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On $\mathfrak{T}$-hypersurfaces of Lorentzian para Kenmotsu manifolds

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ABSTRACT: The main purpose of this paper is to study transversal hypersurface (briefly, $\mathfrak{T}$-hypersurface) of Lorentzian para Kenmotsu manifolds. It is proved that each $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold admits an almost product Lorentzian metric structure (J, G). After that, we show that every $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold also admits a Lorentzian $(f, g, \mu, \nu, \lambda)$-structure and we derive some results allied with relationship between induced almost product Lorentzian metric structure (J, G) and induced Lorentzian $(f, g, \mu, \nu, \lambda)$-structure. Example of $\mathfrak{T}$-hypersurface of Lorentzian para Kenmotsu manifold admitting Lorentzian $(f, g, \mu, \nu, \lambda)$-structure is also illustrated.

Key Words: Lorentzian para Kenmotsu manifolds, Lorentzian-metric, almost product metric structure, transversal hypersurfaces.

Contents

1. Introduction

Hypersurfaces of an almost contact manifold have been studied by D. E. Blair [1], S. S. Eum [2], S. I. Goldberg and K.Yano [3], G. D. Ludden [4] and others. In 1970, S. I. Goldberg and K. Yano [3] defined noninvariant hypersurfaces of almost contact manifolds. Further, the concept of transversal hypersurface is introduced by K. Yano in 1972 [12]. After that transversal hypersurfaces were investigated by several authors in different ambient manifolds such as ([13], [14], [15]) and many others.

In 1976, I. Sato [17] studied a structure similar to the almost contact structure, namely almost paracontact structure. In [16], T. Adati studied hypersurfaces of an almost paracontact manifold. The study of Lorentzian almost paracontact manifolds was initiated by K. Matsumoto in 1989 [18]. Also he introduced the notion of Lorentzian para-Sasakian (for short, LP-Sasakian) manifold. I. Mihai and R. Rosca [21] defined the same notion independently and thereafter many authors ([19], [20], [22], [23]) studied Lorentzian para-Sasakian manifolds and their submanifolds.

The study of hypersurface has ample significance in general theory of relativity, black holes and quantum mechanics ([5], [6], [7]). Therefore, several researchers showed their interest in studying the geometry of hypersurface in different ambient spaces ([8],[9], [10], [11]).

In the present paper, we study transversal hypersurfaces, in short $\mathfrak{T}$-hypersurfaces of Lorentzian almost paracontact manifold, that is, such hypersurfaces of Lorentzian almost paracontact manifold which never contain the vector field ξ. Also, we obtain that every $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold admits an almost product Lorentzian metric structure (J, G). After that, we show that every transversal hypersurface of Lorentzian almost paracontact manifold also admits a Lorentzian $(f, g, \mu, \nu, \lambda)$-structure, then we find some results to obtain the relationship between induced almost product Lorentzian metric structure (J, G) and induced Lorentzian $(f, g, \mu, \nu, \lambda)$-structure. Further, we discuss about some

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properties of Lorentzian para Kenmotsu manifold, along with an example of transversal hypersurface of Lorentzian para Kenmotsu manifold admitting Lorentzian $(f, g, \mu, \nu, \lambda)$-structure.

2. Preliminaries

Let $\mathcal{M}$ be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form on $\mathcal{M}$ such that [25]

$$\phi^2 U = U + \eta(U)\xi, \qquad \eta(\xi) = -1 \tag{2.1}$$

which implies that

$$\phi(\xi) = 0, \qquad \eta \circ \phi = 0, \qquad rank\phi = n - 1 \tag{2.2}$$

If $\mathcal{M}$ admits a Lorentzian metric g such that

$$g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V) \tag{2.3}$$

then M is said to admit a Lorentzian almost paracontact structure (ϕ, ξ, η, g). Also,

$$g(\phi U, V) = g(U, \phi V), \qquad g(U, \xi) = \eta(U), \qquad g(\xi, \xi) = \eta(\xi) = -1 \tag{2.4}$$

Consequently, we call $(\mathcal{M}, \phi, \xi, \eta, g)$ is an Lorentzian almost paracontact manifold with Lorentzian metric g where, ϕ is the structural endomorphism, ξ is the characteristic vector field and η is a 1-form.
Now, we define a manifold called the Lorentzian para-Kenmotsu manifold:

