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On certain integrals involving (p, k)-Mittag-Leffler function

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ABSTRACT: In the present paper, we establish certain integral formulae involving a new generalization of Mittag-Leffler function ${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)$, which are expressed in terms of generalized Wright function and hypergeometric function. Further some interesting special cases of our main findings are also developed.

Key Words: (p, k)-Pochhammer symbol, (p, k)-Mittag-Leffler function, Integral Transform.

Contents

1. Introduction and Preliminaries

In recent years, many authors have developed and studied numerous integral formulas involving a variety of special functions (see [1,2,3,4,5,6,11,12,28]). These integrals play an important role while dealing with several real world problems especially in mathematical physics. Here we present some integral formulas involving generalized Mittag-Leffler function ${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)$. Some interesting special cases of our main results are also considered.

The Gauss hypergeometric function is defined as (see [20])

Definition 1.1 *Let* $|z| < 1,\ z, \delta_1, \delta_2 \in \mathbb{C},\ \delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-,$ *then*

$$ {}_2F_1(\delta_1, \delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n (\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, \tag{1.1} $$

where $(\delta_1)_n$ is familiar Pochhammar symbol (see [25]).

The series ${}_2F_1(\delta_1, \delta_2; \delta_3; z)$ is convergent in the following cases:

1. for $|z| < 1$; the series converges absolutely.
2. for $|z| = 1$; the series converges absolutely for $\Re(\delta_3 - \delta_1 - \delta_2) > 0$.
3. for $|z| = 1\ (z \neq 1)$; the series converges conditionally for $-1 < \Re(\delta_3 - \delta_1 - \delta_2) \leq 0$.

A more generalized form of the hypergeometric function is ${}_rF_s$, defined as (see [20,14]):

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Definition 1.2 *Let* $\delta_1, \cdots, \delta_r \in \mathbb{C}; \omega_1, \cdots, \omega_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then*

$$ {}_rF_s\left[\begin{array}{c} \delta_1, \cdots, \delta_r; \\ \omega_1, \cdots, \omega_s; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\delta_1)_n ...(\delta_r)_n}{(\omega_1)_n ...(\omega_s)_n} \frac{z^n}{n!}. \tag{1.2} $$

The series ${}_rF_s(.)$ is convergent in the following cases (see [20]):

1. Converges absolutely $\forall \;\; z \in \mathbb{C}$, if $r \leq s$.
2. (a) Converges absolutely for $|z| < 1$ and $r = s + 1$.
   (b) Diverges for $|z| > 1$ and $r = s + 1$.
3. Diverges for $z \neq 0$, if $r > s + 1$.
4. Absolutely convergent for $|z| = 1$, when $r = s + 1$ and

$$ \Re\left[\sum_{j=1}^{s} \omega_j - \sum_{i=1}^{r} \delta_i\right] > 0. $$

5. Conditionally convergent for $|z| = 1$ $(z \neq 1)$, if $r = s + 1$ and when

$$ -1 < \Re\left[\sum_{j=1}^{s} \omega_j - \sum_{i=1}^{r} \delta_i\right] \leq 0. $$

where $(\delta)_n$ is the Pochhammer symbol defined by (see [25])

$$ \begin{aligned} (\delta)_n &= \begin{cases} 1 & (n = 0;\ \delta \in \mathbb{C} \setminus \{0\}) \\ \delta(\delta+1)(\delta+2)...(\delta+n-1) & (n \in \mathbb{N};\ \delta \in \mathbb{C}) \end{cases} \\ &= \frac{\Gamma(\delta+n)}{\Gamma(\delta)} \qquad\qquad (n \in \mathbb{N};\ \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{1.3} $$

and $\Gamma(\delta)$ is the familiar Gamma function.

