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New Theorems on Theta-Function Analogues and Explicit Evaluations of Ramanujan's Remarkable Product

S. Vasanth Kumar

ABSTRACT: We define $V_{m,n}$ in this article by using Ramanujan's product of theta functions $\psi(-q)$ and $f(-q^2)$, which are analogues to Ramanujan's amazing product of theta functions. For explicit evaluations of $V_{m,n}$, we prove general theorems.

Key Words: Class invariant, Modular equation, Theta function, Cubic continued fraction.

Contents

1. Introduction

Ramanujan's general theta-function [15] $f(a,b)$ is defined by

$$\begin{aligned} f(a,b) :&= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \end{aligned} \tag{1.1}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

Ramanujan defines [4,15], on page 338 of his first notebook.

$$a_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.5}$$

He then offered a list of eighteen specific values on pages 338 and 339. Berndt, Chan, and Zhang [5] established all eighteen of these values. Naika et al. [7] also proved some general theorems for explicit

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evaluations of the $a_{m,n}$ and discovered some new explicit values from it. For more values of $a_{m,n}$, one can see [9,10]. Nipen Saikia [13] recently established new characteristics of $a_{m,n}$.

Naika et al. [12] have defined $b_{m,n}$ as,

$$b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

They proved some specific values and established general theorems for explicit evaluation of $b_{m,n}$. In terms of the products of $a_{m,n}$ and $b_{m,n}$ defined above, M. S. M. Naika et al. [11] developed generic formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$,

$$V(q) := \frac{q^{1/3}}{1}\underset{+}{}\frac{q+q^2}{1}\underset{+}{}\frac{q^2+q^4}{1}\underset{+}{}\frac{q^3+q^6}{1}\underset{+}{}\cdots, \quad |q|<1, \tag{1.7}$$

and found some particular values of $V(q)$.

$$V_{m,n} = \frac{f(-e^{-2\pi\sqrt{\frac{n}{m}}})\psi(-e^{-\pi\sqrt{mn}})}{e^{\frac{-\pi(1-m)}{24}\sqrt{\frac{n}{m}}}f(-e^{-2\pi\sqrt{mn}})\psi(-e^{-\pi\sqrt{\frac{n}{m}}})}, \tag{1.8}$$

where m and n are real values that are positive. We establish numerous properties of the $V_{m,n}$. We find explicit values for $V_{m,n}$ and prove generic formulas for explicit evaluations.

The complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.9}$$

Then (1.9) induces a modular equation of degree n, which is a relation between k and l. Set $\alpha = k^2$ and $\beta = l^2$ as suggested by Ramanujan. Then we say that β has a degree n over than α.

Define

$$\chi(q) := (-q;q^2)_\infty$$

and

$$G_n := 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\chi(q),$$

where

$$q = e^{-\pi\sqrt{r}}.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{m}{n}}}$ and β has degree n over α, then

$$G_{\frac{n}{m}} = (4\alpha(1-\alpha))^{\frac{-1}{24}} \tag{1.10}$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{-1}{24}}. \tag{1.11}$$

The major goal of this study is to derive numerous general theorems for explicit evaluations of $V_{m,n}$ and Ramanujan's product of theta-function $V_{m,n}$ analogues, as well as several new explicit evaluations.

2. Preliminary Results

We collect various identities in this section that will help us to prove our essential results.

Lemma 2.1 *[2, Ch. 17, Entry 11(ii) and Entry 12(iii), pp. 123–124] We have*

$$2^{1/2}\, e^{-\alpha/8}\, \psi(-e^{-\alpha}) = \sqrt{z_1}\, \{\alpha(1-\alpha)\}^{1/8}, \tag{2.1}$$

$$2^{1/2}\, e^{-m\alpha/8}\, \psi(-e^{-m\alpha}) = \sqrt{z_m}\, \{\beta(1-\beta)\}^{1/8}, \tag{2.2}$$

$$2^{1/3}\, e^{-\alpha/12}\, f(e^{-2\alpha}) = \sqrt{z_1}\, \{\alpha(1-\alpha)\}^{1/12}, \tag{2.3}$$

$$2^{1/3}\, e^{-m\alpha}\, f(e^{-2m\alpha}) = \sqrt{z_m}\, \{\beta(1-\beta)\}^{1/12}. \tag{2.4}$$

