathscr{A}$*. If* $(\mathscr{H}, \psi)$ *is a generalized derivation of* $\mathscr{A}$ *satisfying any one of the conditions*

1. $[\mathscr{H}(a), a] \in \mathscr{T} \ \forall \ a \in \mathscr{A}$ *or* $\mathscr{H}(a) \circ a \in \mathscr{T} \ \forall \ a \in \mathscr{A}$,

2. $[\mathscr{H}(a), b] \pm [a, \mathscr{H}(b)] \in \mathscr{T} \ \forall \ a, b \in \mathscr{A}$,

then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ *or* $\mathscr{A}/\mathscr{T}$ *is commutative.*

Corollary 3.6. *Let* $\mathscr{A}$ *be a semi-prime ring and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$*. If* $(\mathscr{H}, \psi)$ *is a generalized derivation of* $\mathscr{A}$ *satisfying any one of the conditions*

1. $[\mathscr{H}(a), a] = 0 \ \forall \ a \in \mathscr{I}$ *or* $\mathscr{H}(a) \circ a = 0 \ \forall \ a \in \mathscr{I}$,

2. $[\mathscr{H}(a), b] = \pm[a, \mathscr{H}(b)] \ \forall \ a, b \in \mathscr{I}$,

then $\mathscr{A}$ *has a non-zero central ideal.*

Theorem 3.7. *Let* $\mathscr{A}$ *be a ring with* $\mathscr{T}$ *a semi-prime ideal and* $\mathscr{I}$ *an ideal of* $\mathscr{A}$ *and* $\mathscr{A}/\mathscr{T}$ 2*-torsion free. If* $(\mathscr{H}, \psi)$ *is a generalized derivation of* $\mathscr{A}$ *satisfying any one of the conditions*

1. $\mathscr{H}([a, b]) - [\mathscr{H}(a), b] - [\psi(b), a] \in \mathscr{T}$

2. $\mathscr{H}(a \circ b) - (\mathscr{H}(a) \circ b) + (\psi(b) \circ a) \in \mathscr{T}$

3. $[\psi(a), \mathscr{H}(b)] \pm [a, b] \in \mathscr{T}$

4. $[\psi(a), \mathscr{H}(b)] \pm (a \circ b) \in \mathscr{T}$

5. $[\psi(a), \mathscr{H}(b)] \in \mathscr{T}$

$\forall \ a \in \mathscr{I}$*, then* ψ *is* $\mathscr{T}$*-commuting on* $\mathscr{I}$*.*

Proof. (1) Assume that

$$\mathscr{H}([a,b]) - [\mathscr{H}(a), b] - [\psi(b), a] \in \mathscr{T} \tag{3.9}$$

$\forall\ a,b \in \mathscr{I}$. Replacing b by ba in (3.9) and using it, we have

$$2[a,b]\psi(a) - b[\mathscr{H}(a), a] - b[\psi(a), a] \in \mathscr{T} \tag{3.10}$$

$\forall\ a,b \in \mathscr{I}$. Again replacing b by bt in (3.10) and using it, we get $2[a,b]t\psi(a) \in \mathscr{T}$ and so $[a,b]t\psi(a) \in \mathscr{T}$ $\forall\ a,b,t \in \mathscr{I}$. Replacing b by $\psi(a)b$, we obtain

$$[a,\psi(a)]bt\psi(a) \in \mathscr{T} \tag{3.11}$$

$\forall\ a,b,t \in \mathscr{I}$. Replacing t by ta in (3.11), we see that

$$[a,\psi(a)]bta\psi(a) \in \mathscr{T} \tag{3.12}$$

$\forall\ a,b,t \in \mathscr{I}$. Right multiplying (3.11) by a gives

$$[a,\psi(a)]bt\psi(a)a \in \mathscr{T} \tag{3.13}$$

$\forall\ a,b,t \in \mathscr{I}$. Comparing (3.12) and (3.13), we conclude $[a,\psi(a)]bt[a,\psi(a)] \in \mathscr{T}$, and so

$$t[a,\psi(a)]bt[a,\psi(a)] \in \mathscr{T}.$$

Hence, $t[a,\psi(a)] \in \mathscr{T}$. Thus, $[a,\psi(a)]t[a,\psi(a)] \in \mathscr{T}$. This implies that $[a,\psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\mathscr{H}(a \circ b) - (\mathscr{H}(a) \circ b) + (\psi(b) \circ a) \in \mathscr{T} \tag{3.14}$$