The Pochhammer k-symbol due to Diaz and Pariguan [7] is defined as

Definition 1.3 *Let* $\delta \in \mathbb{C}, k \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *then the Pochhemmer* k*-symbol is defined as: (Diaz and Pariguan [7])*

$$ (\delta)_{n,\,k} = \delta(\delta+k)(\delta+2k)\cdots(\delta+(n-1)k). \tag{1.4} $$

and k-Gamma function is defined as

Definition 1.4 *For* $\delta \in \mathbb{C};\ \Re(\delta) > 0;\ k \in \mathbb{R}^+$, *the* k*-gamma function is defined by [7]:*

$$ \Gamma_k(\delta) = \int_0^{\infty} t^{\delta-1} e^{-\frac{t^k}{k}}\, dt. \tag{1.5} $$

Proposition 1.1 *The* k*-gamma function and Pochhammer* k*-symbol are related as (see [7]):*
For $n \in \mathbb{N};\ k \in \mathbb{R}^+$

$$ (\delta)_{n,\,k} = \begin{cases} \dfrac{\Gamma_k(\delta+nk)}{\Gamma_k(\delta)} & (\delta \in \mathbb{C} \setminus \{0\}) \\ \delta(\delta+k)\cdots(\delta+(n-1)k), & (\delta \in \mathbb{C}). \end{cases} \tag{1.6} $$

For $\delta \in \mathbb{C},\ \Re(\delta) > 0;\ k \in \mathbb{R}^+$

$$ \Gamma_k(\delta) = k^{\frac{\delta}{k}-1}\Gamma\left(\frac{\delta}{k}\right). \tag{1.7} $$

Recently, Gehlot [9] introduced a new modification of the k-Gamma function ${}_p\Gamma_k(\delta)$ and is defined as

Definition 1.5 *For $\delta \in \mathbb{C} \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+ \setminus 0$ and $\Re(\delta) > 0$, the (p,k) Gamma function ${}_p\Gamma_k(\delta)$ is given as*

$$ {}_p\Gamma_k(\delta) = \int_0^\infty e^{-\frac{t^k}{p}} t^{\delta-1} d\delta. \tag{1.8} $$

Also, he has defined the (p,k)-Pochhammer symbol ${}_p(\delta)_{n,k}$ which is given by (see [9])

Definition 1.6 *For $\delta \in \mathbb{C}; k, p \in \mathbb{R}^+ \setminus 0$ and $\Re(\delta) > 0, n \in \mathbb{N}$, the (p,k)-Pochhammer Symbol ${}_p(\delta)_{n,k}$ is given as*

$$ \begin{aligned} {}_p(\delta)_{n,k} &= \left(\frac{\delta p}{k}\right) \cdot \left(\frac{\delta p}{k} + p\right) \cdot \left(\frac{\delta p}{k} + 2p\right) \cdots \left(\frac{\delta p}{k} + (n-1)p\right) \\ &= \frac{{}_p\Gamma_k(\delta + nk)}{{}_p\Gamma_k(\delta)}. \end{aligned} \tag{1.9} $$

The following properties hold true for the (p,k)-Gamma function and (p,k)-Pochhammer symbol given as [9]:

Proposition 1.2 *For $x \in \mathbb{C} \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+$ and $\Re(x) > 0, n, q \in \mathbb{N}$, then the following formulas hold true:*

$$ {}_p(\Gamma)_k(x) = \left(\frac{p}{k}\right)^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma\left(\frac{x}{k}\right). \tag{1.10} $$

$$ {}_p(x)_{nq,k} = \left(\frac{p}{k}\right)^{nq} (x)_{nq,k} = p^{nq} \left(\frac{x}{k}\right)_{nq}. \tag{1.11} $$

$$ {}_p(x)_{nq,k} = pq^{nq} \prod_{r=1}^{q} \left(\frac{\frac{x}{k} + r - 1}{q}\right)_n. \tag{1.12} $$

2. Mittag-Leffler function and its generalizations

The Swedish mathematician G. Mittag-Leffler [17] introduced the function $E_\lambda(z)$, is defined in series form as:

$$ E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)}, \qquad z \in \mathbb{C}; \lambda \geq 0. \tag{2.1} $$

The two parameter Mittag-Leffler function $E_{\lambda,\nu}(z)$ [17] studied by Wiman [29], is defined as:

$$ E_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \nu)}, \qquad \lambda, \mu \in \mathbb{C};\ \Re(\lambda) > 0, \Re(\nu) > 0. \tag{2.2} $$

In [19](see also [13]), Prabhakar studied the function $E^\gamma_{\lambda,\nu}(z)$, is defined as:

$$ \begin{gathered} E^\gamma_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \nu)} \frac{z^n}{n!}, \\ \lambda, \mu, \gamma \in \mathbb{C};\ \Re(\lambda) > 0, \Re(\nu) > 0, \Re(\gamma) > 0. \end{gathered} \tag{2.3} $$