Lemma 2.2 *[2, Ch. 16, Entry 27(iii) and (iv), pp. 43] We have*

$$\frac{\psi(-e^{-\alpha})}{\psi(-e^{-\beta})} = \sqrt[4]{\frac{\beta}{\alpha}}\, e^{\frac{\alpha-\beta}{4}} \quad if\ \alpha\beta=\pi^2 \tag{2.5}$$

$$\frac{f(-e^{-2\alpha})}{f(-e^{-2\beta})} = \sqrt[4]{\frac{\beta}{\alpha}}\, e^{\frac{\alpha-\beta}{12}} \quad if\ \alpha\beta=\pi^2. \tag{2.6}$$

Lemma 2.3 *[6, Theorem 2.1] We have,*

$$\frac{f^{12}(-q^2)}{f^{12}(-q^6)} = \frac{\psi^8(q)}{\psi^8(q^3)}\left\{\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}\right\}. \tag{2.7}$$

Lemma 2.4 *[16] [14] We have,*

$$\frac{f^6(-q^2)}{f^6(-q^{10})} = \frac{\psi^4(q)}{\psi^4(q^5)}\left\{\frac{\psi^2(q)-5q\psi^2(q^{10})}{\psi^2(q)-q\psi^2(q^5)}\right\}. \tag{2.8}$$

Lemma 2.5 *[6, Theorem 2.2] We have,*

$$\frac{f^3(-q^2)}{f^3(-q^{18})} = \frac{\psi^2(q)}{\psi^2(q^9)}\left\{\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}\right\}. \tag{2.9}$$

Lemma 2.6 *[2, Chapter 19, entry 5(xii), page 231] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$, then

$$Q+\frac{1}{Q} = 2\sqrt{2}(\frac{1}{P}-P), \tag{2.10}$$

where β is of degree 3 over α.

Lemma 2.7 *[2, Chapter 19, entry 13(xiv), page 282] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ and $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$, then

$$Q+\frac{1}{Q} = 2(\frac{1}{P}-P), \tag{2.11}$$

where β is of degree 5 over α.

Lemma 2.8 *[2, Chapter 19, entry 19(ix), page 315] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/6}$, then

$$Q+\frac{1}{Q}+7 = 2\sqrt{2}(P+\frac{1}{P}), \tag{2.12}$$

where β is of degree 7 over α.

Lemma 2.9 *[1, Theorem 5.1]*
If $P = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $Q = \dfrac{\varphi(q)}{\varphi(q^3)}$, then

$$Q^4(1+P^4) = 9+P^4. \tag{2.13}$$

Lemma 2.10 *[1, Theorem 5.3]*
If $P^2 = \dfrac{\psi^2(-q)}{q\psi^2(-q^5)}$ and $Q^2 = \dfrac{\varphi^2(q)}{\varphi^2(q^5)}$, then

$$Q^2(1+P^2) = 5+P^2. \tag{2.14}$$

Lemma 2.11 *[8, Theorem 3.2]*
If $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ and $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, then

$$Q+PQ = 3+P. \tag{2.15}$$

3. Some Properties of $V_{m,n}$

In this section, we prove some properties of $V_{m,n}$.

Theorem 3.1 *We have*

$$V_{m,n} = V_{n,m}. \tag{3.1}$$

Proof: Employing the equation (2.5) and (2.6), in (1.8), we obtain (3.1). □

Theorem 3.2 *We have*

$$V_{m,n}V_{m,1/n} = 1. \tag{3.2}$$

Proof: Using the equations (2.5) and (2.6) in (1.8), we obtain (3.2). □

Corollary 3.1 *We have*

$$V_{m,1} = 1. \tag{3.3}$$

Proof: Putting $n = 1$ in the equation (3.2), we get (3.3) □

Remark 3.1 *It can be seen that $V_{m,n}$ has a positive real value and that the values of $V_{m,n}$ rise as n grows when $m > 1$ using the definitions of $\psi(-q)$, $f(-q^2)$ and $V_{m,n}$. As a result of the aforementioned corollary, $V_{m,n} > 1$ for any $n > 1$ if $m > 1$.*