$\forall\ a,b \in \mathscr{I}$. Replacing b by ba in (3.14) and using it, we have

$$(a \circ b)\psi(a) + b[\mathscr{H}(a), a] + b(\psi(a) \circ a) - [b,a]\psi(a) \in \mathscr{T} \tag{3.15}$$

$\forall\ a,b \in \mathscr{I}$. Replacing b by tb in (3.15) and using it, we get $2[a,t]b\psi(a) \in \mathscr{T}$ $\forall\ a,b,t \in \mathscr{I}$ and so $[a,t]b\psi(a) \in \mathscr{T}$. Replacing t by $\psi(a)t$, we get $[a,\psi(a)]tb\psi(a) \in \mathscr{T}$ $\forall\ a,b,t, \in \mathscr{I}$. Now, using the same approaches as in (3.11), we get $[a,\psi(a)] \in \mathscr{T}$.

(3) Assume that

$$[\psi(a), \mathscr{H}(b)] \pm [a,b] \in \mathscr{T} \tag{3.16}$$

$\forall\ a,b \in \mathscr{I}$. Replacing a by at in (3.16) and using it, we have

$$\psi(a)[t, \mathscr{H}(b)] + [a, \mathscr{H}(b)]\psi(t) \in \mathscr{T} \tag{3.17}$$

$\forall\ a,b,t \in \mathscr{I}$. Replacing a by ta in (3.17), we get

$$(\psi(t)a + t\psi(a))[t, \mathscr{H}(b)] + t[a, \mathscr{H}(b)]\psi(t) + [t, \mathscr{H}(b)]a\psi(t) \in \mathscr{T} \tag{3.18}$$

$\forall\ a,b,t \in \mathscr{I}$. Left multiplying (3.17) by t, we obtain

$$t\psi(a)[t, \mathscr{H}(b)] + t[a, \mathscr{H}(b)]\psi(t) \in \mathscr{T} \tag{3.19}$$

$\forall\ a,b,t \in \mathscr{I}$. Comparing (3.18) and (3.19), this gives

$$\psi(t)a[t, \mathscr{H}(b)] + [t, \mathscr{H}(b)]a\psi(t) \in \mathscr{T} \tag{3.20}$$

$\forall\ a,b,t \in \mathscr{I}$. By using Lemma 2.2, we get $\psi(t)a[t, \mathscr{H}(b)] \in \mathscr{T}$. Replacing b by bt in the last relation and using it, we have $\psi(t)a[t, b\psi(t)] \in \mathscr{T}$ and so $[t, b\psi(t)]a[t, b\psi(t)] \in \mathscr{T}$ and hence $[t, b\psi(t)] \in \mathscr{T}$. Replacing

b by $\psi(t)b$ in the last relation and using it, we obtain $[t,\psi(t)]b\psi(t)\in\mathscr{T}$ and so $[t,\psi(t)]b[t,\psi(t)]\in\mathscr{T}$ and hence $[t,\psi(t)]\in\mathscr{T}\ \forall\ t\in\mathscr{I}$.