In [24], Shukla and Prajapati studied the function $E^{\gamma;\eta}_{\lambda,\nu}(z)$, is defined as (see also [27]):

$$E_{\lambda,\nu}^{\gamma,\eta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\eta n}}{\Gamma(\lambda n+\nu)}\frac{z^n}{n!},$$
$$\lambda,\nu,\gamma\in\mathbb{C},\ \Re(\lambda)>0,\ \Re(\nu)>0,\ \Re(\gamma)>0\ \text{ and }\ \eta\in(0,1)\cup\mathbb{N},\tag{2.4}$$

where $(\gamma)_{\eta n}=\dfrac{\Gamma(\gamma+\eta n)}{\Gamma(\gamma)}$.

Salim [21] introduced a new generalization of Mittag-Leffler function $E_{\lambda,\nu}^{\gamma,\delta}(z)$ and is defined as:

$$E_{\lambda,\nu}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\nu)}\frac{z^n}{(\delta)_n},$$
$$\lambda,\nu,\gamma,\delta\in\mathbb{C},\ \Re(\lambda)>0,\ \Re(\nu)>0,\ \Re(\gamma)>0,\ \Re(\delta)>0.\tag{2.5}$$

Further, Salim and Faraj [22] introduced the following more generalized Mittag-Leffler function $E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ defined as:

$$E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\eta n}}{\Gamma(\lambda n+\nu)}\frac{z^n}{(\delta)_{\tau n}},$$
$$\lambda,\nu,\gamma,\delta\in\mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0,\quad \tau,\eta\in\mathbb{R}^+,\quad \eta<\Re(\lambda)+\tau.\tag{2.6}$$

Further, the following more generalized Mittag-Leffler function $E_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ is introduced by Gupta and Parihar [10] defined as:

$$E_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\eta n,k}}{\Gamma_k(\lambda n+\nu)}\frac{z^n}{(\delta)_{\tau n,k}},$$
$$\lambda,\nu,\gamma,\delta\in\mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0,\quad k,\tau,\eta\in\mathbb{R}^+,\quad \eta<\Re(\lambda)+\tau.\tag{2.7}$$

Here we introduce a new generalization of Mittag-Leffler function defined as:

$${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)}\frac{z^n}{{}_p(\delta)_{\tau n,k}},$$
$$\lambda,\nu,\gamma,\delta\in\mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0,\quad p,k,\tau,\eta\in\mathbb{R}^+,\quad \eta<\frac{\Re(\lambda)}{k}+\tau.\tag{2.8}$$

where ${}_p(\gamma)_{\eta n,k}$ is the (p,k)-Pochhammer symbol defined in equation (1.9).

The generalized hypergeometric series ${}_pF_q$ (see e.g., [15]) due to Fox [8] and Wright [30,31,32] known as the Fox-Wright function ${}_p\Psi_q(z)$ $(p,q\in\mathbb{N}_0)$ with p numerator and q denominator parameters defined for $a_1,...,a_p\in\mathbb{C}$ and $b_1,...,b_q\in\mathbb{C}\setminus\mathbb{Z}_0^-$ by (For details see [14,16,23,26])

$${}_p\Psi_q\left[\begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p);\\(b_1,\beta_1),...,(b_q,\beta_q);\end{array}z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n)...\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)...\Gamma(b_q+\beta_q n)}\frac{z^n}{n!},\tag{2.9}$$

where the coefficients $\alpha_1,...,\alpha_p,\ \beta_1,...,\beta_q\in\mathbb{R}^+$ are such that

$$1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i\geq 0.\tag{2.10}$$

For $\alpha_i=\beta_j=1$ $(i=1,...,p;j=1,...,q)$, equation (2.9) reduces immediately to the generalized hypergeometric function ${}_pF_q$ $(p,q\in\mathbb{N}_0)$ (see [26]):

$${}_pF_q\left[\begin{array}{c}a_1,...,a_p;\\b_1,...,b_q;\end{array}z\right]=\frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_q)}{}_p\Psi_q\left[\begin{array}{c}(a_1,1),...,(a_p,1);\\(b_1,1),...,(b_q,1);\end{array}z\right].\tag{2.11}$$

For our present study we also require the following Oberhettinger's integral formula [18]

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}dx = 2\zeta a^{-\zeta}\left(\frac{a}{2}\right)^{\xi}\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(1+\zeta+\xi)}, \tag{2.12}$$

provided $0<\Re(\xi)<\Re(\zeta)$.