Theorem 3.3 *We have*

$$\frac{V_{km,n}}{V_{nm,k}} = V_{m,\frac{n}{k}}. \tag{3.4}$$

Proof: Using the $V_{m,n}$ definition, we obtain

$$\frac{V_{km,n}}{V_{nm,k}} = e^{\frac{\pi\left(\sqrt{\frac{k}{mn}}-\sqrt{\frac{n}{mk}}\right)}{24}} \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{mk}}}\right)\psi\left(-e^{\pi\sqrt{\frac{k}{mn}}}\right)}{f\left(-e^{-2\pi\sqrt{\frac{k}{mn}}}\right)\psi\left(-e^{-\pi\sqrt{\frac{n}{mk}}}\right)}. \tag{3.5}$$

Using the Lemma **2.2** in the above equation (3.5) and simplifying using the Theorems **3.1 and 3.2** , we obtain (3.4). □

Corollary 3.2 *We have*

$$V_{m^2,n} = V_{mn,m}V_{m,\frac{n}{m}}. \tag{3.6}$$

Proof: Putting $k = m$ in the Theorem **3.3** and simplifying using (3.2), we obtain (3.6). □

Theorem 3.4 *If $mn = rs$*

$$\frac{V_{m,n}}{V_{kr,ks}} = \frac{V_{r,s}}{V_{km,kn}} \tag{3.7}$$

Proof: Using the $V_{m,n}$ definition and the $mn = rs$ formula for positive real numbers m, n, r, s, and k, we find that

$$\frac{V_{km,kn}}{V_{m,n}} = \frac{V_{kr,ks}}{V_{r,s}}. \tag{3.8}$$

On rearranging the above equation (3.8), we obtain the required result. □

Corollary 3.3 *If $mn = rs$*

$$V_{np,np} = V_{np^2,n}V_{p,p} \tag{3.9}$$

Proof: Letting $m = p^2$, $n = 1$, $r = s = p$ and $k = n$ in above Theorem (3.4), we deduced the equation (3.9). □

Theorem 3.5 *If m, n, r, and s are all positive real values, then*

$$V_{m/n,r/s} = \frac{V_{ms,nr}}{V_{mr,ns}} \tag{3.10}$$

Proof: We find that, for any positive real values m, n, and k, using the equation (3.2) in equation (3.4),

$$V_{m/n,k} = V_{m,nk} V_{n,mk}^{-1}. \tag{3.11}$$

Letting $k = r/s$ and again using the equation (3.4) and (3.1) in (3.11), we get (3.10). □

Theorem 3.6 *We have*

$$V_{m/n,m/n} = V_{n,n} V_{m,m/n^2}. \tag{3.12}$$

Proof: Using the Theorems (3.2) and (3.5), we get (3.12). □

Theorem 3.7 *We have*

$$V_{m,m} V_{m,n^2/m} = V_{n,n} V_{m,n^2/m}. \tag{3.13}$$

Proof: Putting $k = m/n$ in the equation (3.11) and employing Theorems (3.2) and (3.6), we obtain (3.13). □

Theorem 3.8 *We have*

$$V_{m,m} V_{n,m^2 n} = V_{n,n} V_{m,mn^2}. \tag{3.14}$$

Proof: Employing the Theorems (3.1), (3.2), (3.6) and (3.7), we obtain (3.14). □

4. Some General Theorems on $V_{m,n}$ and their explicit evaluations

We develop some general theorems on $V_{m,n}$ and their explicit evaluations in this section.

Theorem 4.1 *If $P := \{G_{n/3} G_{3n}\}^{-3}$ and $Q := V_{3,n}^6$, then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left\{P - \frac{1}{P}\right\}. \tag{4.1}$$

Proof: Using the Lemma (2.1) with the definition of $V_{m,n}$, we obtain

$$V_{m,n} = \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/24}. \tag{4.2}$$

Employing the above equation (4.2) and the definition of class invariant (1.10), (1.11) in the Lemma (2.6) with $m = 3$, we obtain (4.1) □

Corollary 4.1 *We have*

$$V_{3,9} = \left\{\sqrt[3]{2} - 1\right\}^{1/3}. \tag{4.3}$$

Proof: Putting $n = 9$ in the above Theorem (4.1), we obtain

$$V_{3,9}^{6} + V_{3,9}^{-6} = 2\sqrt{2}\left\{G_3^3 G_{27}^3 - G_3^{-3} G_{27}^{-3}\right\}. \tag{4.4}$$

