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Associated Timelike Helices in Minkowski 3-space

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ABSTRACT: In this study, some new types of timelike general helices associated with a non-lightlike curve are introduced in Minkowski 3-space. These new helices are called associated timelike helices. Some special types of associated timelike helices are presented, and by considering the conditions under which the reference curve is a non-lightlike helix or a spacelike slant helix, the position vectors of these new timelike helices are determined.

Key Words: Associated timelike helix; non-lightlike slant helix; position vector.

Contents

1. Introduction

Lorentzian geometry, closely connected with hyperbolic geometry, holds an important place in both mathematics and physics. The German mathematician Felix Klein is known for his contributions to non-Euclidean geometry, which includes Lorentzian geometry. In 1873, he introduced Lorentzian geometry in his paper titled "Ueber die sogenannte Nicht-Euklidische Geometrie" [17]. Henri Poincaré, recognized as both a mathematician and theoretical physicist, presented three-dimensional Lorentzian geometry and described the Lorentz transformations for the first time using their symmetrical form. In 1908, Hermann Minkowski enhanced the understanding of Lorentzian geometry through his work in relativity and Minkowski spaces [20].

From the past to the present, Lorentzian geometry has maintained its appeal, attracting scientists from various fields such as biology and mathematics who are interested in the theory of curves related to this geometry. From the perspective of differential geometry, one of the most fascinating curves is the helix. This curve has been extensively studied in different spaces by mathematicians and biologists interested in explaining the structure of DNA molecules [2,5,10,11,14,16,19,21,23,24].

A geometric non-lightlike curve is classified as a general helix when the ratio of its non-zero curvature to its non-zero torsion is a constant function. Significant characterizations of what constitutes a general helix in Lorentzian geometry have been provided by Barros et al. and Ferrandez et al. [8,14]. Additionally, Ali has investigated position vectors of general helices in Euclidean 3-space E^3 [3], as well as those of spacelike general helices [2]. Subsequently, Ali and Turgut introduced position vectors for timelike general helices [5].

Izumiya and Takeuchi introduced a new type of special curve known as the slant helix in E^3 [15]. Ali has provided position vectors for slant helices in his work [4]. Furthermore, corresponding definitions and characterizations of slant helices in Lorentzian geometry have been presented by Ali and Lopez [1]. A Lorentzian slant helix is characterized by the property that its principal normal vectors maintain a constant angle with respect to a fixed line, referred to as its axis. A non-lightlike slant helix in Minkowski

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3-space E_1^3 has been characterized by the differential equation of its curvature κ and its torsion τ given by $\frac{\kappa^2}{(\tau^2-\kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$ =constant or $\frac{\kappa^2}{(\kappa^2\pm\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$ =constant [1].

In this work, we aim to study a new type of non-lightlike helix associated with a non-lightlike curve, which we term the non-lightlike associated helix. We classify these curves based on their reference curves: if the main curve is a non-lightlike helix (or slant helix), the corresponding associated helix is referred to as a helix-connected (or slant helix-connected) non-lightlike associated helix. Specifically, in this paper, we focus on associated timelike helices and determine position vectors for these helices in several special cases.

2. Preliminaries

This section provides a brief overview of space curves, along with the definitions of general helices, slant helices, and Darboux helices in Minkowski 3-space E_1^3.

The real vector space $\mathbb{R}^3$ endowed with standard flat Lorentzian metric defined by $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$ is called three-dimensional Minkowski space and denoted by E_1^3, where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. The Lorentzian cross product (or vector product) of two vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ in E_1^3 is given by

$$a \times b = \begin{vmatrix} -i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

A vector $a \in E_1^3$ is called a spacelike vector if $\langle a, a\rangle > 0$ or $a = 0$; called a timelike vector if $\langle a, a\rangle < 0$ and called a null(lightlike) vector if $\langle a, a\rangle = 0,\ a \neq 0$. The norm of the vector a is given by $\|a\| = \sqrt{|\langle a, a\rangle|}$. Similarly, a curve $\alpha(t) : I \subset \mathbb{R} \to E_1^3$ is called a spacelike, timelike or null(lightlike) curve if all of its velocity vectors $\alpha'(t)$ are spacelike, timelike or null(lightlike), respectively [18]. The curves in E_1^3 are related to the principles of relativity in physics. Timelike, spacelike, and null curves correspond to paths of an object moving at less than the speed of light, faster than the speed of light, and at the speed of light, respectively [13].

Let $\{T, N, B\}$ denotes the Frenet frame of a non-lightlike curve α. The curve α can be classified as either timelike or spacelike. Spacelike curves with non-lightlike Frenet vectors are categorized into two types based on the Lorentzian causal characteristics of their Frenet vectors. A spacelike curve is referred to as type 1 (respectively, type 2) if its principal normal vector N (respectively, binormal vector B) is timelike while all other Frenet vectors remain spacelike [9]. For the derivatives of the Frenet frame corresponding to a unit speed non-lightlike curve α with arclength parameter s, the Frenet-Serret formulas have been established separately according to the Lorentzian causal characters of space curves [22]. We can express these formulas succinctly for each case as follows:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \varepsilon_B \kappa & 0 & \tau \\ 0 & \varepsilon_T \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where $\varepsilon_Y = \langle Y, Y\rangle,\ \varepsilon_Y^2 = 1,\ Y \in \{T, N, B\},\ \kappa(s)$ is the curvature(or first curvature) and τ is the torsion (or second curvature) of the curve α defined by $\tau(s) = \varepsilon_N \varepsilon_B \langle B', N\rangle$. The Frenet vectors satisfy the relations $B = \varepsilon_T \varepsilon_N T \times N,\ N = \varepsilon_B \varepsilon_T B \times T,\ T = \varepsilon_N \varepsilon_B N \times B,\ \varepsilon_B = -\varepsilon_T \varepsilon_N$.