Definition 2.1 *[24] A Lorentzian almost paracontact manifold $\mathcal{M}$ with structure (ϕ, ξ, η, g) is called* ***Lorentzian para-Kenmotsu manifold*** *if the operator of covariant differentiation $\overline{\nabla}$ with respect to the Lorentzian metric g satisfies the following condition:*

$$(\overline{\nabla}_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi U \tag{2.5}$$

for any vector fields U, V on M.
In the Lorentzian para-Kenmotsu manifold, we have

$$\overline{\nabla}_U \xi = -U - \eta(U)\xi, \tag{2.6}$$

$$(\overline{\nabla}_U \eta)V = -g(U, V) - \eta(U)\eta(V), \tag{2.7}$$

where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Further, on a Lorentzian para-Kenmotsu manifold $\mathcal{M}$, the following relations hold:

1. $g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y)$
2. $R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X$
3. $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$
4. $R(\xi, X)\xi = X + \eta(X)\xi$
5. $S(X, \xi) = (n - 1)\eta(X)$
6. $S(\xi, \xi) = -(n - 1)$
7. $Q\xi = (n - 1)\xi$
8. $S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y)$

for any vector fields X, Y on $\mathcal{M}$.

Definition 2.2 *A differentiable manifold M of dimension n is said to be an* ***almost product manifold****, if it admits a (1,1) tensor field J such that $J^2 = I$.*

3. $\mathfrak{T}$-HYPERSURFACES

Let $\mathcal{M}$ be a Lorentzian almost paracontact manifold equipped with Lorentzian almost paracontact structure $(\phi, \xi, \eta, \tilde{g})$, where $\tilde{g}$ is Lorentzian metric. Let M be an *immersed hypersurface* of $\mathcal{M}$ with induced symmetric tensor field g. In view of casual character of vector fields of manifold, we have three types of hypersurface, namely , pseudo-Riemannian, Riemannian and null (or lightlike) and metric g is a non-degenerate or a degenerate metric according as M is pseudo-Riemannian (Riemannian) hypersurface and lightlike hypersurface respectively. We assume that the structure vector field ξ never belongs to the tangent hyperplane of the hypersurface M. Such a hypersurface is called a **transversal hypersurface**, briefly **$\mathfrak{T}$-hypersurface** of a Lorentzian almost paracontact manifold.

Example 3.1 *Let $\mathcal{M}$ be a 7 dimensional real number space with coordinate system (x, y, z, t, u, v, s). In $\mathcal{M}$, we define*

$$\eta = ds - dx, \qquad \xi = -\frac{\partial}{\partial s}, \qquad \phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} + \frac{\partial}{\partial s},$$
$$\phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}, \qquad \phi(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}, \qquad \phi(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t},$$
$$\phi(\frac{\partial}{\partial u}) = \frac{\partial}{\partial u}, \qquad \phi(\frac{\partial}{\partial v}) = \frac{\partial}{\partial v}, \qquad \phi(\frac{\partial}{\partial s}) = 0$$

and

$$g = dx^2 + dy^2 + dz^2 + dt^2 + du^2 + dv^2 - \eta \otimes \eta$$

then (ϕ, ξ, η, g) is Lorentzian almost paracontact structure in $\mathcal{M}$.

Let M be hypersurface of $\mathcal{M}$ which is defined by $s = x$ with the immersion $i : M \to \mathcal{M}$, then the set $\{\alpha_1 = (1, 0, 0, 0, 0, 0, 1), \alpha_2 = (0, 1, 0, 0, 0, 0, 0), \alpha_3 = (0, 0, 1, 0, 0, 0, 0), \alpha_4 = (0, 0, 0, 1, 0, 0, 0),$ $\alpha_5 = (0, 0, 0, 0, 1, 0, 0), \alpha_6 = (0, 0, 0, 0, 0, 1, 0)\}$ *is a local basis for the tangent hyperplane of M and $N = (1, 0, 0, 0, 0, 0, -1)$ is the normal vector field of the hypersurface. Since $\xi_{i(p)} = \frac{1}{2}(\alpha_1 - N)_{i(p)}$, it can be easily seen that the characteristic vector field $\xi_{i(p)}$, $p \in M$, is not tangent to M. Thus, M is a $\mathfrak{T}$-hypersurface of $\mathcal{M}$ with the characteristic vector field $\xi_{i(p)}$, $p \in M$, which is not tangent to the hypersurface.*

