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Orbits of Random Dynamical Systems

Ahmed Zaou, Otmane Benchiheb and Mohamed Amouch

ABSTRACT: In this paper, we introduce and study the notions of hypercyclicity and transitivity for random dynamical systems and we establish the relation between them in a topological space. We also introduce the notions of mixing and weakly mixing for random dynamical systems and give some of their properties.

Key Words: Hypercyclicity, topological transitivity, Orbit, random dynamical system.

Contents

1. Introduction

Throughout the paper, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ will denote the set of positive integers while $\mathbb{N} = \{1, 2, 3, \ldots\}$ will be the set of nonzero positive integers.

Let X be an F-space that is a complete and metrizable topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let T be a continuous linear operator (operator for short) acting on X. If x is vector of X, then the orbit of x under T is the set denoted by $\text{Orb}(T, x)$ and defined by

$$\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}_0\}.$$

We say that T is hypercyclic if there exists a vector $x \in X$ whose orbit under T is dense in X. In this case, the vector x is called a hypercyclic vector for T. We denote by $HC(T)$ the set of all hypercylic vectors for T. The first example of a hypercyclic operator in the Banach space setting was given by Rolewicz [20], who proved that if $\lambda \in \mathbb{C}$; $|\lambda| > 1$, then λB is hypercyclic, where B is the unilateral backward shift with weights constantly equal to 1. Rolewicz also proved that there are no hypercyclic operators on finite-dimensional space. Thus hypercyclicity is an infinite-dimensional phenomenon. If the space X is a separable space, then the hypercyclicity is equivalent to the notion of topological transitivity, that is; for any pair (U, V) of nonempty and open sets of X, there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

In this case, the set $HC(T)$ is a dense G_δ subset of X, see [12].

A useful general criterion for hypercyclicity was isolated by C. Kitai in a restricted form [14] and then by R. Gethner and J. H. Shapiro in a form close to that given below [16]. The version used here appeared in the Ph.D. thesis of J. Bes [10]: we say that T satisfies the hypercyclicity criterion if there exist an increasing sequence of integers (n_k), two dense sets X_0, $Y_0 \subset X$ and a sequence of maps $S_{n_k} : Y_0 \longrightarrow X$ such that:

(1) $T^{n_k} x \longrightarrow 0$ for any $x \in X_0$;

(2) $S_{n_k} y \longrightarrow 0$ for any $y \in Y_0$;

(3) $T^{n_k} S_{n_k} y \longrightarrow y$ for any $y \in Y_0$.

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Such an operator satisfying the hypercyclicity criterion is hypercyclic, see [10].

It is known that if $T \oplus S$ is hypercyclic on $X \oplus Y$, then T is hypercyclic on X and S is hypercyclic on Y. The converse is not true even in the case $T = S$ see [13]. From [17] if $T \oplus T$ is topologically transitive, then the operator T is called weakly mixing, i.e, $T \oplus T$ is hypercyclic. Clearly a weakly mixing operator is hypercyclic. Moreover, the following are equivalent:

(1) T satisfies the hypercyclicity criterion;

(2) T is hereditarily hypercyclic with respect to an increasing sequence of positive integers (n_k), that is, for any subsequence (m_k) of (n_k), the sequence $(T^{m_k})_{k\in\mathbb{N}_0}$ is hypercyclic;

(3) T is weakly mixing,

see [11]. The notions of hypercyclicity and supercyclicity are well studied in the last few years, see for example K.G. Grosse-Erdmann and A. Peris's book [17] and F. Bayart and E. Matheron's book [9], and the survey article [18] by K.G. Grosse-Erdmann, and the book [15] by Kostić. In [1,2,3,4,5,6,7,8] the authors have studied the dynamics of a set of operators instead of a single operator. In this paper, we introduce the notions of hyperciclycity, topological transitivity, and topological mixing of random dynamical systems and we study some of their properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{T} = \{T_\omega : X \longrightarrow X, \omega \in \Omega\}$ is a collection of measurable maps on a Polish space X. We will refer to $(\Omega, \mathcal{F}, \mathcal{T})$ as random dynamical system and we denote it in the following by $\mathcal{T}$.
By taking, $T^n_{\underline{\omega}} = T_{\omega_n} \circ \cdots \circ T_{\omega_1}$ for any $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega^{\mathbb{N}_0}$, we can relate this random dynamical system to a deterministic dynamical system obtained by defining the following skew-product transformation:

$$\begin{array}{rccc} S & : & \Omega^{\mathbb{N}_0} \oplus X & \longrightarrow & \Omega^{\mathbb{N}_0} \oplus X \\ & & (\underline{\omega}, x) & \longmapsto & (\sigma\underline{\omega}, T_{\omega_1} x), \end{array}$$

where $\sigma : \Omega^{\mathbb{N}_0} \mapsto \Omega^{\mathbb{N}_0}$ is the unilateral shift. It is clear that $S^n(\underline{\omega}, x) = (\sigma^n \underline{\omega}, T^n_{\underline{\omega}} x)$, for any $n \in \mathbb{N}_0$. A probability measure μ on X is stationary if and only if the measure $\mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu$ is invariant under S that is $S^*(\mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu) = \mathbb{P}^{\oplus\mathbb{N}_0} \oplus \mu$, see [19].

Hereinafter, X will be a topological space and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a collection of continuous functions that map X into itself. In this case, the orbit of a point $x \in X$ at some $\underline{\omega} \in \Omega^{\mathbb{N}}_0$ of this random dynamical system is defined by

$$\mathrm{Orb}(x, \mathcal{T}) = \{T^n_{\underline{\omega}} x : n \in \mathbb{N}_0\},$$

where $T^0_{\underline{\omega}} x = x$.

2. Hypercyclic and Topologically Transitive Random Dynamical Systems

In the following, we define the notion of hypercyclicity for a random dynamical system.

Definition 2.1. *Let X be a toplogical space. We say that a random dynamical system $\mathcal{T}$ is hypercyclic on X if there exists $x \in X$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that*

$$\overline{\mathrm{Orb}(x, \mathcal{T})} = X.$$

In such a case, x is called a hypercyclic point for $\mathcal{T}$, and the set of hypercyclic points for $\mathcal{T}$ is denoted by HC($\mathcal{T}$).

Remark 2.2. *Let X be a topological space, and $T : X \to X$ be a continuous map on X. If we take $T_\omega = T$ for any $\omega \in \Omega$, then $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is hypercyclic if only if T is hypercyclic.*

Example 2.3. *We pose $X = [0, 1]$ and $\Omega = \{1, 2\}$, and we consider the maps:*

$$\begin{array}{rccc} T_1 & : & X & \longrightarrow & X \\ & & x & \longmapsto & \begin{cases} 2x & if\ x \in [0, \frac{1}{2}] \\ 2 - 2x & if\ x \in]\frac{1}{2}, 1] \end{cases} \end{array},$$

and

$$\begin{array}{rcccc} T_2 & : & X & \longrightarrow & X \\ & & x & \longmapsto & x+\alpha (mod\ 1), \end{array}$$

where $\alpha \in [0,1[$. *There exists* $x \in X$ *such that* $\overline{\{T_1^n x, n \in \mathbb{N}_0\}} = X$. *Let* $\underline{\omega} = (1,1,1,\dots)$, *then*

$$\overline{\{T_{\underline{\omega}}^n x, n \in \mathbb{N}_0\}} = X.$$

Hence, $\mathcal{T} = \{T_1, T_2\}$ *is hypercyclic on* X.

In the following definition, we introduce the notion of quasi-conjugate for a random dynamical system.

Definition 2.4. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectivly.* $\mathcal{T}$ *is called quasiconjugate to* $\mathcal{S}$ *if there exists a continuous map* $\phi : Y \to X$ *with dense range such that for all* $\omega \in \Omega$, $T_\omega \circ \phi = \phi \circ S_\omega$. *If* ϕ *can be chosen to be a homeomorphism then* $\mathcal{S}$ *and* $\mathcal{T}$ *are called conjugate.*

The property of hypercylicity of a dynamical system is preserved under quasiconjugacy, see [10, Proposition 1.19]. The following proposition proves that the same result holds for a random dynamical system.