The vector W defined by $W(s) = -\varepsilon_B \tau(s) T(s) - \varepsilon_N \kappa(s) B(s)$ is called the Darboux vector of non-lightlike curve α. This vector allows us to write Frenet formulas (2.1) in a different form such as $T' = W \times T,\ N' = W \times N,\ B' = W \times B$.

Unlike Euclidean space, in Minkowski 3-space, four different angle between two non-lightlike vectors have been given in [20] as follows:

Definition 2.1 *Let x and y be spacelike vectors in E_1^3 that span a spacelike vector subspace. Then, we have $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ and hence, there is a unique real number $\theta \geqslant 0$ such that $|\langle x, y\rangle| = \|x\|\,\|y\| \cos\theta$. This number θ is called the Lorentzian spacelike angle between the vectors x and y.*

Definition 2.2 *Let x and y be spacelike vectors in E_1^3 that span a timelike vector subspace. Then, we have $|\langle x,y\rangle| > \|x\|\,\|y\|$ and hence, there is a unique real number $\theta \geqslant 0$ such that $|\langle x,y\rangle| = \|x\|\,\|y\|\cosh\theta$. This number θ is called central angle between the vectors x and y.*

Definition 2.3 *Let x be a spacelike vector and y be a timelike vector in E_1^3. Then, there is a unique non-negative real number θ such that $|\langle x,y\rangle| = \|x\|\,\|y\|\sinh\theta$. This number θ is called the Lorentzian timelike angle between the vectors x and y.*

Definition 2.4 *Let x and y be positive(negative) timelike vectors in E_1^3. Then, there is a unique non-negative real number θ such that $\langle x,y\rangle = \|x\|\,\|y\|\cosh\theta$. This number θ is called the hyperbolic angle between the vectors x and y.*

Let $\psi = \psi(s)$ be a natural parametrization of a non-lightlike unit speed regular curve in E_1^3. Then, ψ is called an i-slant helix if the unit vector $\psi_{i+1} = \frac{\psi_i'}{\|\psi_i'\|}$ makes a constant angle with a constant vector, where $\psi_0(s) = \psi(s)$ [7]. Moreover, in [25], the authors defined the Darboux helix in Euclidean 3-space E^3. We give the corresponding definition for non-lightlike curves as follows: A non-lightlike curve α in E_1^3 is called a non-lightlike Darboux helix if the Darboux vector W of curve α always makes a constant angle with a fixed non-lightlike direction.

3. Associated Timelike Helices in Minkowski 3-space

This section includes the definitions of associated timelike helices and classifications of those curves for some special cases. Also, the position vectors of associated timelike helices are determined.

Let $\alpha : I \to E_1^3$ be a unit speed non-lightlike curve with arc-length parameter s and let $\{T, N, B\}$ and $\kappa(s)$, $\tau(s)$ denote Frenet frame and non-zero curvature functions of α, respectively. Consider non-lightlike curve $\beta : J \to E_1^3$ given by

$$\beta(s) = \alpha(s) + a_1(s)T(s) + a_2(s)N(s) + a_3(s)B(s), \tag{3.1}$$

where $a_i = a_i(s),\ i = 1,2,3$, are smooth functions of s. Then, the curve β is called a non-lightlike associated curve of α. This definition is a generalization for non-lightlike curve pairs and considering with the special properties, parametric forms of well-known curve pairs can be given by (3.1). For instance, if we choose $a_1 = a_3 = 0$ and $a_2 = \lambda$ is a non-zero constant, and assume that the unit principal normals of α and β are linearly dependent, then we obtain the parametric form of a Bertrand mate of non-lightlike curve α [12]. Similarly, with proper chosen of functions $a_i = a_i(s),\ i = 1,2,3$, parametric forms of other famous curve pairs such us Mannheim-partner curves and involute-evolute curves can be given by (3.1).

If the curve β given in (3.1) is a general helix, then it is called a non-lightlike associated helix connected to α. The function $d = d(s)$ defined by $d(s) = \|\beta(s) - \alpha(s)\|$ is called the distance function between the corresponding points of non-lightlike associated curves α and β.

If we differentiate (3.1) with respect to s and consider (2.1), we have

$$\beta'(s) = F_1(s)T(s) + F_2(s)N(s) + F_3(s)B(s), \tag{3.2}$$

where $F_i = F_i(s),\ 1 \leqslant i \leqslant 3$, are given by

$$F_1 = 1 + a_1' + \varepsilon_B a_2\kappa,\ F_2 = a_1\kappa + a_2' + \varepsilon_T a_3\tau,\ F_3 = a_2\tau + a_3'. \tag{3.3}$$

Equations (3.2) and (3.3) allow us to introduce and investigate non-lightlike special associated helices in E_1^3. Privately, in this paper, we study associated timelike helices.