Now, $\mathfrak{T}$-hypersurfaces never contain the structure vector field ξ of the defining Lorentzian almost paracontact structure. Thus, ξ can be considered as affine normal to M. Now, ξ and $X \in TM$ are linearly independent, therefore we may write ϕX as:

$$\phi X = JX + \omega(X)\xi, \tag{3.1}$$

where J is a tensor field of type (1, 1) and ω is a 1-form on M.
Now, operating ϕ on (3.1) and with the help of equation (2.1), we have

$$J^2 = I \tag{3.2}$$

and

$$\omega \circ J = \eta \tag{3.3}$$

which follows that:

$$\eta(JX) = (\omega \circ J)JX$$
$$\eta(JX) = \omega(J^2X)$$
$$(\eta \circ J)X = \omega(X)$$

$$\eta \circ J = \omega. \tag{3.4}$$

In the account of (3.1), we have

Theorem 3.1 *Each $\mathfrak{T}$-hypersurface of a Lorentzian almost paracontact manifold admits an almost product structure J and a 1-form ω.*

Now, we assume that M admits a Lorentzian almost paracontact metric structure (ϕ, ξ, η, g). Then for every $X, Y \in TM$, we obtain the following results:

$$\begin{aligned} g(\phi X, \phi Y) &= g(JX + \omega(X)\xi, JY + \omega(Y)\xi) \\ &= g(JX, JY) + \omega(Y)(\eta \circ J)(X) + \omega(X)(\eta \circ J)(Y) - \omega(X)\omega(Y) \end{aligned}$$

In the account of equation (3.4), we have

$$\begin{aligned} g(\phi X, \phi Y) &= g(JX, JY) + \omega(X)\omega(Y) + \omega(X)\omega(Y) - \omega(X)\omega(Y) \\ &= g(JX, JY) + \omega(X)\omega(Y) \end{aligned}$$

Using equation (2.3), we get

$$g(X, Y) + \eta(X)\eta(Y) = g(JX, JY) + \omega(X)\omega(Y)$$

$$g(JX, JY) = g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y). \tag{3.5}$$

Now, we define a new metric G on the transversal hypersurface given by:

$$G(X, Y) = g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{3.6}$$

So with the help of equation (3.5) we have

$$\begin{aligned} G(JX, JY) &= g(JX, JY) + \eta(JX)\eta(JY) \\ &= g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) + (\eta \circ J)(X)(\eta \circ J)(Y) \\ &= g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) + \omega(X)\omega(Y) \\ &= g(X, Y) + \eta(X)\eta(Y) \\ &= G(X, Y). \end{aligned}$$

Hence, G is Lorentzian metric on M that is (J, G) is an almost product Lorentzian structure on transversal hypersurface M of $\mathcal{M}$.

Consequently, we can state the following theorem:

Theorem 3.2 *Each $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold admits an almost product Lorentzian structure.*

We now assume that M is orientable and consider a unit vector field N of $\mathcal{M}$ which is normal to M. Then, the Gauss and Weingarten formulae are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)N \tag{3.7}$$

and

$$\overline{\nabla}_X N = -HX \tag{3.8}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$ respectively, where $\overline{\nabla}$ is Levi Civita connection on $\mathcal{M}$. Here ∇ represents the Levi-Civita connection on M with respect to Lorentzian metric g induced by $\tilde{g}$ on $\mathcal{M}$ and h is a second fundamental form related to H by

$$h(X, Y) = g(HX, Y). \tag{3.9}$$

Also note that, the hypersurface M is totally geodesic in $\mathcal{M}$ if second fundamental form vanishes identically i.e. $h \equiv 0$.
Further, for any vector field $X \in TM$, we define

$$\phi X = fX + \mu(X)N, \tag{3.10}$$

$$\phi N = -U, \tag{3.11}$$

$$\xi = V + \lambda N, \tag{3.12}$$

$$\eta(X) = \nu(X) \tag{3.13}$$

where, $\eta(N) = \lambda = g(\xi, N)$, f is (1,1)-type tensor field and μ represents a non-zero 1-form and λ is a smooth function on M. Hence, it follows that $\lambda \neq 0$, because if possible $\lambda = 0$, then $\eta(N) = g(N, \xi) = 0$, which shows that ξ is perpendicular to N, so we have $\xi \in TM$, which contradicts the fact that M is a transversal-hypersurface of $\mathcal{M}$.