Solving the above equation (4.4) with from the table of Chapter 34 of Ramanujan notebooks [4, p.189,190] $G_3 = 2^{1/12}$ and $G_{27} = 2^{1/12}\left(\sqrt[3]{2} - 1\right)^{-1/3}$, we obtain (4.3). □

Theorem 4.2 *If* $P := \{G_{n/5} G_{5n}\}^2$ *and* $Q := V_{5,n}^3$, *then*

$$Q + \frac{1}{Q} = 2\left\{P - \frac{1}{P}\right\}. \tag{4.5}$$

Proof: Using the equation (4.2) and the definition of class invariant (1.10), (1.11) in the Lemma (2.7) with $m = 5$, we obtain (4.5). □

Theorem 4.3 *If* $P := \{G_{n/7} G_{7n}\}^3$ *and* $Q := V_{7,n}^4$, *then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left\{P + \frac{1}{P}\right\}. \tag{4.6}$$

Proof: Using the equation (4.2) and definition of class invariant (1.10), (1.11) in the Lemma (2.8) with $m = 7$, we obtain (4.6). □

Theorem 4.4 *If*

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q := \frac{f(-q^2)}{q^{1/6} f(-q^6)}, \quad \textit{then} \tag{4.7}$$

$$V_{3,n}^{12} = \frac{P^4 + 9}{P^4(1 + P^4)}, \qquad P^4 \neq -1. \tag{4.8}$$

Proof: Employing the definition of $V_{m,n}$ with $m = 3$, we get

$$V_{3,n} = \frac{f(-q^2)\psi(-q^3)}{q^{-1/12} f(-q^6)\psi(-q)}. \tag{4.9}$$

Using (4.9) in the Lemma (2.3) we obtain (4.8). □

Corollary 4.2 *We have*

$$V_{3,3} = \left\{2 - \sqrt{3}\right\}^{1/6}. \tag{4.10}$$

Proof: Putting $n = 3$ in the equation (4.7) and from Notebooks [4, p. 327], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3} - 9}. \tag{4.11}$$

Employing the equation (2.13) and (4.11), we obtain

$$P := \frac{\psi(-e^{-\pi})}{q^{1/4}\psi(-e^{-3\pi})} = \sqrt[4]{9 + 6\sqrt{3}}. \tag{4.12}$$

Substituting (4.12) in (4.8), we obtain the required result. □

Theorem 4.5 *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad and \quad Q := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}, \quad then \tag{4.13}$$

$$V_{5,n}^6 = \frac{P^2+5}{P^2(P^2+1)}, \qquad P^2 \neq -1. \tag{4.14}$$

Proof: Employing the definition of $V_{m,n}$ with $m = 5$, we get

$$V_{5,n} = \frac{f(-q^2)\psi(-q^5)}{q^{-1/6}f(-q^{10})\psi(-q)}. \tag{4.15}$$

Using (4.15) in the Lemma (2.7) we obtain (4.14). □

Corollary 4.3 *We have*

$$V_{5,5} = 9 - 4\sqrt{5}. \tag{4.16}$$

Proof: Putting $n = 5$ in the equation (4.13) and from Notebooks [4, p. 327], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5}-10}. \tag{4.17}$$

Employing the equation (2.14) and (4.17), we obtain

$$P := \frac{\psi(-e^{-\pi})}{q^{1/2}\psi(-e^{-5\pi})} = \sqrt{5\sqrt{5}+10}. \tag{4.18}$$

Substituting (4.18) in (4.14), we obtain the required result. □

Theorem 4.6 *If*

$$P := \frac{\psi(-q)}{q\psi(-q^9)} \quad and \quad Q := \frac{f(-q^2)}{q^{2/3}f(-q^{18})}, \quad then \tag{4.19}$$

$$V_{9,n}^3 = \left\{\frac{P+3}{P(P+1)}\right\}, \qquad P \neq -1. \tag{4.20}$$

Proof: Employing the definition of $V_{m,n}$ with $m = 9$, we get

$$V_{9,n} = \frac{f(-q^2)\psi(-q^9)}{q^{-1/3}f(-q^{18})\psi(-q)}. \tag{4.21}$$

Using (4.21) in the Lemma (2.8) we obtain (4.20). □

Corollary 4.4 *We have*

$$V_{9,9} = \left\{\frac{2-u}{u+u^2}\right\}^{1/3}. \tag{4.22}$$

where $u = \sqrt[3]{2\sqrt{3}+2}$

Proof: Putting $n = 9$ in the equation (4.19) and from Notebooks [4, p. 327], we have

$$Q := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1+\left\{2(\sqrt{3}+1)\right\}^{1/3}}. \tag{4.23}$$