3.1. Helix-Connected Associated(HCA) Timelike Helices

Let α be a non-lightlike general helix and β is a non-lightlike associated curve of α. The curve β is called helix-connected non-lightlike associated curve or HC-NL-associated curve. Since we assume α is a non-lightlike general helix, we can write $\tau(s) = m\kappa(s)$, where m is a non-zero constant. Let now consider special case that β is a timelike curve such that tangent vector β' of β is linearly dependent with unit tangent vector T of α. Then, we have that α is also a timelike curve. For this case, we have $F_1 \neq 0,\ F_2 = F_3 = 0$, and from (3.3), the following system is obtained

$$1 + a_1' + a_2\kappa \neq 0,\ a_1\kappa + a_2' - a_3\tau = 0,\ a_2\tau + a_3' = 0. \tag{3.4}$$

Now, the following special cases can be given:

Case (1): $a_1 = 0$. In this case, system (3.4) becomes,

$$1 + a_2\kappa \neq 0,\ a_2' - a_3\tau = 0,\ a_2\tau + a_3' = 0, \tag{3.5}$$

and solving these equations gives

$$a_1 = 0,\ a_2 = \sin\left(\int \tau(s)ds\right),\ a_3 = \cos\left(\int \tau(s)ds\right). \tag{3.6}$$

By writing (3.6) in (3.1), the position vector of associated timelike curve β is

$$\beta(s) = \alpha(s) + \sin\left(\int \tau(s)ds\right)N(s) + \cos\left(\int \tau(s)ds\right)B(s), \tag{3.7}$$

which gives the followings:

Theorem 3.1 *The associated curve β given in (3.7) is a timelike helix if and only if α is a timelike helix.*

Remark 3.1 *The curve (3.7) can be named by: Helix-connected associated timelike helix of type 1 or HCA-timelike helix of type 1.*

Case (2): $a_2 = 0$. In this case, from (3.4) we have $a_1 = c\frac{\tau}{\kappa}$ and $a_3 = c$, where c is a constant. If we take $c = 0$, it follows $a_i = 0,\ i = 1, 2, 3$, which gives $\beta(s) = \alpha(s)$. So, we take into account the condition that c is non-zero and from (3.1), the parametric form of β is

$$\beta(s) = \alpha(s) + c\left(\frac{\tau(s)}{\kappa(s)}T(s) + B(s)\right), \tag{3.8}$$

and the followings are obtained:

Theorem 3.2 *The associated curve β given in (3.8) is a timelike helix if and only if α is a timelike helix.*

Remark 3.2 *The curve (3.8) can be named by: Helix-connected associated timelike helix of type 2 or HCA-timelike helix of type 2.*

Case (3): $a_3 = 0$. In this case, from system (3.4) we have $a_1 = a_2 = 0$ and from (3.1), it follows $\beta(s) = \alpha(s)$.

On the other hand, we know that if α is a spacelike general helix, its binormal vector B makes a constant angle with a constant vector. For this situation, assume that α is a spacelike curve of type 2. Then, unit binormal vector B is timelike and we can consider the situation that the tangent of associated curve β is linearly dependent with timelike binormal vector B of spacelike curve α. For this case, from (3.3), it follows

$$a_2\tau + a_3' \neq 0,\ 1 + a_1' - a_2\kappa = 0,\ a_1\kappa + a_2' + a_3\tau = 0, \tag{3.9}$$

and following special cases can be written:

Case (4) : $a_1 = 0$. Then, system (3.9) reduces to

$$a_2\tau + a_3' \neq 0,\ 1 - a_2\kappa = 0,\ a_2' + a_3\tau = 0,$$

and the solution is

$$a_1 = 0,\ a_2 = \frac{1}{\kappa(s)},\ a_3 = -\frac{1}{\tau(s)}\left(\frac{1}{\kappa(s)}\right)'.$$

Considering (3.1), the parametric form of β is

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)}N(s) - \frac{1}{\tau(s)}\left(\frac{1}{\kappa(s)}\right)' B(s), \tag{3.10}$$

and followings can be given:

Theorem 3.3 *The associated curve β given in (3.10) is a timelike helix if and only if α is a spacelike helix with unit timelike binormal vector B.*

Remark 3.3 *The curve (3.10) can be named by: Helix-connected associated timelike helix of type 3 or HCA-timelike helix of type 3.*

Case (5) : $a_2 = 0$. In this case, from (3.9) we have,

$$a_3' \neq 0,\ 1 + a_1' = 0,\ a_1\kappa + a_3\tau = 0,$$

and the solution is

$$a_1 = v - s,\ a_2 = 0,\ a_3 = (s - v)\frac{\kappa(s)}{\tau(s)},$$

where v is integration constant. Then, from (3.1), we can write

$$\beta(s) = \alpha(s) + (v - s)\left(T(s) - \frac{\kappa(s)}{\tau(s)}B(s)\right). \tag{3.11}$$

Now, we can write followings:

Theorem 3.4 *The associated curve β given in (3.11) is a timelike helix if and only if α is a spacelike helix with unit timelike binormal vector B.*

Remark 3.4 *The curve (3.11) can be named by: Helix-connected associated helix of type 4 or HCA-timelike helix of type 4.*

Case (6) : $a_3 = 0$. For this case, system (3.9) gives us

$$a_2\tau \neq 0,\ 1 + a_1' - a_2\kappa = 0,\ a_1\kappa + a_2' = 0. \tag{3.12}$$

From the last equation, we obtain

$$a_1' = -\frac{1}{\kappa}a_2'' - \left(\frac{1}{\kappa}\right)' a_2', \tag{3.13}$$

and by writing this result in the second equation of system (3.12), it follows

$$a_2'' + \kappa\left(\frac{1}{\kappa}\right)' a_2' + \kappa^2 a_2 - \kappa = 0. \tag{3.14}$$