The well known Legendre duplication formula (see [25]) is defined as:

$$(\delta)_{2n} = 2^{2n}\left(\frac{\delta}{2}\right)_n\left(\frac{\delta+1}{2}\right)_n \qquad (n\in\mathbb{N}_0) \tag{2.13}$$

3. Main Results

Theorem 3.1 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\frac{k\Gamma\left(\frac{\delta}{k}\right)\Gamma(2\xi)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)}{}_4\Psi_4\left[\begin{array}{ll}\left(\frac{\gamma}{k},\eta\right),\ (\zeta-\xi,1),\ (1+\zeta,1),\ (1,1); & \dfrac{zp^{\eta-\tau-\frac{\lambda}{k}}}{a}\\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right),\left(\frac{\delta}{k},\tau\right),(1+\zeta+\xi,1),\ (\zeta,1); & \end{array}\right].\end{aligned} \tag{3.1}$$

Proof: Denote left hand side of equation (3.1) by $\mathcal{I}$, then interchanging order of integration and summation and using equation (2.8), we have

$$\mathcal{I}=\sum_{n=0}^\infty\frac{{}_p(\gamma)_{\eta n,k}z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-(\zeta+n)}dx, \tag{3.2}$$

evaluating the above integral in view of equation (2.12) and after little simplification, we get

$$\mathcal{I}=2^{1-\xi}a^{\xi-\zeta}\frac{k\Gamma\left(\frac{\delta}{k}\right)\Gamma(2\xi)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^\infty\frac{\Gamma\left(\frac{\gamma}{k}+\eta n\right)\Gamma(\zeta-\xi+n)\Gamma(1+\zeta+n)\Gamma(1+n)}{\Gamma\left(\frac{\nu}{k}+\frac{\lambda n}{k}\right)\Gamma\left(\frac{\delta}{k}+\tau n\right)\Gamma(1+\zeta+\xi+n)\Gamma(\zeta+n)}\left(\frac{zp^{\eta-\tau-\frac{\lambda}{k}}}{a}\right)^n\frac{1}{n!}, \tag{3.3}$$

in view of equation (2.9), we get the required result (3.1). □

Theorem 3.2 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\frac{k\Gamma\left(\frac{\delta}{k}\right)\Gamma(\zeta-\xi)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)}{}_4\Psi_4\left[\begin{array}{ll}\left(\frac{\gamma}{k},\eta\right),\ (2\xi,2),\ (1+\zeta,1),\ (1,1); & \dfrac{zp^{\eta-\tau-\frac{\lambda}{k}}}{2}\\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right),\left(\frac{\delta}{k},\tau\right),(1+\zeta+\xi,2),(\zeta,1); & \end{array}\right].\end{aligned} \tag{3.4}$$

Proof: The proof of Theorem 3.2 is similar to that of Theorem 3.1, so we omit the details. □

Theorem 3.3 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{{}_p\Gamma_k(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}\\
&\times{}_{\eta+3}F_{\frac{\lambda}{k}+\tau+2}\left[\begin{array}{lllll}\Delta\left(\eta;\frac{\gamma}{k}\right), & \zeta-\xi, & 1+\zeta, & 1; & \frac{zp^{\eta-\tau-\frac{\lambda}{k}}\eta^\eta k^{\frac{\lambda}{k}}}{a\tau^\tau\lambda^{\frac{\lambda}{k}}}\\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right), \Delta\left(\tau;\frac{\delta}{k}\right), 1+\zeta+\xi, \zeta; & & & & \end{array}\right].
\end{aligned}\tag{3.5}$$

Proof: By using equation (2.8), (1.10), (1.11), (1.12) and in view of equation (2.12) and (2.13) we get the required result (3.5). □