Proposition 2.5. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is quasiconjugate to* S *with respect to* ϕ. *If* $\mathcal{S}$ *is hypercyclic on* Y, *then* $\mathcal{T}$ *is hypercyclic on* X. *Furthermore,*

$$\phi(HC(\mathcal{S})) \subset HC(\mathcal{T}).$$

Proof. Suppose that $\mathcal{S}$ is hypercyclic, then there exists some $x \in Y$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that $\{S_{\underline{\omega}}^n x, n \in \mathbb{N}_0\}$ visits every nonempty open subset of Y. Let U be a nonempty open subset of X, then $\phi^{-1}(U)$ is a nonempty open subset of Y, implies that there exists some $n \in \mathbb{N}_0$ such that $S_{\underline{\omega}}^n x \in \phi^{-1}(U)$. This implies that $T_{\underline{\omega}}^n(\phi(x)) \in U$. Thus,

$$\overline{\{T_{\underline{\omega}}^n \phi(x) : n \in \mathbb{N}_0\}} = X.$$

Hence, $\mathcal{T}$ is hypercyclic and $\phi(x) \in HC(\mathcal{T})$. □

Corollary 2.6. *Let* X *and* Y *be topological spaces,* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is conjugate to* S *with respect to* ϕ. *Then* $\mathcal{T}$ *is hypercyclic on* X *if only if* $\mathcal{S}$ *is hypercyclic on* Y. *Furthermore,*

$$\phi(HC(\mathcal{S})) = HC(\mathcal{T})$$

Let $\{X\}_{i=1}^p$ be a family of topological spaces and let $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ be a random dynamical system on X_i, for all $i = 1,2,\dots,p$. Let

$$\oplus_{i=1}^p X_i = X_1 \oplus X_2 \oplus \cdots \oplus X_p = \{(x_1, x_2, \dots, x_p) : x_i \in X_i,\ i = 1,2,\dots,p\}$$

and define the random dynamical system $\oplus_{i=1}^p \mathcal{T}_i = \{(\oplus_{i=1}^p T_i)_\omega, \omega \in \Omega\}$ on $\oplus_{i=1}^p X_i$ by,

$$(\oplus_{i=1}^p T_i)_\omega : \oplus_{i=1}^p X_i \to \oplus_{i=1}^p X_i,\ (x_1, x_2, \dots, x_p) \mapsto (T_{1,\omega}x_1, T_{2,\omega}x_2, \dots, T_{p,\omega}x_p).\ (\forall \omega \in \Omega)$$

Remark 2.7. *For all* $\underline{\omega} \in \Omega^{\mathbb{N}_0}$, *for all* $n \in \mathbb{N}_0$, *and for all* $(x_1, x_2, \dots, x_p)$,

$$(\oplus_{i=1}^n T_i)_{\underline{\omega}}^n (x_1, x_2, \dots, x_p) = (T_{1,\underline{\omega}}^n x_1, T_{2,\underline{\omega}}^n x_2, \dots, T_{p,\underline{\omega}}^n x_p).$$

Proposition 2.8. *Let* $\{X\}_{i=1}^p$ *be a family of topological spaces and let* $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ *be a random dynamical system on* X_i *for all* $i = 1,2,\dots,p$. *If* $\oplus_{i=1}^p \mathcal{T}_i$ *is hypercyclic on* $\oplus_{i=1}^p X_i$, *then* $\mathcal{T}_i$ *is hypercyclic in* X_i *for all* $i = 1,2,\dots,p$. *Moreover if* $(x_1, x_2, \dots, x_p) \in HC(\oplus_{i=1}^p \mathcal{T}_i)$, *then* $x_i \in HC(\mathcal{T}_i)$ *for all* $i = 1,2,\dots,p$.