Making the variable change $\mu = \int \kappa(s)ds$, the homogeneous part of (3.14) becomes

$$\frac{d^2 a_2}{d\mu^2} + a_2 = 0,$$

and it has the solution

$$a_{2h}(s) = c_1 \cos\left(\int \kappa(s)ds\right) + c_2 \sin\left(\int \kappa(s)ds\right), \tag{3.15}$$

where $c_1,\ c_2$ are real constants. Now, considering (3.15), the solution of (3.14) is

$$\begin{aligned} a_2(s) \quad = \quad & c_1 \cos\left(\int \kappa(s)ds\right) + c_2 \sin\left(\int \kappa(s)ds\right) \\ & - \cos\left(\int \kappa(s)ds\right) \int \left[\sin\left(\int \kappa(s)ds\right)\right] ds \\ & + \sin\left(\int \kappa(s)ds\right) \int \left[\cos\left(\int \kappa(s)ds\right)\right] ds. \end{aligned} \tag{3.16}$$

By direct computation from (3.16) and (3.12), it follows

$$\begin{aligned} a_1(s) \quad = \quad & \sin\left(\int \kappa(s)ds\right) \left[\int \left[\sin\left(\int \kappa(s)ds\right)\right] ds - c_1\right] \\ & - \cos\left(\int \kappa(s)ds\right) \left[\int \left[\cos\left(\int \kappa(s)ds\right)\right] ds + c_2\right]. \end{aligned} \tag{3.17}$$

Then, we can write

$$\beta(s) = \alpha(s) + a_1(s)T(s) + a_2(s)N(s), \tag{3.18}$$

where a_1 and a_2 are given in (3.17) and (3.16), respectively. Now, we have the followings:

Theorem 3.5 *The associated curve β given in (3.18) is a timelike helix if and only if α is a spacelike helix with unit timelike binormal vector B.*

Remark 3.5 *The curve (3.18) can be named by: Helix-connected associated timelike helix of type 5 or HCA-timelike helix of type 5.*

Furthermore, let $d_i(s),\ 1 \leqslant i \leqslant 5$, denotes the distance function for HCA-timelike helix of type i. Then, for $d_5(s)$, we have $d_5(s) = \sqrt{a_1^2(s) + a_2^2(s)}$. It is clear that d_5 is constant if and only if $a_1 a_1' + a_2 a_2' = 0$. On the other hand, multiplying the second and third equalities in (3.12) with a_1 and a_2, respectively, and adding the results gives $a_1 a_1' + a_2 a_2' = -a_1$. Then, it follows that d_5 is constant if and only if $a_1 = 0$. Similarly, by considering the differentiation of other distance functions, we can give the followings:

Corollary 3.1 *(i) The distance functions d_1 and d_2 are constants at all points.*

(ii) The distance function d_3 is constant if and only if κ is constant or $\frac{1}{\kappa}\left(\frac{1}{\kappa}\right)' = m^2 s + c$, where c is integration constant.

(iii) The distance function d_4 is constant and equal to zero at the intersection points $\beta(v) = \alpha(v)$ and non-constant at all other points.

(iv) The distance function d_5 is constant if and only if $a_1 = 0$.

Let now determine the position vector of a HCA-timelike helix. For this, first assume that main curve α is a timelike helix with $\left|\frac{\tau(s)}{\kappa(s)}\right| < 1$. Without loss of generality, we can choose the spacelike base vector $e_3 = (0, 0, 1)$ as the axis of timelike helix α. Then, we have the following theorem:

Theorem 3.6 *([5]) The position vector ψ of a timelike general helix whose tangent vector makes a constant Lorentzian angle with a fixed spacelike straight line in the space, is computed in the natural representation form:*

$$\psi(s) = \sqrt{1+n^2} \int \left(\cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], m\right) ds, \tag{3.19}$$

or in the parametric form as follows:

$$\psi(\theta) = \int \frac{\sqrt{1+n^2}}{\kappa(\theta)} \left(\cosh\left[\sqrt{1-m^2}\theta\right], \sinh\left[\sqrt{1-m^2}\theta\right], m \right) d\theta, \tag{3.20}$$

where $\theta(s) = \int \kappa(s)ds$, $m = \frac{n}{\sqrt{1+n^2}}$, $n = \sinh\varphi$ *and* φ *is the constant angle between the fixed spacelike straight line* e_3 *(axis of timelike general helix) and the tangent vector of the curve* ψ*.*

From (3.19), the Frenet vectors of timelike helix α are calculated as follows:

$$T(s) = \sqrt{1+n^2}\left(\cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], m \right),$$

$$N(s) = \left(\sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], \cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], 0 \right),$$

$$B(s) = \sqrt{1+n^2}\left(m\cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], m\sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right], 1 \right).$$

Now, we can give the followings for the special cases 1 and 2:

Theorem 3.7 *The position vector* $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ *of the HCA-timelike helix of type 1 with spacelike axis* e_3*, is obtained as*

$$\begin{aligned}\beta_1(s) &= \sqrt{1+n^2}\int \cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] ds \\ &\quad + \sin\left[m\int \kappa(s)ds\right] \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] \\ &\quad + n\cos\left[m\int \kappa(s)ds\right] \cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right],\end{aligned}$$

$$\begin{aligned}\beta_2(s) &= \sqrt{1+n^2}\int \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] ds \\ &\quad + \sin\left[m\int \kappa(s)ds\right] \cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] \\ &\quad + n\cos\left[m\int \kappa(s)ds\right] \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right],\end{aligned}$$

$$\beta_3(s) = (ns+l) + \sqrt{1+n^2}\cos\left[m\int \kappa(s)ds\right],$$

where l *is integration constant.*

Theorem 3.8 *The position vector* $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ *of the HCA-timelike helix of type 2 with spacelike axis* e_3*, is obtained as*