Theorem 3.4 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\zeta\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{{}_p\Gamma_k(\nu)\Gamma(1+\zeta+\xi)}\\
&\times{}_{\eta+4}F_{\frac{\lambda}{k}+\tau+3}\left[\begin{array}{llllll}\Delta\left(\eta;\frac{\gamma}{k}\right), & 1+\zeta, & \xi, & \xi+\frac{1}{2}, & 1; & \frac{zp^{\eta-\tau-\frac{\lambda}{k}}\eta^\eta k^{\frac{\lambda}{k}}}{2\tau^\tau\lambda^{\frac{\lambda}{k}}}\\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right), \Delta\left(\tau;\frac{\delta}{k}\right), \zeta, & \frac{1+\zeta+\xi}{2}, & \frac{2+\zeta+\xi}{2}; & & & \end{array}\right].
\end{aligned}\tag{3.6}$$

where $\Delta(q;\alpha)$ is q-tuple $\frac{\alpha}{q},\frac{\alpha+1}{q},\frac{\alpha+2}{q},...,\frac{\alpha+q-1}{q}$.

Proof: By using equation (2.8), (1.10), (1.11), (1.12) and in view of equation (2.12) and (2.13) we get the required result (3.6). □

3.1. Special Cases

On setting $p=1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.1 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{k\Gamma\left(\frac{\delta}{k}\right)\Gamma(2\xi)}{\Gamma\left(\frac{\gamma}{k}\right)}{}_4\Psi_4\left[\begin{array}{lllll}\left(\frac{\gamma}{k},\eta\right), & (\zeta-\xi,1), & (1+\zeta,1), & (1,1); & \frac{z}{a}\\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), & \left(\frac{\delta}{k},\tau\right), & (1+\zeta+\xi,1), & (\zeta,1); & \end{array}\right].
\end{aligned}\tag{3.7}$$

Corollary 3.2 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{k\Gamma\left(\frac{\delta}{k}\right)\Gamma(\zeta-\xi)}{\Gamma\left(\frac{\gamma}{k}\right)}{}_4\Psi_4\left[\begin{array}{lllll}\left(\frac{\gamma}{k},\eta\right), & (2\xi,2), & (1+\zeta,1), & (1,1); & \frac{z}{2}\\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), & \left(\frac{\delta}{k},\tau\right), & (1+\zeta+\xi,2), & (\zeta,1); & \end{array}\right].
\end{aligned}\tag{3.8}$$

Corollary 3.3 *Let $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$
\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma_k(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}\\
&\times{}_{\eta+3}F_{\frac{\lambda}{k}+\tau+2}\left[\begin{array}{ll}\Delta\left(\eta;\frac{\gamma}{k}\right),\quad \zeta-\xi,\quad 1+\zeta,\quad 1; & \dfrac{z\eta^\eta k^{\frac{\lambda}{k}}}{a\tau^\tau\lambda^{\frac{\lambda}{k}}}\\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right),\Delta\left(\tau;\frac{\delta}{k}\right),1+\zeta+\xi,\zeta; & \end{array}\right].
\end{aligned}
\tag{3.9}
$$

Corollary 3.4 *Let* $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0,\;\; k,\tau,\eta\in\mathbb{R}^+,\quad \eta<\frac{\Re(\lambda)}{k}+\tau$, *then for* $0<\Re(\xi)<\Re(\zeta)$, *the following formula holds true:*

$$
\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\zeta\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma_k(\nu)\Gamma(1+\zeta+\xi)}\\
&\times{}_{\eta+4}F_{\frac{\lambda}{k}+\tau+3}\left[\begin{array}{ll}\Delta\left(\eta;\frac{\gamma}{k}\right),\quad 1+\zeta,\quad \xi,\quad \xi+\frac{1}{2},\quad 1; & \dfrac{z\eta^\eta k^{\frac{\lambda}{k}}}{2\tau^\tau\lambda^{\frac{\lambda}{k}}}\\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right),\Delta\left(\tau;\frac{\delta}{k}\right),\zeta,\dfrac{1+\zeta+\xi}{2},\dfrac{2+\zeta+\xi}{2}; & \end{array}\right].
\end{aligned}
\tag{3.10}
$$