Proof. Suppose that $\oplus_{i=1}^{p}\mathcal{T}_i$ is hypercyclic on $\oplus_{i=1}^{p}X_i$. Let $(x_1, x_2, \dots, x_p) \in HC(\oplus_{i=1}^{p}\mathcal{T}_i)$, then there exists $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that

$$\overline{\{(\oplus_{i=1}^{p}T_i)_{\underline{\omega}}^{n}(x_1, x_2, \dots, x_p), n \in \mathbb{N}_0\}} = \oplus_{i=1}^{p}X_i,$$

For all $i = 1, 2, \dots, n$, let U_i be a nonempty open subset of X_i, then $U_1 \oplus U_2 \oplus \cdots \oplus U_p$ is a nonempty open subset of $\oplus_{i=1}^{p}X_i$, implies that there exists some $p \in \mathbb{N}_0$ such that

$$(\oplus_{i=1}^{p}T_i)_{\underline{\omega}}^{n}(x_1, x_2, \dots, x_p) = (T_{1,\underline{\omega}}^{n}x_1, T_{2,\underline{\omega}}^{n}x_2, \dots, T_{p,\underline{\omega}}^{n}x_p) \in U_1 \oplus U_2 \oplus \cdots \oplus U_p,$$

that is $T_{i,\underline{\omega}}^{p}x_i \in U_i$ for all $i = 1, 2, \dots, p$, it follows that

$$\overline{\{T_{i,\underline{\omega}}^{n}x_i, n \in \mathbb{N}_0\}} = X_i,$$

Hence $\mathcal{T}_i$ is hypercyclic in X_i and $x_i \in HC(\mathcal{T}_i)$, for all $i = 1, 2, \dots, p$.

□

Remark 2.9. *The converse of Proposition 2.8 is not true in general. Indeed, let $X = \{z \in \mathbb{C} : |z| = 1\}$ and $\Omega = \{0, 1\}$. We consider the maps $T_0 : X \to X$, $z \mapsto e^{i\alpha}z$, where α is irrational in $[0, 2\pi[$, and $T_1 = Id_X$. There exists $x \in X$, such that $\overline{\{T_0^n x, n \in \mathbb{N}_0\}} = X$, see [17]. Take $\underline{\omega} = (0, 0, 0, \dots)$, then $\overline{\{T_{\underline{\omega}}^n x, n \in \mathbb{N}_0\}} = X$, implies that, the random dynamical system $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is hypercyclic. But $\mathcal{T} \oplus \mathcal{T}$ is not hypercyclic.*

In the following definition, we introduce the notion of topological transitivity for a random dynamical system.

Definition 2.10. *Let X be a topological space, and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. We say that $\mathcal{T}$ is topologically transitive on X if: for any U and V nonempty open subsets of X, there exists $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $n \in \mathbb{N}_0$, such that*

$$T_{\underline{\omega}}^{n}(U) \cap V \neq \varnothing.$$

Remark 2.11. *Let X be a topological space, and $T : X \to X$ be a continuous map on X. Take $T_\omega = T$ for any $\omega \in \Omega$. Then $\{T_\omega\}_{\omega\in\Omega}$ is topologically transitive on X if only if T is a topologically transitive operator on X.*

Example 2.12. *Let $X = \{x \in \mathbb{C} : |x| = 1\}$ and $\Omega = \{0, 1\}$. Consider the maps: $T_0 : X \to X$, $x \mapsto e^{i\alpha}x$, where $\alpha \in \mathbb{R} - \mathbb{Q}$ and $T_1 : X \to X$, $x \mapsto T_1(x) = x^2$. For any U and V nonempty open subsets of X, there exists some $n \in \mathbb{N}_0$ such that $T_1^n(U) \cap V \neq \varnothing$. Take $\underline{\omega} = (1, 1, 1, \dots)$, then for any pair (U, V) of nonempty open subsets of X there exists some $n \in \mathbb{N}_0$, such that*

$$T_{\underline{\omega}}^{n}(U) \cap V \neq \varnothing.$$

Thus, the random dynamical $\mathcal{T} = \{T_0, T_1\}$ is topologically transitive on X.