(4) Assume that

$$[\psi(a),\mathscr{H}(b)]\pm(a\circ b)\in\mathscr{T} \tag{3.21}$$

$\forall\ a,b\in\mathscr{I}$. Replacing a by at in (3.21) and using it, we have

$$\psi(a)[t,\mathscr{H}(b)]+a[\psi(t),\mathscr{H}(b)]+[a,\mathscr{H}(b)]\psi(t)\pm a[t,b]\in\mathscr{T} \tag{3.22}$$

$\forall\ a,b,t\in\mathscr{I}$. Replacing a by ta in (3.22) and using it, we get

$$\begin{aligned}\psi(t)a[t,\mathscr{H}(b)]+t\psi(a)[t,\mathscr{H}(b)]+ta[\psi(t),\mathscr{H}(b)]+t[a,\mathscr{H}(b)]\psi(t)\\+[t,\mathscr{H}(b)]a\psi(t)\pm ta[t,b]\in\mathscr{T}\end{aligned} \tag{3.23}$$

$\forall\ a,b,t\in\mathscr{I}$. Left multiplying (3.22), we obtain

$$t\psi(a)[t,\mathscr{H}(b)]+ta[\psi(t),\mathscr{H}(b)]+t[a,\mathscr{H}(b)]\psi(t)\pm ta[t,b]\in\mathscr{T} \tag{3.24}$$

$\forall\ a,b,t\in\mathscr{I}$. Comparing (3.23) and (3.24), we see that

$$\psi(t)a[t,\mathscr{H}(b)]+[t,\mathscr{H}(b)]a\psi(t)\in\mathscr{T}.$$

Now, using the same approaches as in Eq.(3.20), we get $[t,\psi(t)]\in\mathscr{T}\ \forall\ t\in\mathscr{I}$.

(5) Using the same approaches as in the proof of (4), we get the required result. □

Corollary 3.8. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{T}$ 2-torsion free. If $(\mathscr{H},\psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the condition*

1. $\mathscr{H}([a,b])-[\mathscr{H}(a),b]-[\psi(b),a]\in\mathscr{T}$
2. $\mathscr{H}(a\circ b)-(\mathscr{H}(a)\circ b)+(\psi(b)\circ a)\in\mathscr{T}$
3. $[\psi(a),\mathscr{H}(b)]\pm[a,b]\in\mathscr{T}$
4. $[\psi(a),\mathscr{H}(b)]\pm(a\circ b)\in\mathscr{T}$
5. $[\psi(a),\mathscr{H}(b)]\in\mathscr{T}$

$\forall\ a\in\mathscr{A}$, then $\psi(\mathscr{A})\subseteq\mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 3.9. *Let $\mathscr{A}$ be a 2-torsion free semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $(\mathscr{H},\psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the condition*

1. $\mathscr{H}([a,b])=[\mathscr{H}(a),b]+[\psi(b),a]$
2. $\mathscr{H}(a\circ b)=(\mathscr{H}(a)\circ b)-(\psi(b)\circ a)$
3. $[\psi(a),\mathscr{H}(b)]=\pm[a,b]$
4. $[\psi(a),\mathscr{H}(b)]=\pm(a\circ b)$
5. $[\psi(a),\mathscr{H}(b)]=0$

$\forall\ a\in\mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 3.10. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $(\mathscr{H},\psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\mathscr{H}(a^2)\pm a^2\in\mathscr{T}$

2. $\psi(a)\mathscr{H}(b) \pm ab \in \mathscr{T}$

$\forall\ a \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.

Proof. (1) Assume that

$$\mathscr{H}(a^2) - a^2 \in \mathscr{T} \tag{3.25}$$

$\forall\ a \in \mathscr{I}$. By linearizing (3.25) and using it, we have

$$\mathscr{H}(a \circ b) - (a \circ b) \in \mathscr{T} \tag{3.26}$$

$\forall\ a, b \in \mathscr{I}$. Substituting ba for b in (3.26) and using it, we get

$$(a \circ b)\psi(a) \in \mathscr{T} \tag{3.27}$$

$\forall\ a, b \in \mathscr{I}$. Now, replace b by $\psi(a)b$ in (3.27), to get $\psi(a)(a \circ b)\psi(a) + [a, \psi(a)]b\psi(a) \in \mathscr{T}$ and by using (3.27) in the last relation, we see that $[a, \psi(a)]b\psi(a) \in \mathscr{T}$. Putting b by ba then b by ab in the last relation, we get $[a, \psi(a)]b[a, \psi(a)] \in \mathscr{T}$. Hence, $[a, \psi(a)] \in \mathscr{T}$. In case $\mathscr{H}(a^2) + a^2 \in \mathscr{T}$ using the same approaches as in the above proof, we get the required result.