We obtain an induced Lorentzian structure $(f, g, \mu, \nu, \lambda)$-structure [8] on transversal-hypersurface such that

$$f^2 = I + \mu \otimes U + \nu \otimes V \tag{3.14}$$

$$fU = -\lambda V \quad \textit{and} \quad fV = \lambda U \tag{3.15}$$

$$\mu \circ f = \lambda \nu \quad \textit{and} \quad \nu \circ f = -\lambda \mu \tag{3.16}$$

$$\mu(U) = -\lambda^2 - 1, \quad \nu(V) = -1 - \lambda^2 \quad \textit{and} \quad \nu(U) = 0 = \mu(V) \tag{3.17}$$

$$g(fX, fY) = g(X, Y) + \nu(X)\nu(Y) - \mu(X)\mu(Y) \tag{3.18}$$

$$g(fX, Y) = g(X, fY), \quad g(X, U) = -\mu(X) \quad \textit{and} \quad g(X, V) = \nu(X) \tag{3.19}$$

for all $X, Y \in TM$, where $\lambda = \eta(N)$.
Thus, we have obtained the following result:

Theorem 3.3 *Every $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold also admits a Lorentzian $(f, g, \mu, \nu, \lambda)$-structure.*

Now, we find a relationship between the induced almost product Lorentzian structure (J, G) and induced $(f, g, \mu, \nu, \lambda)$-structure on transversal hypersurface of Lorentzian almost paracontact manifold.

Theorem 3.4 *If M be a $\mathfrak{T}$-hypersurface of Lorentzian almost paracontact manifold $\mathcal{M}$ equipped with Lorentzian almost paracontact structure (ϕ, η, ξ, g), then we have*

$$\omega = \frac{1}{\lambda}\mu, \tag{3.20}$$

$$J = f - \frac{1}{\lambda}\mu \otimes V, \tag{3.21}$$

$$JU = \frac{1}{\lambda}V, \tag{3.22}$$

$$\mu \circ J = \mu \circ f = \lambda \nu, \tag{3.23}$$

$$JV = fV = \lambda U, \tag{3.24}$$

$$\nu \circ J = \frac{1}{\lambda} \mu \tag{3.25}$$

and

$$G(X, JY) = g(X, fY). \tag{3.26}$$

Proof: As we know that,
$\phi X = JX + \omega(X)\xi$ and $\xi = V + \lambda N$
which implies that

$$\phi X = JX + \omega(X)(V + \lambda N)$$
$$\implies \phi X = JX + \omega(X)V + \omega(X)\lambda N.$$

In account of equation (3.10), we have

$$fX + \mu(X)N = JX + \omega(X)V + \omega(X)\lambda N.$$

On comparing normal and tangential parts respectively, we get

$$\mu(X) = \lambda\omega(X)$$
$$\implies \omega(X) = \frac{1}{\lambda}\mu(X)$$

or

$$\omega = \frac{1}{\lambda}\mu, \quad \forall X \in TM$$

which is equation (3.20) and

$$fX = JX + \omega(X)V$$

using above result i.e. equation (3.20), we have

$$JX = fX - \frac{1}{\lambda}\mu(X)V$$
$$or \qquad J = f - \frac{1}{\lambda}\mu \otimes V, \qquad \forall X \in TM$$

which is equation (3.21).
Now, from equations (3.21),(3.15) and (3.17) we have

$$JU = fU - \frac{1}{\lambda}\mu(U)V$$
$$= \lambda V - \frac{1}{\lambda}(-\lambda^2 - 1)V$$
$$\implies JU = \frac{1}{\lambda}V$$

which is equation (3.22).
Now, from equation (3.21), (3.16) and (3.17) we get

$$(\mu \circ J)(X) = (\mu \circ f)(X) - \frac{1}{\lambda}\mu(X)\mu(V)$$
$$(\mu \circ J)(X) = (\mu \circ f)(X) = \lambda\nu(X) \qquad \{\because \mu(V) = 0\}$$
$$\implies (\mu \circ J) = (\mu \circ f) = \lambda\nu, \qquad \forall X \in TM$$