On setting $p=k=1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.5 *Let* $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0,\;\; \tau,\eta\in\mathbb{R}^+,\quad \eta<\Re(\lambda)+\tau$, *then for* $0<\Re(\xi)<\Re(\zeta)$, *the following formula holds true:*

$$
\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(\delta)\Gamma(2\xi)}{\Gamma(\gamma)}{}_4\Psi_4\left[\begin{array}{ll}(\gamma,\eta),\;(\zeta-\xi,1),\;(1+\zeta,1),\;(1,1); & \dfrac{z}{a}\\ (\nu,\lambda),\left(\frac{\delta}{k},\tau\right),(1+\zeta+\xi,1),\;(\zeta,1); & \end{array}\right].
\end{aligned}
\tag{3.11}
$$

Corollary 3.6 *Let* $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0,\;\; \tau,\eta\in\mathbb{R}^+,\; \eta<\Re(\lambda)+\tau$, *then for* $0<\Re(\xi)<\Re(\zeta)$, *the following formula holds true:*

$$
\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(\delta)\Gamma(\zeta-\xi)}{\Gamma(\gamma)}{}_4\Psi_4\left[\begin{array}{ll}(\gamma,\eta),\;(2\xi,2),\;(1+\zeta,1),\;(1,1); & \dfrac{z}{2}\\ (\nu,\lambda),(\delta,\tau),(1+\zeta+\xi,2),(\zeta,1); & \end{array}\right].
\end{aligned}
\tag{3.12}
$$

Corollary 3.7 *Let* $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0,\;\; \tau,\eta\in\mathbb{R}^+,\quad \eta<\Re(\lambda)+\tau$, *then, for* $0<\Re(\xi)<\Re(\zeta)$, *the following formula holds true:*

$$
\begin{aligned}
&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\
&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}\\
&\times{}_{\eta+3}F_{\lambda+\tau+2}\left[\begin{array}{ll}\Delta(\eta;\gamma),\quad \zeta-\xi,\quad 1+\zeta,\quad 1; & \dfrac{z\eta^\eta}{a\tau^\tau\lambda^\lambda}\\ \Delta(\lambda;\nu),\Delta(\tau;\delta),1+\zeta+\xi,\zeta; & \end{array}\right].
\end{aligned}
\tag{3.13}
$$

Corollary 3.8 *Let* $x>0;\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0,\;\; \tau,\eta\in\mathbb{R}^+,\quad \eta<\Re(\lambda)+\tau$, *then, for* $0<\Re(\xi)<\Re(\zeta)$, *the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\zeta\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)} \tag{3.14}$$
$$\times\ {}_{\eta+4}F_{\lambda+\tau+3}\left[\begin{array}{lc} \Delta(\eta;\gamma), \quad 1+\zeta, \quad \xi, \quad \xi+\frac{1}{2}, \quad 1; & \dfrac{z\eta^\eta}{2\tau^\tau\lambda^\lambda} \\ \Delta(\lambda;\nu), \Delta(\tau;\delta), \zeta, \dfrac{1+\zeta+\xi}{2}, \dfrac{2+\zeta+\xi}{2}; & \end{array}\right].$$

On setting $p=k=\eta=\tau=1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.9 *Let $x>0; \lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma,\delta}_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(\delta)\Gamma(2\xi)}{\Gamma(\gamma)}\,{}_4\Psi_4\left[\begin{array}{lc} (\gamma,1), \ (\zeta-\xi,1), \ (1+\zeta,1), \ (1,1); & \dfrac{z}{a} \\ (\nu,\lambda), (\delta,1), (1+\zeta+\xi,1), \ (\zeta,1); & \end{array}\right]. \tag{3.15}$$

Corollary 3.10 *Let $x>0; \lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma,\delta}_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(\delta)\Gamma(\zeta-\xi)}{\Gamma(\gamma)}\,{}_4\Psi_4\left[\begin{array}{lc} (\gamma,1), \quad (2\xi,2), \quad (1+\zeta,1), \quad (1,1); & \dfrac{z}{2} \\ (\nu,\lambda), (\delta,1), (1+\zeta+\xi,2), (\zeta,1); & \end{array}\right]. \tag{3.16}$$