The topological transitivity of a dynamical system is preserved under quasiconjugacy, see [10]. The following proposition proves that the same result holds for a random dynamical system.

Proposition 2.13. *Let X and Y be topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be random dynamical systems on X and Y respectivly, such that $\mathcal{T}$ is quasiconjugate to S. If $\mathcal{S}$ is topologically transitive on Y, then $\mathcal{T}$ is topologically transitive on X.*

Proof. Suppose that $\mathcal{S}$ is topologically transitive. Let U and V be nonempty open subsets of X, then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open of Y. Hence there exists $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $n \in \mathbb{N}_0$, such that

$$S_{\underline{\omega}}^{n}(\phi^{-1}(U)) \cap \phi^{-1}(V) \neq \varnothing.$$

This implies that

$$T_{\underline{\omega}}^{n}(U) \cap V \neq \varnothing.$$

Thus $\mathcal{T}$ is toplogically transitive. □

Corollary 2.14. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectivly, such that $\mathcal{T}$ is conjugate to S. Then $\mathcal{S}$ is topologically transitive on Y if only if $\mathcal{T}$ is topologically transitive on X.*

Proposition 2.15. *Let $\{X_i\}_{i=1}^n$ be a family of topological spaces and let $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ be a random dynamical system on X_i, for all $i = 1, 2, \dots, n$. If $\oplus_{i=1}^n \mathcal{T}_i$ is topologically transitive in $\oplus_{i=1}^n X_i$, then $\mathcal{T}_i$ is topologically transitive in X_i, for all $i = 1, 2, \dots, n$.*

Proof. Suppose that $\oplus_{i=1}^n \mathcal{T}_i$ is topologically transitive. Let U_i and V_i be nonempty open subsets of X_i; $1 \le i \le n$. Then, $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ are nonempty open subsets of $\oplus_{i=1}^n X_i$, which implies that there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $p \in \mathbb{N}_0$ such that

$$(\oplus_{i=1}^n T_i)^p_{\underline{\omega}}(U_1 \oplus U_2 \oplus \cdots \oplus U_n) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing$$

then

$$(T^p_{1,\underline{\omega}}(U_1) \oplus T^p_{2,\underline{\omega}}(U_2) \oplus \cdots \oplus T^p_{n,\underline{\omega}}(U_n)) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing,$$

it follows that

$$T^p_{i,\underline{\omega}}(U_i) \cap V_i \neq \varnothing \text{ for any } i = 1, 2, \dots, n.$$

Thus, $\mathcal{T}_i$ is topologically transitive on X_i, for all $i = 1, 2, \dots, n$.

□

Remark 2.16. *The converse is not true. Let $X = \{z \in \mathbb{C} : |z| = 1\}$ and $\Omega = \{0, 1\}$. We consider the maps $T_0 : X \to X$, $z \mapsto e^{i\alpha} z$, where α is irrational in $[0, 2\pi[$, and $T_1 = Id_X$. There exists $x \in X$, such that*

$$\{\overline{T_0^n x, n \in \mathbb{N}_0}\} = X.$$

Then the random dynamical system $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is hypercyclic on X, but $\mathcal{T} \oplus \mathcal{T}$ is not hypercyclic in $X \oplus X$.

By the Birkhoff's transitivity theorem [12], if X is a separable F-space, then a continuous map on X is hypercyclic if and only if it is topologically transitive. For $\mathcal{T}$ a random dynamical system we have the following remark. Recall that

$$\begin{array}{cccccc} \sigma & : & \Omega^{\mathbb{N}_0} & \longrightarrow & \Omega^{\mathbb{N}_0} \\ & & (\omega_1, \omega_2, \dots) & \longmapsto & \sigma\underline{\omega} = (\omega_2, \omega_3, \dots). \end{array}$$

the full shift in $\Omega^{\mathbb{N}_0}$.