which is equation (3.23).
Similarly, with the help of equations (3.21), (3.16) and (3.17) we get

$$\nu \circ J = \frac{1}{\lambda}\mu$$

which is equation (3.25).
Now, from (3.21) and (3.17), we get

$$JV = fV - \frac{1}{\lambda}\mu(V)V$$
$$\implies JV = fV = \lambda U$$

which is equation (3.24).
Now, using equations (3.4), (3.13), (3.19) and (3.20) in equation (3.6), we have

$$\begin{aligned} G(X, JY) &= g(X, JY) + \eta(X)\eta(JY) \\ &= g(X, JY) + \eta(X)\omega(Y) \\ &= g(X, fY - \frac{1}{\lambda}\mu(Y)V) + \eta(X)\omega(Y) \\ &= g(X, fY) - \frac{1}{\lambda}\mu(Y)\nu(X) + \frac{1}{\lambda}\mu(Y)\nu(X) \qquad \{\because \omega = \frac{1}{\lambda}\mu, \eta(X) = \nu(X)\} \\ &= g(X, fY) \end{aligned}$$

which is equation (3.26). □

4. Some Properties of $\mathfrak{T}$-Hypersurfaces

Firstly, we state the following lemma:

Lemma 4.1 *Let M be a $\mathfrak{T}$-hypersurface with Lorentzian $(f, g, \mu, \nu, \lambda)$-structure of Lorentzian almost paracontact manifold $\mathcal{M}$, then*

$$(\overline{\nabla}_X\phi)Y = ((\nabla_X f)Y - \mu(Y)HX + h(X, Y)U) + ((\nabla_X\mu)Y + h(X, fY))N, \tag{4.1}$$

$$\overline{\nabla}_X\xi = (\nabla_X V - \lambda HX) + (h(X, V) + X\lambda)N, \tag{4.2}$$

$$(\overline{\nabla}_X\phi)N = (-\nabla_X U + f(HX)) + (-h(X, U) + \mu(HX))N, \tag{4.3}$$

since $h(X, U) + \mu(HX) = 0$, we have $(\overline{\nabla}_X\phi)N = (-\nabla_X U + f(HX)) + 2\mu(HX)N$
or, $(\overline{\nabla}_X\phi)N = (-\nabla_X U + f(HX)) - 2h(X, U)N$
and

$$(\overline{\nabla}_X\eta)Y = (\nabla_X\nu)Y - \lambda h(X, Y), \qquad \forall X, Y \in TM. \tag{4.4}$$

Theorem 4.1 *Let M be a $\mathfrak{T}$-hypersurface with induced Lorentzian $(f, g, \mu, \nu, \lambda)$-structure of Lorentzian para Kenmotsu manifold $\mathcal{M}$, then we have*

$$(\nabla_X f)Y - \mu(Y)HX + h(X,Y)U = -g(fX,Y)V - \nu(Y)fX, \tag{4.5}$$

$$(\nabla_X \mu)Y + h(X, fY) = -\nu(Y)\mu(X) - \lambda g(fX,Y), \tag{4.6}$$

$$\nabla_X V - \lambda HX = -X - \nu(X)V, \tag{4.7}$$

$$h(X,V) + X\lambda = -\lambda\nu(X), \tag{4.8}$$

$$-\nabla_X U + f(HX) = -\lambda fX + \mu(X)V, \qquad h(X,U) = 0 \quad i.e. \quad \mu(HX) = 0 \tag{4.9}$$

and

$$(\nabla_X \nu)Y = \lambda h(X,Y) - g(X,Y) - \nu(X)\nu(Y) \qquad \forall X, Y \in TM. \tag{4.10}$$

Proof: Using equations (4.1), (2.5), (3.12), (3.10) and (3.13), we have

$$\begin{aligned} -g(fX,Y)(V+\lambda N) - \nu(Y)f(X) - \nu(Y)\mu(X)N = {} & ((\nabla_X f)Y - \mu(Y)HX + h(X,Y)U) \\ & + ((\nabla_X \mu)Y + h(X, fY))N. \end{aligned}$$