$$\beta_1(s) = \sqrt{1+n^2}\int \cosh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] ds,$$

$$\beta_2(s) = \sqrt{1+n^2}\int \sinh\left[\sqrt{1-m^2}\int \kappa(s)ds\right] ds,$$

$$\beta_3(s) = (ns+l) + c(nm - \sqrt{1+n^2}),$$

where l *is integration constant.*

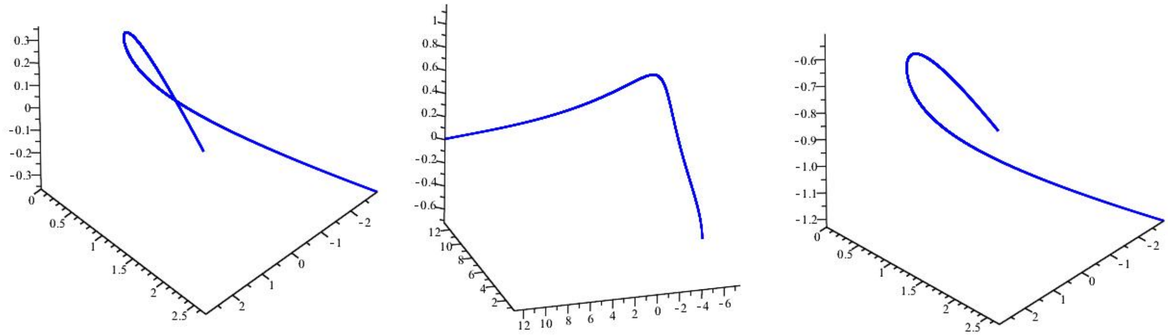

Figure 1: From left to right, timelike helix α and HCA-timelike helices of type 1 and 2, respectively.

For $n = \frac{\sqrt{3}}{3}$, $m = 1/2$, the graphs of timelike general helix α with curvatures $\kappa = 6$, $\tau = 3$, and HCA-timelike helices of type 1 and of type 2 with $c = 1$ are given in left side, middle and right sight of Fig. 1, respectively.

In Theorem 3.6, if the curve ψ is a timelike helix with a timelike axis e_1, then the natural representation form of ψ is

$$\psi(s) = \sqrt{n^2-1}\int\left(m, \cos\left[\sqrt{m^2-1}\int\kappa(s)ds\right], \sin\left[\sqrt{m^2-1}\int\kappa(s)ds\right]\right)ds,$$

where $m = \frac{n}{\sqrt{n^2-1}}$, $n = \cosh\varphi$ and φ is the constant Lorentzian angle between the fixed timelike straight line e_1(axis of timelike general helix) and the tangent vector of the curve ψ [5]. Then, corresponding theorems for HCA-timelike helices of type 1 and 2 can be given similarly.

If the curve ψ is a spacelike helix with a timelike binormal vector, then there exists two situations [2]:

ii) If ψ is a spacelike curve of type 2 with $|\tau(s)/\kappa(s)| > 1$, the axis is spacelike and constant angle between the unit tangent of curve and axis is a timelike angle, then the natural representation form of ψ is

$$\psi(s) = \sqrt{n^2-1}\int\left(\cosh\left[\sqrt{m^2-1}\int\kappa(s)ds\right], \sinh\left[\sqrt{m^2-1}\int\kappa(s)ds\right], m\right)ds,$$

where $m = \frac{n}{\sqrt{n^2-1}}$, $n = \cosh\varphi$ and φ is the constant Lorentzian angle between the fixed spacelike straight line e_3(axis of spacelike general helix of type 1) and the tangent vector of the curve ψ.

iii) If ψ is a spacelike curve of type 2 with $|\tau(s)/\kappa(s)| < 1$, the axis is timelike and constant angle between the unit tangent of curve and axis is a timelike angle, then the natural representation form of ψ is

$$\psi(s) = \sqrt{n^2+1}\int\left(m, \cos\left[\sqrt{1-m^2}\int\kappa(s)ds\right], \sin\left[\sqrt{1-m^2}\int\kappa(s)ds\right]\right)ds,$$

where $m = \frac{n}{\sqrt{n^2+1}}$, $n = \sinh\varphi$ and φ is the timelike angle between the fixed timelike straight line e_1(axis of spacelike general helix of type 2) and the tangent vector of the curve ψ. Then, from (3.10), (3.11) and (3.18), the position vectors of HCA-timelike helices of types 3, 4 and 5 can be determined similarly.

3.2. Slant helix-connected associated (SHCA) timelike helices

Suppose that the vector field β' of the non-lightlike curve β is linearly dependent with unit vector field N of main curve α. For Lorentzian causal character of β we have followings:

i) β is a timelike curve if and only if α is a spacelike curve of type 1.

ii) β is a spacelike curve if and only if α is a timelike curve or a spacelike curve of type 2.