(2) Assume that

$$\psi(a)\mathscr{H}(b) \pm ab \in \mathscr{T} \tag{3.28}$$

$\forall\ a, b \in \mathscr{I}$. Replacing b by ba in (3.28) and using it, we have $\psi(a)b\psi(a) \in \mathscr{T}$ and so $a\psi(a)b\psi(a) \in \mathscr{T}$ also $\psi(a)ab\psi(a) \in \mathscr{T}$ and hence $[\psi(a), a]b\psi(a) \in \mathscr{T}$ similarity, $[\psi(a), a]b[\psi(a), a] \in \mathscr{T}$. Thus, $[\psi(a), a] \in \mathscr{T}$. □

Corollary 3.11. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\mathscr{H}(a^2) \pm a^2 \in \mathscr{T}$

2. $\psi(a)\mathscr{H}(b) \pm ab \in \mathscr{T}$

$\forall\ a \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 3.12. *Let $\mathscr{A}$ be a semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\mathscr{H}(a^2) \pm a^2 = 0$

2. $\psi(a)\mathscr{H}(b) \pm ab = 0$

$\forall\ a \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 3.13. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal and $\mathscr{A}/\mathscr{T}$ 2-torsion free. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\psi(a) \circ \mathscr{H}(b) \pm (a \circ b) \in \mathscr{T}$

2. $\psi(a) \circ \mathscr{H}(b) \pm [a, b] \in \mathscr{T}$

3. $\psi(a) \circ \mathscr{H}(b) \in \mathscr{T}$

$\forall\ a, b \in \mathscr{A}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{A}$.

Proof. (1) Assume that

$$\psi(a) \circ \mathscr{H}(b) \pm (a \circ b) \in \mathscr{T} \tag{3.29}$$

$\forall\, a, b \in \mathscr{A}$. Replacing b by ba in (3.29) and using it, we have $-\mathscr{H}(b)[\psi(a), a] + (\psi(a) \circ b)\psi(a) \in \mathscr{T}$, that is,

$$-\mathscr{H}(b)[\psi(a), a] + \psi(a)b\psi(a) + b\psi(a)^2 \in \mathscr{T} \tag{3.30}$$

$\forall\, a, b \in \mathscr{A}$. Replacing b by $b[a, \psi(a)]$ in (3.30), we get

$$-\mathscr{H}(b)[a, \psi(a)][\psi(a), a] - b[a, \psi^2(a)][\psi(a), a] + \psi(a)b[a, \psi(a)]\psi(a) + b[a, \psi(a)]\psi(a)^2 \in \mathscr{T} \tag{3.31}$$

$\forall\, a, b \in \mathscr{A}$. Right multiplication of (3.30) by $[a, \psi(a)]$ gives

$$-\mathscr{H}(b)[\psi(a), a][a, \psi(a)] + \psi(a)b\psi(a)[a, \psi(a)] + b\psi(a)^2[a, \psi(a)] \in \mathscr{T} \tag{3.32}$$

$\forall\, a, b \in \mathscr{A}$. Subtracting (3.32) from (3.31), we obtain

$$b[a, \psi^2(a)][\psi(a), a] + \psi(a)b[\psi(a), [a, \psi(a)]] + b[\psi(a)^2, [a, \psi(a)]] \in \mathscr{T} \tag{3.33}$$

$\forall\, a, b \in \mathscr{A}$. Taking $b = ab$ in (3.33), we get

$$ab[a, \psi^2(a)][\psi(a), a] + \psi(a)ab[\psi(a), [a, \psi(a)]] + ab[\psi(a)^2, [a, \psi(a)]] \in \mathscr{T} \tag{3.34}$$

$\forall\, a, b \in \mathscr{A}$. Left multiplying (3.33) by a and then subtracting from (3.34), we get

$$[a, \psi(a)]b[\psi(a), [a, \psi(a)]] \in \mathscr{T} \tag{3.35}$$

$\forall\, a, b \in \mathscr{A}$. Putting b by $\psi(a)b$ in (3.35) and then left multiplying (3.35) by $\psi(a)$, we see that

$$[\psi(a), [a, \psi(a)]]b[\psi(a), [a, \psi(a)]] \in \mathscr{T}.$$

Hence, $[\psi(a), [a, \psi(a)]] \in \mathscr{T}$, and by Lemma 2.5, we get $[a, \psi(a)] \in \mathscr{T}$.