Now, on comparing tangential and normal parts in above equation, we get the results (4.5) and (4.6) respectively.
Now, from equations (4.2), (2.6), (3.13) and (3.12), we get

$$\begin{aligned} -X - \eta(X)\xi &= (\nabla_X V - \lambda HX) + (h(X,V) + X\lambda)N \\ \implies -X - \nu(X)(V+\lambda N) &= (\nabla_X V - \lambda HX) + (h(X,V) + X\lambda)N, \end{aligned}$$

on comparing tangential and normal parts, we have the required results (4.7) and (4.8) respectively.
Similarly, using equations (2.5), (3.12) and (3.10) in (4.3), we get

$$\begin{aligned} (\overline{\nabla}_X \phi)N &= -g(\phi X, N)\xi - \eta(N)\phi X \\ \implies (-\nabla_X U + f(HX)) - 2h(X,U)N &= -g(X, \phi N)(V+\lambda N) - \lambda(fX + \mu(X)N) \\ &= \mu(X)(V+\lambda N) - \lambda fX - \lambda\mu(X)N \\ &= \mu(X)V + \lambda\mu(X)N - \lambda fX - \lambda\mu(X)N \\ &= \mu(X)V - \lambda fX \end{aligned}$$

On comparing both sides we can easily obtain equation (4.9).
Further, since

$$\begin{aligned} (\overline{\nabla}_X \eta)Y &= \overline{\nabla}_X\{\eta(Y)\} - \eta(\overline{\nabla}_X Y) \\ &= \overline{\nabla}_X\{g(Y,\xi)\} - \eta(\overline{\nabla}_X Y) \\ &= g(\overline{\nabla}_X Y, \xi) + g(Y, \overline{\nabla}_X \xi) - \eta(\overline{\nabla}_X Y) \\ &= g(Y, \overline{\nabla}_X \xi) \\ &= g(Y, -X - \eta(X)\xi) \\ &= -g(Y,X) - \eta(X)\eta(Y). \end{aligned}$$

Hence, we get the required equation (4.10) by using above the result and equation (4.4). □

Theorem 4.2 *If M is a $\mathfrak{T}$-hypersurface with Lorentzian $(f, g, \mu, \nu, \lambda)$-structure of a Lorentzian para Kenmotsu manifold $\mathcal{M}$, then 2-form F on M is given by:*

$$F(X, Y) = g(X, fY)$$

satisfies the following condition:

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 2[-\{g(fY, X)\nu(Z) + g(fZ, Y)\nu(X) + g(fX, Z)\nu(Y)\} \\ + \{\mu(Z)h(X, Y) + \mu(Y)h(X, Z) + \mu(X)h(Y, Z)\}]$$

and consequently, 2-form F is not closed on M.

Proof: In the view of equations (3.9), (3.19), (4.5), we get

$$\begin{aligned}(\nabla_X F)(Y, Z) &= g(Y, (\nabla_X f)Z) \\ &= g(Y, -g(fX, Z)V - \nu(Z)fX + \mu(Z)HX - h(X, Z)U) \\ &= -g(fX, Z)g(Y, V) - g(Y, fX)\nu(Z) + \mu(Z)g(Y, HX) - g(Y, U)h(X, Z) \\ &= -g(fX, Z)\nu(Y) - g(Y, fX)\nu(Z) + \mu(Z)h(X, Y) + \mu(Y)h(X, Z)\end{aligned}$$

Similarly,

$$(\nabla_Y F)(Z, X) = -g(fY, X)\nu(Z) - g(Z, fY)\nu(X) + \mu(X)h(Y, Z) + \mu(Z)h(Y, X)$$

and

$$(\nabla_Z F)(X, Y) = -g(fZ, Y)\nu(X) - g(X, fZ)\nu(Y) + \mu(Y)h(Z, X) + \mu(X)h(Z, Y)$$

which gives

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 2[-\{g(fY, X)\nu(Z) + g(fZ, Y)\nu(X) + g(fX, Z)\nu(Y)\} \\ + \{\mu(Z)h(X, Y) + \mu(Y)h(X, Z) + \mu(X)h(Y, Z)\}]$$

Now, we can easily see that
$dF = (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) \neq 0$
for any $X, Y, Z \in TM$.
Hence, 2-form F is not closed on M. □

5. Example of $\mathfrak{T}$-Hypersurface of Lorentzian Para Kenmotsu Manifold admitting a $(f, g, \mu, \nu, \lambda)$-structure