Now, from (3.2), we have $\beta'(s) = F_2(s)N(s)$ and the arc-length parameter s_β of curve β is obtained as $ds_\beta = \pm F_2(s)ds$. Using that, the Frenet vectors of β are calculated as follows:

$$\begin{cases} T_\beta(s) = \pm N(s),\ N_\beta(s) = \frac{1}{\sqrt{|\varepsilon_T\kappa^2(s)+\varepsilon_B\tau^2(s)|}}(\varepsilon_B\kappa(s)T(s) + \tau(s)B(s)), \\ B_\beta(s) = \frac{\mp 1}{\sqrt{|\varepsilon_T\kappa^2(s)+\varepsilon_B\tau^2(s)|}}(-\varepsilon_B\tau(s)T(s) - \varepsilon_N\kappa(s)B(s)) = \frac{\mp W(s)}{\sqrt{|\varepsilon_T\kappa^2(s)+\varepsilon_B\tau^2(s)|}}, \end{cases} \tag{3.21}$$

where $W(s)$ is the Darboux vector of main curve α. Now, we have the following theorem:

Theorem 3.9 *Let β be a non-lightlike curve such that tangent vector β' of β is linearly dependent with unit principal normal vector $N(s)$ of main curve α. The followings are equivalent,*
(i) β is a non-lightlike helix.
(ii) α is a non-lightlike slant helix.
(iii) α is a non-lightlike Darboux helix.

Remark 3.6 *The helix β associated to slant helix α can be named by: Slant helix-connected associated non-lightlike helix or SHCA-non-lightlike helix.*

Now, we will consider the case that β is a timelike curve and introduce special types of SHCA-timelike helices. For this purpose, let assume that α is a spacelike curve of type 1, i.e., principal normal $N(s)$ is timelike. Then, from (3.2) we have that $F_2 \neq 0$, $F_1 = F_3 = 0$ and from (3.3), the following system is obtained

$$a_1\kappa + a_2' + a_3\tau \neq 0,\ 1 + a_1' + a_2\kappa = 0,\ a_2\tau + a_3' = 0, \tag{3.22}$$

and some special cases are given as follows:

Case (A): $a_1 = 0$. In this case, system (3.22) reduces to

$$a_2' + a_3\tau \neq 0,\ 1 + a_2\kappa = 0,\ a_2\tau + a_3' = 0. \tag{3.23}$$

Solving these equations, it follows

$$a_1 = 0,\ a_2 = -\frac{1}{\kappa(s)},\ a_3 = \int \frac{\tau(s)}{\kappa(s)}ds. \tag{3.24}$$

By writing (3.24) in (3.1), the position vector of associated timelike curve β is obtained as

$$\beta(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s) + \left(\int \frac{\tau(s)}{\kappa(s)}ds\right)B(s), \tag{3.25}$$

and following results are obtained:

Theorem 3.10 *The associated curve β given in (3.25) is a timelike helix if and only if α is a spacelike slant helix with timelike principal normal N.*

Remark 3.7 *The curve β given in (3.25) can be named by: Slant helix-connected associated timelike helix of type 1 or SHCA-timelike helix of type 1.*

Case (B): $a_2 = 0$. Then, system (3.22) becomes

$$a_1\kappa + a_3\tau \neq 0,\ 1 + a_1' = 0,\ a_3' = 0, \tag{3.26}$$

and it has the solution

$$a_1 = \xi - s,\ a_2 = 0,\ a_3 = \zeta, \tag{3.27}$$

where ξ and ζ are real constants. By writing (3.27) in (3.1), the position vector of timelike curve β is

$$\beta(s) = \alpha(s) + (\xi - s)T(s) + \zeta B(s), \tag{3.28}$$

and following theorem is obtained:

Theorem 3.11 *The associated curve β given in (3.28) is a timelike helix if and only if α is a spacelike slant helix with timelike principal normal N.*

Remark 3.8 *The curve (3.28) can be named by: Slant helix-connected associated timelike helix of type 2 or SHCA-timelike helix of type 2.*

Case (C): $a_3 = 0$. From (3.22), it follows

$$a_1\kappa + a_2' \neq 0,\ 1 + a_1' + a_2\kappa = 0,\ a_2\tau = 0, \tag{3.29}$$

which has the solution

$$a_1 = \omega - s,\ a_2 = 0,\ a_3 = 0, \tag{3.30}$$

where ω is integration constant. Writing (3.30) in (3.1) gives the position vector of β as follows,

$$\beta(s) = \alpha(s) + (\omega - s)T(s). \tag{3.31}$$

Then, the following theorem is given:

Theorem 3.12 *The associated curve β given in (3.31) is a timelike helix if and only if α is a spacelike slant helix with timelike principal normal N.*

Remark 3.9 *The curve (3.31) can be named by: Slant helix-connected associated timelike helix of type 3 or SHCA-timelike helix of type 3.*

Furthermore, by considering (3.25), (3.28) and (3.31), the followings can be given:

Corollary 3.2 *SHCA-timelike helix of type 2 is also a SHCA-timelike helix of type 3 if and only if $\zeta = 0$.*

Let $d_1,\ d_2$, and d_3 denote the distance functions for timelike associated curves (3.25), (3.28) and (3.31), respectively. Then, we have

$$\begin{aligned} d_1(s) &= \sqrt{\left| -\left(\frac{1}{\kappa(s)}\right)^2 + \left(\int \frac{\tau(s)}{\kappa(s)} ds\right)^2 \right|}, \\ d_2(s) &= \sqrt{(\xi - s)^2 + \zeta^2},\ d_3(s) = \sqrt{(\omega - s)^2}. \end{aligned}$$

By differentiating these functions, the following corollary can be given:

Corollary 3.3 *(i) For the distance function $d_1(s)$ the followings are equivalent:*
(a) $d_1 = c$ is a constant. (b) $\tau\left(\int \frac{\tau}{\kappa} ds\right) - \left(\frac{1}{\kappa}\right)' = 0$. (c) $\tau = \frac{g'}{\sqrt{g^2 + c^2}}$, where $g = \frac{1}{\kappa}$.
(ii) d_2 is constant and equal to ζ at points $\alpha(\zeta)$ and non-constant at all other points.
(iii) d_3 is constant and equal to zero at the intersection points $\beta(\omega) = \alpha(\omega)$ and non-constant at all other points.