Employing the equation (2.15) and (4.23), we obtain

$$P := \frac{\psi(-e^{-\pi})}{q\psi(-e^{-9\pi})} = \frac{3u}{2-u}. \tag{4.24}$$

Substituting (4.24) in (4.20), we obtain the required result. □

Theorem 4.7 *We have*

$$V_{m,n} = \left\{ \frac{G_{n/m}}{G_{mn}} \right\}. \tag{4.25}$$

Proof: Employing the Lemma **2.1** in the definition of $V_{m,n}$, we obtain

$$V_{m,n} = \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/24}. \tag{4.26}$$

Using the equation (1.10) and (1.11), we get

$$\frac{G_{nm}}{G_{n/m}} = \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{4.27}$$

By observing the equations (4.26) and (4.27), we obtain (4.25). □

Corollary 4.5 *We have*

$$V_{n,n} = G_{n^2}^{-1}. \tag{4.28}$$

Proof: Setting $m = n$ in the above Theorem (4.7) with the value $G_1 = 1$, we obtain required result. □

Corollary 4.6 *We have*

$$V_{2,2} = \frac{2^{3/16}}{\left\{1+\sqrt{2}\right\}^{1/4}}, \tag{4.29}$$

$$V_{3,3} = \left\{2-\sqrt{3}\right\}^{1/6}, \tag{4.30}$$

$$V_{5,5} = \frac{\sqrt{5}-1}{2}, \tag{4.31}$$

$$V_{9,9} = \left\{ \frac{1+(2(-1+3^{1/3}))^{1/3}}{1-(2(1+3^{1/3}))^{1/3}} \right\}^{-1/3}. \tag{4.32}$$

Proof: For (i), we use the values of G_4 from [4, p.114, Theorem 6.2.2(ii)]. For $(ii)-(iv)$, we use corresponding values of G_n from [2, p.189-193]. □

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References

1. C. Adiga, Taekyun Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, Indian J. pure appl. math., 35(9) , 1047–1062, (2004).
2. B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, (1991).
3. B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York, (1994).
4. B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, (1997).
5. B. C. Berndt, H. H. Chan and L.-C. Zhang, Ramanujan's remarkable product of the theta-function, Proc. Edinburgh Math. Soc., 40, 583-612, (1997).
6. M. S. Mahadeva Naika, Some theorems on Ramanujan's cubic continued fraction and related identities, Tamsui Oxf. J. Math. Sci., 24(3) , 243–256, (2008).

7. M. S. Mahadeva Naika and B. N. Dharmendra, On some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-function, Ramanujan J., 15(3), 349-366, (2008).

8. M. S. Mahadeva Naika, K. Sushan Bairy and S. Chandankumar, On Some Explicit evaluation of the ratios of Ramanujan's theta–function, Bull. Allahabad Math. Soc., 29(1), 35–86, (2014).

9. M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankara, On some new explicit evaluations of Ramanujan's remarkable product of theta-function, South East Asian J. Math. Math. Sci., 5(1), 107-119, (2006)

10. M. S. Mahadeva Naika and M. C. Maheshkumar, Explicit evaluations of Ramanujan's remarkable product of theta-function, Adv. Stud. Contemp. Math., 13(2), 235-254, (2006).

11. M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, General formulas for explicit evaluations of Ramanujan's cubic continued fraction, Kyungpook Math. J., 49(3) , 435–450, (2009).

12. M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, On some remarkable product of theta-function, Aust. J. Math. Anal. Appl., 5(1), 1-15, (2008).

13. Nipen Saikia, Some Properties, Explicit Evaluation and Applications of Ramanujan's Remarkable Product of Theta-Functions, Acta Math Vietnam Journal of Mathematics, DOI 10.1007/s40306-014-0106-8, (2015).

14. S. -Y. Kang, Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook, Ramanujan J., 3(1) , 91–11, (1999).

15. S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, (1957).

16. S. Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi, (1988).

Dr Vasanth Kumar S,

Head and Associate Professor,

Department of Mathematics,

Seshadripuram Institute of Technology, Mysuru,

India.

E-mail address: svkmaths86@gmail.com