Let us consider a 5-dimensional manifold $\mathcal{M}^5$ which is defined as follows:

$$\mathcal{M}^5 = \{(x_1, x_2, x_3, x_4, t) \in \mathbb{R}^5 : t > 0, 0 < x_1 < 1\},$$

where (x_1, x_2, x_3, x_4, t) are the standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on $\mathcal{M}^5$ given by
$e_1 = t\frac{\partial}{\partial x_1}$, $e_2 = t\frac{\partial}{\partial x_2}$, $e_3 = t\frac{\partial}{\partial x_3}$, $e_4 = t\frac{\partial}{\partial x_4}$, $e_5 = t\frac{\partial}{\partial t} = \xi$
which are linearly independent at each point of $\mathcal{M}^5$. Let Lorentzian metric tensor, $\tilde{g}$ on $\mathcal{M}^5$ is defined as follows:
$\tilde{g}(e_1, e_1) = 1$, $\tilde{g}(e_2, e_2) = 1$, $\tilde{g}(e_3, e_3) = 1$
$\tilde{g}(e_4, e_4) = 1$, $\tilde{g}(e_5, e_5) = -1$, $\tilde{g}(e_1, e_2) = 0$
$\tilde{g}(e_1, e_3) = 0$, $\tilde{g}(e_1, e_4) = 0$, $\tilde{g}(e_1, e_5) = 0$
$\tilde{g}(e_2, e_3) = 0$, $\tilde{g}(e_2, e_4) = 0$, $\tilde{g}(e_2, e_5) = 0$
$\tilde{g}(e_3, e_4) = 0$, $\tilde{g}(e_3, e_5) = 0$, $\tilde{g}(e_4, e_5) = 0$
Let η be the 1-form such that $\eta(X) = \tilde{g}(X, e_5) = \tilde{g}(X, \xi), \forall X \in \Gamma(T\mathcal{M}^5)$. Now, we define the tensor field

ϕ of (1,1) type such that
$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = -e_4, \quad \phi e_4 = -e_3, \quad \phi e_5 = 0.$
Then, we can easily see that

$$\tilde{g}(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0,$$
$$\eta(e_5) = \eta(\xi) = \tilde{g}(\xi,\xi) = \tilde{g}(t\frac{\partial}{\partial t}, t\frac{\partial}{\partial t}) = -1, \qquad \{\because e_5 = \xi\}$$
$$\phi^2 X = X + \eta(X)\xi$$

and

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X,Y) + \eta(X)\eta(Y), \qquad \forall X, Y \in \Gamma(T\mathcal{M}^5).$$

Thus, $\mathcal{M}^5(\phi,\xi,\eta,\tilde{g})$ defines a Lorentzian almost paracontact manifold. By ∇, we denote the Levi-Civita connection on $\mathcal{M}^5$, then by direct computations, we get
$[e_1,e_2]=0 \quad [e_1,e_3]=0 \quad [e_1,e_4]=0 \quad [e_1,e_5]=-e_1$
$[e_2,e_1]=0 \quad [e_2,e_3]=0 \quad [e_2,e_4]=0 \quad [e_2,e_5]=-e_2$
$[e_3,e_1]=0 \quad [e_3,e_2]=0 \quad [e_3,e_4]=0 \quad [e_3,e_5]=-e_3$
$[e_4,e_1]=0 \quad [e_4,e_2]=0 \quad [e_4,e_3]=0 \quad [e_4,e_5]=-e_4$
$[e_5,e_1]=e_1 \quad [e_5,e_2]=e_2 \quad [e_5,e_3]=e_3 \quad [e_5,e_4]=e_4.$

The Riemannian connection ∇ of Lorentzian metric $\tilde{g}$ is given by:

$$2\tilde{g}(\nabla_X Y, Z) = X\tilde{g}(Y,Z) + Y\tilde{g}(Z,X) - Z\tilde{g}(X,Y) - \tilde{g}(X,[Y,Z]) + \tilde{g}(Y,[Z,X]) + \tilde{g}(Z,[X,Y])$$