Let now determine the position vector of a SHCA-timelike helix. For this, first assume that main curve α is a spacelike slant helix with timelike principal normal N. Without loss of generality, we can choose the spacelike base vector $e_3 = (0, 0, 1)$ as the axis of spacelike slant helix α. Then, we have the following theorem:

Theorem 3.13 *([6]). The position vector $\psi(s)$ of a spacelike slant helix, whose timelike principal normal vector makes a constant Lorentzian angle with a fixed spacelike straight line e_3 is computed in the natural representation form:*

$$\begin{aligned} \psi(s) = &\left(\frac{n}{m}\int\left[\int \kappa(s)\cosh[A]\,ds\right]ds, \frac{n}{m}\int\left[\int \kappa(s)\sinh[A]\,ds\right]ds,\right. \\ &\left. n\int\left[\int \kappa(s)ds\right]ds\right), \end{aligned} \tag{3.32}$$

where $A = \frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)$, $\tau(s) = \pm\frac{m\kappa(s)\int\kappa(s)ds}{\sqrt{1-m^2\left(\int\kappa(s)ds\right)^2}}$, $m = \frac{n}{\sqrt{1+n^2}}$, $n = \sinh\varphi$ *and* φ *is the constant Lorentzian angle between the axis of spacelike slant helix and timelike principal normal vector of the curve* ψ.

From (3.32) the Frenet vectors of slant helix α are computed as follows:

$$T(s) = \left(\frac{n}{m}\int\kappa(s)\cosh[A]\,ds, \frac{n}{m}\int\kappa(s)\sinh[A]\,ds, n\int\kappa(s)ds\right), \tag{3.33}$$

$$N(s) = \left(\frac{n}{m}\cosh[A], \frac{n}{m}\sinh[A], n\right), \tag{3.34}$$

$$\begin{aligned} B(s) \;=\; & \left(\frac{n^2}{m}\sinh[A]\int\kappa(s)ds - \frac{n^2}{m}\int\kappa(s)\sinh[A]\,ds\right. \\ & +\frac{n^2}{m}\cosh[A]\int\kappa(s)ds - \frac{n^2}{m}\int\kappa(s)\cosh[A]\,ds \\ & \left.+\frac{n^2}{m^2}\sinh[A]\int\kappa(s)\cosh[A]\,ds - \frac{n^2}{m^2}\cosh[A]\int\kappa(s)\sinh[A]\,ds\right). \end{aligned} \tag{3.35}$$

Now, by writing (3.33)-(3.35) in (3.25), (3.28) and (3.31), respectively, we can give the following theorems:

Theorem 3.14 *The position vector* $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ *of the SHCA-timelike helix of type 1 with spacelike axis* e_3 *and constant spacelike angle* φ *between the unit tangent of SHCA-timelike helix and* e_3*, is obtained as*

$$\begin{aligned} \beta_1(s) \;=\; & \frac{n}{m}\int\left[\int\kappa(s)\cosh[A]\,ds\right]ds - \frac{n}{m\kappa(s)}\cosh[A] \\ & -\frac{n^2}{m}(ms+c_1)\left(\int\kappa(s)\sinh[A]\,ds - \sinh[A]\int\kappa(s)ds\right), \end{aligned}$$

$$\begin{aligned} \beta_2(s) \;=\; & \frac{n}{m}\int\left[\int\kappa(s)\sinh[A]\,ds\right]ds - \frac{n}{m\kappa(s)}\sinh[A] \\ & -\frac{n^2}{m}(ms+c_2)\left(\int\kappa(s)\cosh[A]\,ds - \cosh[A]\int\kappa(s)ds\right), \end{aligned}$$

$$\begin{aligned} \beta_3(s) \;=\; & n\int\left[\int\kappa(s)ds\right]ds - \frac{n}{\kappa(s)} \\ & +\frac{n^2}{m^2}(ms+c_3)\left(\sinh[A]\int\kappa(s)\cosh[A]\,ds - \cosh[A]\int\kappa(s)\sinh[A]\,ds\right), \end{aligned}$$

where c_i, $1 \leqslant i \leqslant 3$ *are integration constants.*

Theorem 3.15 *The position vector* $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ *of the SHCA-timelike helix of type 2 with spacelike axis* e_3 *and constant spacelike angle* φ *between the unit tangent of SHCA-timelike helix and* e_3*, is obtained as*

$$\begin{aligned} \beta_1(s) \;=\; & \frac{n}{m}\int\left[\int\kappa(s)\cosh[A]\,ds\right]ds + \frac{n}{m}(\xi - s)\int\kappa(s)\cosh[A]\,ds \\ & +\zeta\frac{n^2}{m}\left(\sinh[A]\int\kappa(s)ds - \int\kappa(s)\sinh[A]\,ds\right), \end{aligned}$$