Corollary 3.11 *Let $x>0; \lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, then, for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma,\delta}_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}\,{}_4F_{\lambda+3}\left[\begin{array}{lc} \gamma, \quad \zeta-\xi, \quad 1+\zeta, \quad 1; & \dfrac{z}{a\lambda^\lambda} \\ \Delta(\lambda;\nu), \delta, 1+\zeta+\xi, \zeta; & \end{array}\right]. \tag{3.17}$$

Corollary 3.12 *Let $x>0; \lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, then, for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma,\delta}_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\zeta\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)}\,{}_5F_{\lambda+4}\left[\begin{array}{lc} \gamma, \quad 1+\zeta, \quad \xi, \quad \xi+\frac{1}{2}, \quad 1; & \dfrac{z}{2\lambda^\lambda} \\ \Delta(\lambda;\nu), \delta, \zeta, \dfrac{1+\zeta+\xi}{2}, \dfrac{2+\zeta+\xi}{2}; & \end{array}\right]. \tag{3.18}$$

On setting $p=k=\eta=\tau=\delta=1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.13 *Let $x>0; \lambda,\nu,\gamma\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E^{\gamma}_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)}{\Gamma(\gamma)}\,{}_3\Psi_3\left[\begin{array}{lc} (\gamma,1), \ (\zeta-\xi,1), \ (1+\zeta,1); & \dfrac{z}{a} \\ (\nu,\lambda), (1+\zeta+\xi,1), \ (\zeta,1); & \end{array}\right]. \tag{3.19}$$

Corollary 3.14 *Let $x>0;\lambda,\nu,\gamma\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E^{\gamma}_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(\zeta-\xi)}{\Gamma(\gamma)}{}_3\Psi_3\left[\begin{array}{ll}(\gamma,1),\ \ (2\xi,2),\ \ (1+\zeta,1); & \frac{z}{2}\\ (\nu,\lambda),(1+\zeta+\xi,2),(\zeta,1); & \end{array}\right].\end{aligned}\tag{3.20}$$

Corollary 3.15 *Let $x>0;\lambda,\nu,\gamma\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, then, for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E^{\gamma}_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}{}_3F_{\lambda+2}\left[\begin{array}{ll}\gamma,\ \ \zeta-\xi,\ \ 1+\zeta; & \frac{z}{a\lambda^\lambda}\\ \Delta(\lambda;\nu),1+\zeta+\xi,\zeta; & \end{array}\right].\end{aligned}\tag{3.21}$$

Corollary 3.16 *Let $x>0;\lambda,\nu,\gamma\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, $\Re(\gamma)>0$, then, for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E^{\gamma}_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\zeta\frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)}{}_4F_{\lambda+3}\left[\begin{array}{ll}\gamma,\ \ 1+\zeta,\ \ \xi,\ \ \xi+\frac{1}{2}; & \frac{z}{2\lambda^\lambda}\\ \Delta(\lambda;\nu),\zeta,\frac{1+\zeta+\xi}{2},\frac{2+\zeta+\xi}{2}; & \end{array}\right].\end{aligned}\tag{3.22}$$

On setting $p=k=\eta=\tau=\gamma=\delta=1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.17 *Let $x>0;\lambda,\nu\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\Gamma(2\xi){}_3\Psi_3\left[\begin{array}{ll}(\zeta-\xi,1),\ (1+\zeta,1),\ \ (1,1); & \frac{z}{a}\\ (\nu,\lambda),(1+\zeta+\xi,1),\ (\zeta,1); & \end{array}\right].\end{aligned}\tag{3.23}$$

Corollary 3.18 *Let $x>0;\lambda,\nu\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, then for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\Gamma(\zeta-\xi){}_3\Psi_3\left[\begin{array}{ll}(2\xi,2),\ \ (1+\zeta,1),\ \ (1,1); & \frac{z}{2}\\ (\nu,\lambda),(1+\zeta+\xi,2),(\zeta,1); & \end{array}\right].\end{aligned}\tag{3.24}$$

Corollary 3.19 *Let $x>0;\lambda,\nu\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\nu)>0$, then, for $0<\Re(\xi)<\Re(\zeta)$, the following formula holds true:*

$$\begin{aligned}&\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta}E_{\lambda,\nu}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=2^{1-\xi}a^{\xi-\zeta}\frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)\Gamma(\zeta)}{}_3F_{\lambda+2}\left[\begin{array}{ll}\zeta-\xi,\ \ 1+\zeta,\ \ 1; & \frac{z}{a\lambda^\lambda}\\ \Delta(\lambda;\nu),1+\zeta+\xi,\zeta; & \end{array}\right].\end{aligned}\tag{3.25}$$