(2) Assume that

$$\psi(a) \circ \mathscr{H}(b) \pm [a, b] \in \mathscr{T} \tag{3.36}$$

$\forall\, a, b \in \mathscr{A}$. Replacing b by ba in (3.36) and using it, we have $-\mathscr{H}(b)[\psi(a), a] + (\psi(a) \circ b)\psi(a) \in \mathscr{T}$, that is,

$$-\mathscr{H}(b)[\psi(a), a] + \psi(a)b\psi(a) + b\psi(a)^2 \in \mathscr{T} \tag{3.37}$$

$\forall\, a, b \in \mathscr{A}$. Now, using the same approaches as in Eq.(3.30), we get $[a, \psi(a)] \in \mathscr{T}$.

(3) Using the same approaches as in the proof of (2),, we get the required result. □

Corollary 3.14. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{T}$ 2-torsion free. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\psi(a) \circ \mathscr{H}(b) \pm (a \circ b) \in \mathscr{T}$
2. $\psi(a) \circ \mathscr{H}(b) \pm [a, b] \in \mathscr{T}$
3. $\psi(a) \circ \mathscr{H}(b) \in \mathscr{T}$

$\forall\, a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.

Corollary 3.15. *Let $\mathscr{A}$ be a 2-torsion free semi-prime ring. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying any one of the conditions*

1. $\psi(a) \circ \mathscr{H}(b) = \pm(a \circ b)$
2. $\psi(a) \circ \mathscr{H}(b) = \pm[a, b]$
3. $\psi(a) \circ \mathscr{H}(b) = 0$

$\forall\ a, b \in \mathscr{A}$, then $\mathscr{A}$ has a non-zero central ideal.

Theorem 3.16. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semi-prime ideal, $\mathscr{I}$ an ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{T}$ 2-torsion free. Let $(\mathscr{H}, \psi)$ be a generalized derivation of $\mathscr{A}$ satisfying $\mathscr{H}(ab) = a\mathscr{H}(b) + \psi(a)b\ \forall\ a, b \in \mathscr{A}$. Further, if $\mathscr{H}([a, b]) \pm [\psi(a), \mathscr{H}(b)] \in \mathscr{T}\ \forall\ a, b \in \mathscr{I}$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.*

Proof. Assume that

$$\mathscr{H}([a, b]) - [\psi(a), \mathscr{H}(b)] \in \mathscr{T} \tag{3.38}$$

$\forall\ a, b \in \mathscr{I}$. Replacing b by ab in (3.38) and using it, we have

$$\psi(a)[a, b] - [\psi(a), a]\mathscr{H}(b) - \psi(a)[\psi(a), b] \in \mathscr{T} \tag{3.39}$$

$\forall\ a, b \in \mathscr{I}$. Again replace b by ba in (3.39) and using it, we get $[\psi(a), a]b\psi(a) + \psi(a)b[\psi(a), a] \in \mathscr{T}$ and by Lemma 2.2, $\psi(a)b[\psi(a), a] \in \mathscr{T}$ and so $[\psi(a), a]b[\psi(a), a] \in \mathscr{T}$ and hence $[\psi(a), a] \in \mathscr{T}$. In case $\mathscr{H}([a, b]) + [\psi(a), \mathscr{H}(b)] \in \mathscr{T}$ using the same approaches as in the above proof, we get the required result. □