Remark 2.17. *Let X be a topological space without isolated points and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. It is easy to see that if $x \in HC(\mathcal{T})$ with $\underline{\omega}$, then so is every $T_{\sigma^p \underline{w}} x$ $(p \ge 1)$. As a result, we have*

$$\operatorname{Orb}(x, \mathcal{T}) \subset HC(\mathcal{T})$$

and this shows that $HC(\mathcal{T})$ is dense in X.

In the following proposition, we prove that if $\mathcal{T}$ is hypercyclic then it is topologically transitive.

Proposition 2.18. *Let X be a topological space and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X, such that for any $\omega \in \Omega$, T_ω is a continuous map on X. If $\mathcal{T}$ is hypercyclic on X, then it is topologically transitive on X.*

Proof. Suppose that $\mathcal{T}$ is hypercyclic. Then there exists some $x \in X$ and $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that

$$\overline{\{T^n_{\underline{\omega}} x, n \in \mathbb{N}_0\}} = X.$$

Let U and V be two nonempty open subsets of X, then there exists some $p, n \in \mathbb{N}_0$, such that $T^p_{\underline{\omega}}x \in U$ and $T^n_{\underline{\omega}}x \in V$. Suppose that $n \geqslant p$, then $T^n_{\underline{\omega}}x = T^{n-p}_{\sigma^p\underline{\omega}} \circ T^p_{\underline{\omega}}x$, which implies that,

$$T^{n-p}_{\sigma^p\underline{\omega}}(U) \cap V \neq \varnothing.$$

Hence, $\mathcal{T}$ is topologically transitive.

□

With some additional assumptions on the topological space, the following theorem shows that we have the equivalence between the properties of hypercyclicity and topological transitivity.

Theorem 2.19. *Let X be a separable complete metric space X without isolated points. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. Then $\mathcal{T}$ is topologically transitive on X if only if it is hypercyclic on X.*

Proof. Let $\{U_k\}_{k\geq 1}$ be a countable base for the topology of X. Then there is some $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ such that for any nonempty open set V in X and each fixed $k \geq 1$, there is some $n \geq 0$ such that

$$T^n_{\underline{\omega}}(V) \cap U_k \neq \varnothing$$

or equivalently

$$V \cap T^{-n}_{\underline{\omega}}(U_k) \neq \varnothing$$

This shows that $\bigcup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$ is dense in X and hence, since X is a Baire space, the set $\cap_{k\geq 1} \cup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$ is also dense in X. Now, if we define the set

$$D_{\underline{\omega}}(\mathcal{T}) = \{x \in X : \overline{\{T^n_{\underline{\omega}}x : n \in \mathbb{N}_0\}} = X\},$$

then it is easy to see that

$$D_{\underline{\omega}}(\mathcal{T}) = \bigcap_{k\geq 1}\bigcup_{n\geq 0} T^{-n}_{\underline{\omega}}(U_k)$$

Thus, is a dense G_δ set in X and in particular nonempty. So, $\mathcal{T}$ is hypercyclic and we are done. □

3. Topological mixing and Weakly Topological Mixing Random Dynamical Systems

In the following definition, we introduce the notion of topological mixing for a random dynamical system.

Definition 3.1. *Let X be a topological space, and $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ be a random dynamical system on X. We say that $\mathcal{T}$ is topologically mixing on X if, for any U and V nonempty open subsets of X, there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that*

$$T^n_{\underline{\omega}}(U) \cap V \neq \varnothing, \text{ for all } n \geqslant N.$$

Remark 3.2. *Let X be a topological space, and $T : X \to X$ be a continuous map on X. Take $T_\omega = T$ for any $\omega \in \Omega$. Then $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ is topologically mixing on X if only if T is a topologically mixing operator on X.*