Corollary 3.20 *Let $x > 0; \lambda, \nu \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\nu) > 0$, then, for $0 < \Re(\xi) < \Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E_{\lambda,\nu}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right) dx = 2^{1-\xi} a^{\xi-\zeta} \zeta \frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(\nu)\Gamma(1+\zeta+\xi)} {}_4F_{\lambda+3}\left[\begin{array}{c} 1+\zeta, \quad \xi, \quad \xi+\frac{1}{2}, \quad 1; \\ \Delta(\lambda;\nu), \zeta, \frac{1+\zeta+\xi}{2}, \frac{2+\zeta+\xi}{2}; \end{array} \frac{z}{2\lambda^\lambda}\right]. \tag{3.26}$$

On setting $p = k = \eta = \tau = \gamma = \delta = \nu = 1$, results presented in Theorem 3.1–Theorem 3.4 reduce to the following form.

Corollary 3.21 *Let $x > 0; \lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$ then for $0 < \Re(\xi) < \Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E_{\lambda}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right) dx = 2^{1-\xi} a^{\xi-\zeta} \Gamma(2\xi) {}_3\Psi_3\left[\begin{array}{c} (\zeta-\xi,1), \ (1+\zeta,1), \ (1,1); \\ (1,\lambda), (1+\zeta+\xi,1), \ (\zeta,1); \end{array} \frac{z}{a}\right]. \tag{3.27}$$

Corollary 3.22 *Let $x > 0; \lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$, then for $0 < \Re(\xi) < \Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E_{\lambda}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right) dx = 2^{1-\xi} a^{\xi-\zeta} \Gamma(\zeta-\xi) {}_3\Psi_3\left[\begin{array}{c} (2\xi,2), \ (1+\zeta,1), \ (1,1); \\ (1,\lambda), (1+\zeta+\xi,2), (\zeta,1); \end{array} \frac{z}{2}\right]. \tag{3.28}$$

Corollary 3.23 *Let $x > 0; \lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$, then, for $0 < \Re(\xi) < \Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E_{\lambda}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right) dx = 2^{1-\xi} a^{\xi-\zeta} \frac{\Gamma(2\xi)\Gamma(1+\xi)\Gamma(\zeta-\xi)}{\Gamma(1+\zeta+\xi)\Gamma(\zeta)} {}_3F_{\lambda+2}\left[\begin{array}{c} \zeta-\xi, \quad 1+\zeta, \quad 1; \\ \Delta(\lambda;1), 1+\zeta+\xi, \zeta; \end{array} \frac{z}{a\lambda^\lambda}\right]. \tag{3.29}$$

Corollary 3.24 *Let $x > 0; \lambda \in \mathbb{C}$, $\Re(\lambda) \geq 0$, then, for $0 < \Re(\xi) < \Re(\zeta)$, the following formula holds true:*

$$\int_0^\infty x^{\xi-1}(x+a+\sqrt{x^2+2ax})^{-\zeta} E_{\lambda}\left(\frac{xz}{x+a+\sqrt{x^2+2ax}}\right) dx = 2^{1-\xi} a^{\xi-\zeta} \zeta \frac{\Gamma(2\xi)\Gamma(\zeta-\xi)}{\Gamma(1+\zeta+\xi)} {}_4F_{\lambda+3}\left[\begin{array}{c} 1+\zeta, \quad \xi, \quad \xi+\frac{1}{2}, \quad 1; \\ \Delta(\lambda;1), \zeta, \frac{1+\zeta+\xi}{2}, \frac{2+\zeta+\xi}{2}; \end{array} \frac{z}{2\lambda^\lambda}\right]. \tag{3.30}$$

4. Conclusion

In this paper, certain integral formulae involving a new generalization of Mittag-Leffler function ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ were estalished, which are expressed in terms of generalized Wright function and hypergeometric function. Further some interesting special cases of our main findings are also developed. All the findings and their special cases are new and general in nature.

Competing interests

The author declares to have no competing interests.

Authors Contributions

All authors contribute equally in the present investigation.

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