Corollary 3.17. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a prime ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{T}$ 2-torsion free. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying $\mathscr{H}(ab) = a\mathscr{H}(b) + \psi(a)b\ \forall\ a, b \in \mathscr{A}$. Further, if $\mathscr{H}([a, b]) \pm [\psi(a), \mathscr{H}(b)] \in \mathscr{T}\ \forall\ a, b \in \mathscr{A}$, then $\psi(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Corollary 3.18. *Let $\mathscr{A}$ be a 2-torsion free semi-prime ring and $\mathscr{I}$ an ideal of $\mathscr{A}$. If $(\mathscr{H}, \psi)$ is a generalized derivation of $\mathscr{A}$ satisfying $\mathscr{H}(ab) = a\mathscr{H}(b) + \psi(a)b\ \forall\ a, b \in \mathscr{A}$. Further, if $\mathscr{H}([a, b]) = \pm[\psi(a), \mathscr{H}(b)]\ \forall\ a, b \in \mathscr{I}$, then $\mathscr{A}$ has a non-zero central ideal.*

4. Examples

The following example prove that the condition that $\mathscr{A}$ semi-prime ring and $\mathscr{T}$ prime ideal are not superfluous:

Example 4.1. *Let $\mathscr{A} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and let $\mathscr{I} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}$ and $\mathscr{T} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Define $\mathscr{H}, \psi : \mathscr{A} \to \mathscr{A}$ and $\mathscr{H}(a) = 2e_{11}a - ae_{11}$, $\psi(a) = e_{11}a - ae_{11}$. Note that $[\mathscr{H}(a), a] \in \mathscr{T}$, $\mathscr{H}(a) \circ a \in \mathscr{T}$, $\mathscr{H}([a, b]) - [\psi(a), \mathscr{H}(b)] \in \mathscr{T}$, $\psi(a)\mathscr{H}(b) - ab \in \mathscr{T}$, $\mathscr{H}(a^2) - a^2 \in \mathscr{T}$, $\mathscr{H}([a, b]) - [\mathscr{H}(a), b] - [\psi(b), a]$ and $\mathscr{H}(a \circ b) - (\mathscr{H}(a) \circ b) + (\psi(b) \circ a) \in \mathscr{T}\ \forall\ a, b \in \mathscr{I}$. However $\mathscr{A}$ has no a non-zero central ideal, also $\mathscr{A}/\mathscr{T}$ is non-commutative and $\psi(\mathscr{A}) \nsubseteq \mathscr{T}$.*

Example 4.2. *Consider $\mathscr{D}$ be a ring such that $s^2 = 0\ \forall\ s \in \mathscr{D}$, but the product of some elements of $\mathscr{D}$ is non-zero. Since $s^2 = 0$, so $(s + t)^2 = 0\ \forall\ s, t \in \mathscr{D}$ this implies that $s \circ t = 0\ \forall\ s, t \in \mathscr{D}$. Suppose $\mathscr{A} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathscr{D} \right\}$ and let $\mathscr{I} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathscr{D} \right\}$ and $\mathscr{T} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Define $\mathscr{H} = \psi : \mathscr{A} \to \mathscr{A}$ such that $\psi \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$. Then $\mathscr{H}$ is a generalized derivation of $\mathscr{A}$ associated with the derivation ψ of $\mathscr{A}$. Note that $[\psi(a), \mathscr{H}(b)] \pm [a, b] \in \mathscr{T}$, $[\psi(a), \mathscr{H}(b)] \pm (a, \circ b) \in \mathscr{T}$ and $[\psi(a), \mathscr{H}(b)] \in \mathscr{T}\ \forall\ a, b \in \mathscr{I}$. Also $\psi(a) \circ \mathscr{H}(b) \pm (a, \circ b) \in \mathscr{T}$ and $\psi(a) \circ \mathscr{H}(b) \in \mathscr{T}\ \forall\ a, b \in \mathscr{A}$. However $\mathscr{A}$ has no a non-zero central ideal, also $\mathscr{A}/\mathscr{T}$ is non-commutative, $\psi(\mathscr{A}) \nsubseteq \mathscr{T}$.*

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