Example 3.3. *We pose $X = [0, 1]$ and $\Omega = \{0, 1\}$, and we consider the maps: $T_1 : X \to X$,*

$$x \mapsto \begin{cases} 2x & if\ x \in [0, \frac{1}{2}] \\ 2 - 2x & if\ x \in]\frac{1}{2}, 1] \end{cases}$$

and $T_2 : X \to X$,

$$x \mapsto T_2(x) = x + \alpha (mod\ 1),$$

with $\alpha \in [0, 1[$. For any U and V of nonempty open subsets of X there exists some $N \in \mathbb{N}_0$, such that

$$T_1^n(U) \cap V \neq \varnothing, \text{ for all } n \geqslant N,$$

see [17]. Take $\underline{\omega} = (1,1,1,\dots)$*, then for any pair* (U,V) *of nonempty open subsets of* X *there exists some* $N \in \mathbb{N}_0$*, such that* $T_{\underline{\omega}}^n(U) \cap V \neq \varnothing$*, for all* $n \geqslant N$ *, hence* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *is topologically mixing on* X*.*

Proposition 3.4. *Let* X *and* Y *be two topological spaces. Let* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be two random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is quasi-conjugate to* S *with respect to* ϕ*. If* $\mathcal{S}$ *is topologically mixing on* Y*, then* $\mathcal{T}$ *is topologically mixing on* X*.*

Proof. Suppose that $\mathcal{S}$ is topologically mixing on X. Let U and V be two nonempty open subsets of X, then $\phi^{-1}(V)$ and $\phi^{-1}(U)$ are nonempty and open in Y. Hence there exist $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that

$$S_{\underline{\omega}}^n(\phi^{-1}(U)) \cap \phi^{-1}(V) \neq \varnothing \text{ for all } n \geqslant N,$$

which implies that

$$T_{\underline{\omega}}^n(U) \cap V \neq \varnothing \text{ for all } n \geqslant N.$$

Thus $\mathcal{T}$ is topologically mixing. □

Corollary 3.5. *Let* X *and* Y *be two topological spaces. Let* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *and* $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ *be two random dynamical systems on* X *and* Y *respectively, such that* $\mathcal{T}$ *is conjugate to* S*. Then* $\mathcal{T}$ *is topologically mixing on* X *if only if* $\mathcal{S}$ *is topologically mixing on* Y*.*

Proposition 3.6. *Let* $\{X_i\}_{i=1}^n$ *be a family of topological spaces, and let* $\mathcal{T}_i = \{T_{i,\omega} : \omega \in \Omega\}$ *be a random dynamical system on* X_i*, for all* $i = 1,2,\dots,n$*. If* $\oplus_{i=1}^n \mathcal{T}_i$ *is topologically mixing on* $\oplus_{i=1}^n X_i$*, then* $\mathcal{T}_i$ *is topologically mixing in* X_i *for all* $i = 1,2,\dots,n$*.*

Proof. Suppose that $\oplus_{i=1}^n \mathcal{T}_i$ is topologically mixing. Let U_i and V_i be nonempty open subsets of X_i; $1 \leq i \leq n$. Then $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ are nonempty open subsets of $\oplus_{i=1}^n X_i$, which implies that there exists $\underline{\omega} \in \Omega^{\mathbb{N}_0}$ and $N \in \mathbb{N}_0$, such that

$$(\oplus_{i=1}^n T_i)_{\underline{\omega}}^p(U_1 \oplus U_2 \oplus \cdots \oplus U_n) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing, \text{ for all } p \geqslant N.$$

Then

$$(T_{1,\underline{\omega}}^p(U_1) \oplus T_{2,\underline{\omega}}^p(U_2) \oplus \cdots \oplus T_{n,\underline{\omega}}^p(U_n)) \cap (V_1 \oplus V_2 \oplus \cdots \oplus V_n) \neq \varnothing, \text{ for all } p \geqslant N.$$

It follows that

$$T_{i,\underline{\omega}}^p(U_i) \cap V_i \neq \varnothing,$$

for all $p \geqslant N$, for any $i = 1,2,\dots,n$. Thus, $\mathcal{T}_i$ is topologically mixing on X_i for all $i = 1,2,\dots,n$. □

In the following definition, we introduce the notion of weakly topologically mixing for a random dynamical system.