which is *Koszul's formula*. Using *Koszul's formula*, we can easily find that:
$\nabla_{e_1}e_1=-e_5 \quad \nabla_{e_1}e_2=0 \quad \nabla_{e_1}e_3=0 \quad \nabla_{e_1}e_4=0 \quad \nabla_{e_1}e_5=-e_1$
$\nabla_{e_2}e_1=0 \quad \nabla_{e_2}e_2=-e_5 \quad \nabla_{e_2}e_3=0 \quad \nabla_{e_2}e_4=0 \quad \nabla_{e_2}e_5=-e_2$
$\nabla_{e_3}e_1=0 \quad \nabla_{e_3}e_2=0 \quad \nabla_{e_3}e_3=-e_5 \quad \nabla_{e_3}e_4=0 \quad \nabla_{e_3}e_5=-e_3$
$\nabla_{e_4}e_1=0 \quad \nabla_{e_4}e_2=0 \quad \nabla_{e_4}e_3=0 \quad \nabla_{e_4}e_4=-e_5 \quad \nabla_{e_4}e_5=-e_4$
$\nabla_{e_5}e_1=0 \quad \nabla_{e_5}e_2=0 \quad \nabla_{e_5}e_3=0 \quad \nabla_{e_5}e_4=0 \quad \nabla_{e_5}e_5=0$

Also, let $X = \sum_{i=1}^{5} X^i e_i$ and we can easily verify that :

$$\nabla_X \xi = -X - \eta(X)\xi, \qquad (\nabla_X \phi)Y = -\tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X$$

Hence, $\mathcal{M}^5(\phi,\xi,\eta,\tilde{g})$ is a *Lorentzian para Kenmotsu manifold.*
Now, consider $(\mathcal{M}^4, g)$, where $\mathcal{M}^4 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : 0 < x_1 < 1\}$, be a hypersurface of $\mathcal{M}^5$ which is given by $\chi : \mathcal{M}^4 \to \mathcal{M}^5$ such that $\chi(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4,logx_1)$. Then, the local basis of tangent hyperplane of $\mathcal{M}^4$ is given by:

$$X_1 = \frac{\partial}{\partial x_1} + \frac{1}{x_1}\frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x_2}, \qquad X_3 = \frac{\partial}{\partial x_3}, \qquad X_4 = \frac{\partial}{\partial x_4}$$

and the unit normal vector field N of hypersurface is given by

$$N = \frac{t}{\sqrt{1-x_1^2}}(\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial t}).$$

Here, it is clear that

$$\chi_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{x_1} & 0 & 0 & 0 \end{bmatrix}$$

which implies that $Rank\ d\chi = Rank\ \chi_* = 4 = dim\ of\ \mathcal{M}^4$. Further, we can see that $\xi_p, p \in \mathcal{M}^4$ is not tangent to the hypersurface. Therefore, $\mathcal{M}^4$ is a transversal hypersurface of $\mathcal{M}^5$. Also, we have

$$\eta(N) = -\frac{x_1}{\sqrt{1-x_1^2}} = \lambda,$$

$$V = \frac{tx_1}{1-x_1^2}\frac{\partial}{\partial x_1} + \frac{t}{1-x_1^2}\frac{\partial}{\partial t}$$

and

$$U = \frac{t}{\sqrt{1-x_1^2}}\frac{\partial}{\partial x_2}.$$

Further, any tangent vector field of the transversal hypersurface $\mathcal{M}^4$ can be expressed as $X = \sum_{i=1}^{4} c_i X_i$, where $c_i, 1 \leq i \leq 4$ are smooth functions. Operating ϕ on both sides we obtain

$$\phi X = -c_2(1+\frac{tx_1}{\sqrt{1-x_1^2}})\frac{\partial}{\partial x_1} - c_1\frac{\partial}{\partial x_2} - c_4\frac{\partial}{\partial x_3} - c_3\frac{\partial}{\partial x_4} - c_2\frac{t{x_1}^2}{\sqrt{1-x_1^2}}\frac{\partial}{\partial t} + c_2 x_1 N = fX + \mu(X)N$$

where, $\mu(X) = c_2 x_1$ and f is given by

$$f = \begin{bmatrix} 0 & -(1+\frac{tx_1}{\sqrt{1-x_1^2}}) & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{t{x_1}^2}{\sqrt{1-x_1^2}} & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\mathcal{M}^4$ is a $\mathfrak{T}$-hypersurface or transversal hypersurface of $\mathcal{M}^5$ which admits $(f, g, \mu, \nu, \lambda)$-structure.

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