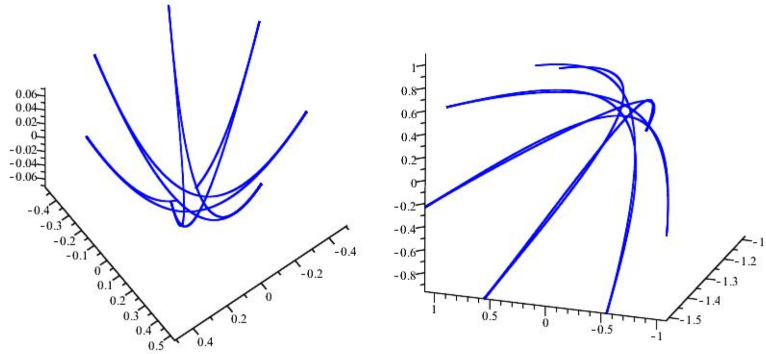

Figure 2: Spacelike slant helix α (left) and SHCA-timelike helix of type 1(right).

$$\begin{aligned}\beta_2(s) &= \frac{n}{m}\int\left[\int \kappa(s)\sinh[A]\,ds\right]ds+\frac{n}{m}(\xi-s)\int\kappa(s)\sinh[A]\,ds\\ &\quad+\zeta\frac{n^2}{m}\left(\cosh[A]\int\kappa(s)ds-\int\kappa(s)\cosh[A]\,ds\right),\end{aligned}$$

$$\begin{aligned}\beta_3(s) &= n\int\left[\int \kappa(s)ds\right]ds+n(\xi-s)\int\kappa(s)ds\\ &\quad+\zeta\frac{n^2}{m^2}\left(\sinh[A]\int\kappa(s)\cosh[A]\,ds-\cosh[A]\int\kappa(s)\sinh[A]\,ds\right).\end{aligned}$$

Theorem 3.16 *The position vector $\beta(s)=(\beta_1(s),\beta_2(s),\beta_3(s))$ of the SHCA-timelike helix of type 3 with spacelike axis e_3 and constant spacelike angle φ between the unit tangent of SHCA-timelike helix and e_3, is obtained as*

$$\beta_1(s)=\frac{n}{m}\int\left[\int \kappa(s)\cosh[A]\,ds\right]ds+\frac{n}{m}(\omega-s)\int\kappa(s)\cosh[A]\,ds,$$

$$\beta_2(s)=\frac{n}{m}\int\left[\int \kappa(s)\sinh[A]\,ds\right]ds+\frac{n}{m}(\omega-s)\int\kappa(s)\sinh[A]\,ds,$$

$$\beta_3(s)=n\int\left[\int \kappa(s)ds\right]ds+n(\omega-s)\int\kappa(s)ds.$$

For and $N=\frac{2\sqrt{3}}{3}$ and $m=2$; the graphs of spacelike slant helix α with timelike principal normal N and curvatures $\kappa(t)=1$, $\tau(t)=\cot\left(\frac{2\sqrt{3}}{3}t\right)$ and SHCA-timelike helix of type 1 are given in left side and right side of Fig. 2, respectively. The graphs of SHCA-timelike helices of type 2 and type 3 are given in left side and right side of Fig. 3, respectively.

Moreover, if ψ is a spacelike slant helix with timelike principal normal N, the axis is timelike vector e_1 and the constant angle between the timelike vectors N and e_1 is Lorentzian timelike angle, the natural representation form of ψ is

$$\psi(s)=n\int\left(\int\kappa(s)ds,\frac{1}{m}\int\kappa(s)\cos[A]\,ds,\frac{1}{m}\int\kappa(s)\sin[A]\,ds\right)ds,$$

where $A=\frac{1}{n}\arcsin\left(m\int\kappa(s)ds\right)$, $\tau(s)=\pm\frac{m\kappa(s)\int\kappa(s)ds}{\sqrt{m^2(\int\kappa(s)ds)^2-1}}$, $m=\frac{n}{\sqrt{n^2-1}}$, $n=\cosh\varphi$ and φ is the constant Lorentzian timelike angle between the fixed timelike straight line e_1(axis of timelike slant helix)

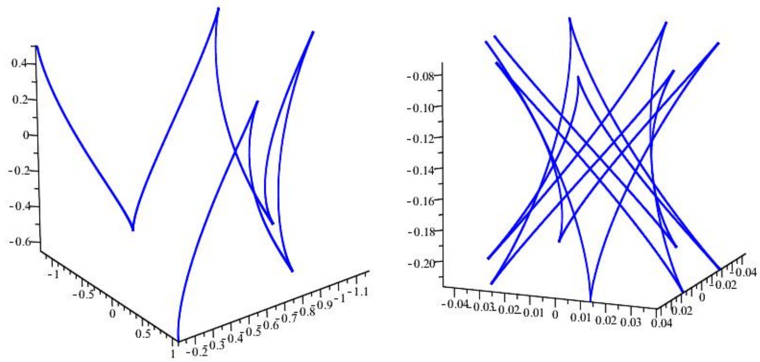

Figure 3: SHCA-timelike helices of type 2(left) and type 3(right).

and the tangent vector of the curve ψ [6]. Then, using (3.25), (3.28) and (3.31), the position vectors of SHCA-timelike helices of types 1, 2 and 3 can be written similarly as given in Theorems 5.6-5.8.

4. Conclusions

A new type of non-lightlike general helix called the associated non-lightlike helix has been introduced. These are special helices related to a non-lightlike reference curve and have different types according to the reference curve. If the main curve is a timelike helix or a spacelike helix with a timelike binormal (respectively, if it is a spacelike slant helix with a timelike principal normal), then the associated timelike helix is called helix-connected associated (HCA) (respectively, slant helix-connected associated (SHCA)) timelike helix. Furthermore, some special types of HCA- and SHCA-timelike helices have been defined, and the position vectors of these curves have been introduced. A similar definition can be given for associated spacelike helices, and corresponding theorems can be established by considering the Lorentzian characteristics of these associated curves.

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