Definition 3.7. *Let* X *be a topological space. A random dynamical system* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *is called weakly topologically mixing on* X*, if* $\mathcal{T} \oplus \mathcal{T}$ *is topologically transitive on* $X \oplus X$*.*

Proposition 3.8. *Let* X *be a topological space, and* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *be a random dynamical system on* X*. If* $\mathcal{T}$ *is weakly topologically mixing on* X*, then it is topologically transitive on* X*.*

Proof. This is a consequence of Proposition 2.15. □

Remark 3.9. *Let* X *be a topological space, and* $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ *be a random dynamical system on* X*, then*

topologically mixing $\Rightarrow$ *weak topologically mixing* $\Rightarrow$ *topologically transitive.*

Furthermore, if X is a separable complete metric space without isolated points, then

$$\text{topologically transitive} \Leftrightarrow \text{hypercyclic}.$$

Proposition 3.10. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectivly such that $\mathcal{T}$ is quasiconjugate to $\mathcal{S}$. If $\mathcal{S}$ is weakly topologically mixing on Y then $\mathcal{T}$ is weakly topologically mixing on X.*

Proof. Suppose that $\mathcal{S}$ is weakly topologically mixing on Y, then $\mathcal{S} \oplus \mathcal{S}$ is topologically transitive in X. Let $\phi : Y \to X$ be a continuous map with dense range such that for all $\omega \in \Omega$, $T_\omega \circ \phi = \phi \circ S_\omega$. Take $\psi = \phi \oplus \phi$, then ψ defines a continuous map with dense range from $Y \oplus Y$ to $X \oplus X$. Furthermore, for all $\omega \in \Omega$ we have $\psi \circ (S \oplus S)_\omega = (T \oplus T)_\omega \circ \psi$. That is $\mathcal{S} \oplus \mathcal{S}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{T}$ via ψ. Hence by Proposition (2.13), we deduce that $\mathcal{T} \oplus \mathcal{T}$ is topologically transitive on $X \oplus X$. Thus $\mathcal{T}$ is weakly topologically mixing on X.

□

Corollary 3.11. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectively such that $\mathcal{T}$ is conjugate to $\mathcal{S}$. Then $\mathcal{S}$ is weakly topologically mixing on X if only if $\mathcal{T}$ is weakly topologically mixing on Y.*

Proposition 3.12. *Let X and Y be two topological spaces. Let $\mathcal{T} = \{T_\omega\}_{\omega\in\Omega}$ and $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ be two random dynamical systems on X and Y respectively. If $\mathcal{T} \oplus \mathcal{S}$ is weakly topologically mixing on $X \oplus Y$, then $\mathcal{T}$ and $\mathcal{S}$ are topologically weakly mixing on X and Y respectively.*

Proof. Suppose that $\mathcal{T} \oplus \mathcal{S}$ is weakly mixing. We consider the maps, $\phi : X \oplus Y \to X$, $(x, y) \mapsto x$ and $\psi : X \oplus Y \to X$, $(x, y) \mapsto y$. For all $\omega \in \Omega$ we have $\phi \circ (T \oplus S)_\omega = T_\omega \circ \phi$ and $\psi \circ (T \oplus S)_\omega = S_\omega \circ \psi$, then $\mathcal{T}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{S}$ via ϕ and $\mathcal{S}$ is quasiconjugate to $\mathcal{T} \oplus \mathcal{S}$ via ψ. Thus, by Proposition 3.10 we deduce that $\mathcal{T}$ and $\mathcal{S}$ are weakly topologically mixing on X and Y respectively.

□

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Ahmed Zaou, Otmane Benchiheb and Mohamed Amouch,
Department of Mathematics,
University Chouaib Doukkali,
Faculty of science,
Eljadida, Morocco .
E-mail address: ahmed.zaou1991@gmail.com
E-mail address: otmane.benchiheb@gmail.com
E-mail address: amouch.m@ucd.ac.ma